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EQUIVALENCE THEOREMS

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1. Introduction. This is a revised and slightly expanded version of a lecture given at the workshop on abelian groups held at the University of Connecticut in October 1989. The lecture was actually a general departmental colloquium held during the workshop, and therefore the exposition is as nontechnical as possible. Because it is intended for a more general audience, this paper is more basic and perhaps a little less formal than most publications.

To begin, I would like to ask my audience and readers, in order to be properly motivated, to accept principally the premise set forth in the following quotation from Thomas Hungerford's *Algebra*. "Ideally the goal in studying groups is to classify all groups up to isomorphism, which in practice means finding necessary and sufficient conditions for two groups to be isomorphic."

If the above level of achievement is thought of as the pinnacle of the theory of groups, there certainly are many other high points of interest that surround and support the pinnacle. The whole mass, as it were from the foot of the mountain to the top, dealing directly with the question of when two groups are isomorphic or are related in some weaker sense is known as structure theory. This is the oldest and most settled side of the mountain (of abelian group theory), particularly for torsion groups. The other (some say softer) side of the mountain is the homological face, which was first explored seriously by D. Harrison [**11**] in 1959; it should be understood that we are going to be dealing exclusively with abelian groups. However, before we decree that all groups are abelian, let us mention in passing that although the strategies and techniques employed by abelian group theorists and nonabelian group theorists are quite different, the two camps share the common goal, if on different paths, of reaching the pinnacle identified above. Consider, for example, the effort in the 60's and 70's to classify finite groups.

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Henceforth, all groups discussed here are abelian and are written additively. Finite groups were classified over a hundred years ago, and H. Prüfer laid a large part of the foundation of the structure theory of infinite abelian groups in the 1920's. But it was H. Ulm and L. Zippin who first reached the mountain's pinnacle (for countable p-groups) in the 1930's; see, for example, [**8**].

Typically, we try to show that two groups are isomorphic if and only if they have certain numerical invariants that agree. It should be noted here that we are heavily indebted to I. Kaplansky and G. Mackey for identifying and mapping out this route. Not only did they reach Ulm's theorem in a straight way, but nearly all who have journeyed after them toward the classification problem have followed this path—at least for a portion of the way. Indeed, current standards for the classification of groups virtually insist on numerical invariants [**16**].

What are these numerical invariants and how do we find them? It is, it seems, almost a characteristic feature of abelian groups that they possess or produce a large variety of vector spaces over the various prime fields, and these spaces in many cases are hereditarily linked to the sustaining group (in the sense that they retain or reveal structural characteristics). Principally for the benefit of the nonspecialist, in the next section we identify some of the vector spaces derived from an abelian group.

2. Numerical invariants. Let p denote a prime. Then p and its powers operate on the abelian group G in an obvious and natural way. Indeed, the set $pG = \{px : x \in G\}$ is a subgroup, and we can extend even to transfinite powers by the identities $p^{\alpha+1}G = p(p^\alpha G)$ and $p^\beta G = \cap_{\alpha<\beta}\, p^\alpha G$ when β is a limit. In contrast to pG, we define the dual $G[p] = \{x \in G : px = 0\}$. Note that $p^\alpha G[p]$, which always means $(p^\alpha G)[p]$, is a vector space over the prime field of characteristic p. The cardinal number $\delta_\alpha = \dim\,(p^\alpha G[p]/p^{\alpha+1}G[p])$ is known as the α^{th} Ulm invariant of a p-group G, but it would probably be more accurate to call it the α^{th} Kaplansky-Mackey number. At any rate, these numbers uniquely determine not only countable p-groups, but this same pinnacle of success for the isomorphism problem has been reached for a much wider class of p-groups known as simply presented groups. A group is simply presented if it can be presented with generators and relations with each relation involving at most two generators. Recognition of the fact that the numerical invariants $\dim(p^\alpha G[p]/p^{\alpha+1}G[p])$ singularly determine such a wide class of p-groups seems to invite the following philosophical question.

Problem 1. Do all the vector spaces that descend from the group G determine it? In other words, if the dimensions of all the corresponding vector spaces of G and G' always match, must G and G' be isomorphic?

A more practical problem than the one stated above is that of identifying important vector-space descendants of G, that is, of finding those vector spaces associated with G that collectively codify all or a significant part of the structure of G. Aside from the Ulm spaces $p^\alpha G[p]/p^{\alpha+1}G[p]$, we refer to [**15**] for a description of additional spaces that determine a class of p-groups more general than simply presented groups. It should be pointed out that vector spaces associated not with G alone but with G and a fixed subgroup H of G are sometimes useful. Indeed, these relative invariants were essential in the proof that the Ulm invariants determine simply presented groups. To be specific, the α^{th} Ulm invariant of G relative to H is

$$\dim(p^\alpha G[p]/p^\alpha G[p] \cap (p^{\alpha+1}G[p] + H)).$$

Obviously, some of the vector spaces associated with a group G give only minimal information about the group, for example, G/pG or $G[p]$. It should also be noted that, with the exception of G/pG, none of the preceding vector spaces reveals anything at all about a torsion-free group.

We shall now try to identify some vector spaces associated with torsion-free groups that are relevant to the structure of the group. First, however, we need a little more notation. For a prime p, we write $|x|_p = \alpha$ if $x \in p^\alpha G \backslash p^{\alpha+1}G$; in case the containing group G is not clear from the context, we use the notation $|x|_p^G$. The sequence $\{|x|_p\}$ where p ranges over the set $\mathbf{P}$ of all primes (written in ascending order) will be denoted simply by $|x|$ (or by $|x|^G$ if necessary for clarity). A function s from $\mathbf{P}$ to the set of nonnegative integers with ∞ adjointed is called a height sequence. A prime p, and by induction any positive integer n, operates on s in a natural way: only one term is changed and it is increased by 1. For a height sequence $s = \{s_p\}$, we define as usual two invariant subgroups of G as follows:

$$G(s) = \{x \in G : |x|_p \geq s_p \text{ for each prime } p\},$$

and

$$G(s^*) = \langle x \in G(s) : \sum_{p \in P} (|x_p| - s_p) \geq \infty \rangle.$$

Vector spaces of interest for a torsion-free group G certainly include $G(s)/G(ps)$ and $G \otimes Q$. One might say, however, that these spaces go about as far in uniquely determining a group as age and gender go in determining the identity of a person. Other, perhaps more sensitive, invariants include (the dimensions of) the vector spaces $G(s)/(G(ps) + G(s^*))$ and $G(s)/G(s^*) \otimes Q$. I suspect that there is much fruitful work that can be done in discovering important vector spaces associated with torsion-free groups. Not forgetting the history of torsion groups, I suggest also that relative spaces should not be overlooked. One useful relative space seems to be the following. If H is a subgroup of G, for a height sequence s and a prime p, define

$$V_{s,p} = (H \cap G(s) + G(s^*) + G(ps))/(G(s^*) + G(ps)).$$

The dimension of $V_{s,p}$ is denoted by $\delta_{s,p}$.

We have barely tapped the large pool of vector spaces associated with a group G. For more examples (in the torsion-free case) and for a discussion of what are "good" invariants, see the paper by D. Arnold [**1**].

3. Structure versus relative structure. We have been considering the problem of determining when two groups are isomorphic (via numerical invariants) and have set this determination as our highest goal. Let us now refine this problem and consider when two *subgroups* are equivalent—equivalent not just as groups but as subgroups.

Definition 1. Two subgroups H and H' of a group G are equivalent if there is an automorphism of G that maps H onto H'.

Our primary interest, in this section, is the following equivalence problem.

Problem 2 (Baer-Fuchs). Find necessary and sufficient conditions for two subgroups H and H' of G to be equivalent.

Remark . The preceding problem restricted to p-groups is Problem 52 in [**6**], which is credited to Baer.

In the more general context of sets, the above problem has a simple solution:

$$H \cong H' \tag{1}$$

and

$$G/H \cong G/H'. \tag{2}$$

(Notice that we have, for the convenience of uniformity, used the notation G/H to denote the complement of H in G when the equivalence problem is considered in the category of sets–a singular occurrence.) The simple solution to the Baer-Fuchs problem given by conditions (1) and (2) is also valid for vector spaces and even for cyclic groups, but certainly not for groups in general. There is already a counter-example in a finite group of order 64 (or of order p^6 for an arbitrary prime p). To see this, let

$$G = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle,$$

where a has order 8, b has order 4, and c has order 2. Consider $H = \langle 2a + c \rangle \oplus \langle 2b \rangle$ and $H' = \langle 2a \rangle \oplus \langle c \rangle$. It is easy to verify that conditions (1) and (2) are satisfied, but H and H' are not equivalent subgroups of G.

In the context where the isomorphism problem is the ultimate objective, one might criticize the attention given to the equivalence problem at this time as being a diversion, but such criticism is not justified. Indeed, we intend to use equivalences, in certain instances, to solve the isomorphism problem. However, it would be appropriate to insert an additional clause in the Baer-Fuchs problem.

Problem 3. Solve the Baer-Fuchs problem without assuming *a priori* that H and H' are isomorphic.

To be specific and to relate to a well-known class of groups, we remark that the isomorphism problem for Warfield's S-groups (or more generally for the A-groups classified in [**15**]) can be solved most efficiently with an appropriate equivalence theorem. This phenomenon has set the trend for more far-reaching results on the isomorphism problem, some of which will be described in more detail later. Let us turn first, however, to some other applications or benefits of equivalence theorems.

The Baer-Fuchs problem in particular and relative structure in general illuminate what I consider to be some misconceptions. I will briefly discuss two of these.

Misconception 1. Everything is known about countable torsion groups.

We could make our point more dramatic by claiming that, in fact, not everything is known about finite (abelian) groups! Consider, for example, the Baer-Fuchs problem. One could argue, of course, that two subgroups H and H' of the finite group G are equivalent if and only if they are isomorphic as *valuated* groups, but this is simply to answer a question with one. When are two finite valuated groups isomorphic? In regard to the latter, we rest on the authority of Hunter, Richman, and Walker [**27**]: "The study of finite valuated groups ... is only in its initial stages."

I have chosen the familiar and historic ground of finite groups to distinguish as clearly as possible the difference between structure and relative structure. Relative structure encompasses structure but not the other way around. There have been two major approaches toward relative structure, manifested by valuated groups on one hand and equivalence theorems on the other. The main distinction between the two approaches, at least in intuitive terms, is that in valuated groups the containing group is only present in spirit, while for equivalence theorems it is present in the flesh. Incidentally, valuated groups being more flexible and general would seem destined to play a more important role, but equivalence theorems probably because they are more concrete, if ad hoc, have certainly up to now had a larger direct impact on structure theory.

Now let me mention another misconception (perhaps more controversial than the first).

Misconception 2. Ext (B, A) represents the extensions of A by B.

Let p denote an odd prime and let $C(n)$ denote the cyclic group of order n. Suppose that $A = C(p) = B$. The fact that Ext $(B, A) = C(p)$ would suggest that there are p different extensions of $C(p)$ by $C(p)$. But this is wrong. In no real sense are there but two. The explanation, of course, if that there are $p - 1$ artificial distinctions of the extension

$$0 \to C(p) \to C(p^2) \to C(p) \to 0$$

associated with the automorphisms of $C(p)$. Before I go any further, let me hasten to add that I am not suggesting that the definition of Ext (B, A) should be changed. There are too many advantages that accrue to the artificial distinctions (like those mentioned above) that result from the narrow concept of the equality of extensions that has been used from the beginning. Not the least of these advantages is that Ext (B, A) becomes a group in a natural way and fits into the homological scheme. What then? If I am not suggesting that Ext (B, A) be modified, what is the point? Simply that Ext (B, A) should not be misinterpreted. Maybe I could best illustrate my point this way. Recall in the introduction when we referred to the structure side of the mountain and the homological face as the "other side." In this context, there is an amusing irony. The structure of Ext (B, A) is not of much interest to the people on the "structure" side of the mountain. To be sure, they are keenly interested in whether Ext (B, A) is vital or vanishes but are not (or should not be) obsessed with its size or structure. Basically, the point we are trying to make here is that there is merit in considering again an old problem. How many nonisomorphic extensions of A by B are there? Actually, a better question probably is: after the subgroup A is identified, how many nonisomorphic extensions are there? This question is directly associated with the "equivalence" diagram:

$$\begin{array}{ccccc} A & \longrightarrow & G & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & G' & \longrightarrow & B \end{array}$$

We mention [**3**] as an example of a recent solution to the former problem in a special case.

4. Equivalence theory: The initial phase. First, I will make a few remarks about the early history of equivalence theorems and about my own modest contribution to that theory with a brief indication of how my interest evolved in taking this approach to the isomorphism problem. The concept of two subgroups being equivalent is primitive and has been around for a long time, and the idea is in no way restricted to abelian group theory. (Sylow's Second Theorem is a special kind of equivalence theorem.) Nevertheless, until fairly recently there were only a few equivalence theorems in the literature. This may prove embarrassing, but the only equivalence theorem that I recall off-hand from the standard sources is a theorem of J. Erdös concerning subgroups of free groups.

Theorem (Erdös). *Suppose that F is a free group and that K and L are pure subgroups of F. Then K and L are equivalent subgroups of F if and only if:*

(i) $\operatorname{rank}(K) = \operatorname{rank}(L)$ *and*

(ii) $F/K \cong F/L$.

Erdös' theorem is a beautiful (yet superficial) example of a case where the equivalent subgroups are not assumed *a priori* to be isomorphic. In other words, his theorem is a solution to Problem 3 for pure subgroups of free groups. Whether this theorem, however, should be distinguished as the first equivalence theorem—even for free groups—depends on the standards for classifying a result as an equivalence theorem. Should we require that the theorem explicitly refer to a map (through the term "equivalence" or in some other way), or is it enough for the existence of the desired map to be a rather immediate consequence of the stated result? If the latter suffices, the invariant factor theorem for a finitely generated free group could be considered an equivalence theorem: if G is a free group of finite rank, then two subgroups H and H' are equivalent if and only if $G/H \cong G/H'$. In any case, we *now* recognize the invariant factor theorem as being an equivalence theorem.

Leaving precedent aside, it is of interest to analyze Erdös' theorem in a broad context that includes motivation and consequences. As to the former, there is little, if any, evidence that Erdös was particularly interested in equivalence theorems, in the sense of *relative* structure.

Let us not forget that he proved the above theorem (and related results) as a means to an end, the end being the structure of torsion-free groups. Indeed, he was attempting to do nothing less than to reach the pinnacle in grand style by resolving the isomorphism problem for all torsion-free groups. The title of his paper indicates that he might have thought at first that he had succeeded, but it was soon realized that the isomorphism problem had only been translated to another problem (concerning the equivalence of infinite matrices) of equal difficulty [**6**]. Misinterpretations of this kind concerning the classification of groups make a strong case in favor of using numerical invariants. Although the preceding equivalence theorem is obviously not strong enough to serve as a key to the classification (using numerical invariants) of all torsion-free groups, the theorem does have substance. What the theorem essentially amounts to is that a torsion-free group A has a unique free resolution (modulo trivialities): $K \rightarrowtail F \twoheadrightarrow A$. (One of the trivialities is adding a superfluous summand to both K and F.) In 1970, 13 years after Erdös' theorem appeared, I published in [**12**] my first equivalence theorem, which incidentally was of no great consequence; nevertheless, it was the forerunner of more important results that followed years later. It is noted that when I proved this theorem I placed emphasis on the fact that my result meant that the automorphism group of G is transitive on a certain set of (basic) subgroups of G as if the theorem had more to do with the structure of the automorphism group of G than it did with G itself. The opposite view, of course, would ultimately prevail.

Theorem [**12**]. *Two basic subgroups B and B' of a countable p-group G are equivalent if and only if $G/B \cong G/B'$.*

At about the same time the preceding theorem appeared, D. Tarwater and E. Walker [**31**] also published an equivalence theorem about basic subgroups, an early sign perhaps that equivalence theorems would accumulate to form a recognized aspect of structure theory. Actually, a preprint of this paper (minus Walker's name) had been circulated prior to my paper. The original version apparently contained a flawed proof, but nevertheless could be considered the beginning of the equivalence theory for p-groups. A few years later, Warfield [**32**] proved a more substantial equivalence theorem concerning certain subgroups of arbitrary simply presented p-groups. Warfield's theorem is equivalent to the following, where a subgroup H of a p-group G is said to be λ-dense in G if $H + p^\alpha G = G$ whenever $\alpha < \lambda$, λ being a limit ordinal.

Theorem [**32**]. *Let G be a simply presented p-group of length λ, λ a limit ordinal. Let H and K be two λ-dense, isotype subgroups of G. Then H and K are equivalent if and only if $G/H \cong G/K$.*

In private communications, Warfield later indicated to me an interest in the equivalence-theorem approach to structure, and he discussed results of this nature outside of group theory. Unfortunately, after the result about basic subgroups cited above, I did nothing more worth mentioning with equivalence theorems until the early 1980's. In 1981 D. Cutler [**5**] proved that any two high subgroups of certain groups G are isomorphic. Recalling my equivalence theorem for basic subgroups and Warfield's theorem, I had a hunch that the high subgroups in Cutler's theorem might in fact be equivalent, and I proceeded to prove this conjecture. Recall that H is high in the p-group G if H is maximal with respect to $H \cap p^\omega G = 0$.

Theorem [**13**]. *If G is a p-group and $G/p^{\omega+1}G$ is simply presented, then any two high subgroups of G are equivalent.*

Notice that the above theorem is a slight variance from the preceding equivalence theorems inasmuch as the containing group G is not required to be simply presented. It should also be observed that in all nontrivial cases of all of the above equivalence theorems dealing with p-groups, the quotient groups with respect to the equivalent subgroups are divisible as well as isomorphic. Therefore, the next theorem breaks new ground.

Theorem [**14**]. *If H and K are balanced subgroups of a simply presented p-group G, then H and K are equivalent if and only if they have the same Ulm invariants and $G/H \cong G/K$.*

The next theorem essentially encompasses and improves the preceding one.

Theorem [**18**]. *If H and K are almost balanced subgroups of a simply presented p-group G, then H and K are equivalent if and only if they have the same Ulm invariants and* $G/H \cong G/K$.

Somewhat arbitrarily we call this the end of the initial phase of the development of an equivalence theory partly because the above was developed piecemeal whereas the rest is more systematic but mainly because the next stage will have a direct impact on structure.

5. Equivalence theory: The second phase. Although considerable insight about equivalence theorems for p-groups was gained during the initial and developmental phase discussed in the preceding section, it is the next result which is definitive and which has significant implications for structure. Indeed, the next theorem provides a solution to the isomorphism problem for a new class of groups. In order to understand this theorem, one needs to be familiar with the concept of coset valuation due to Fuchs [**9**]. If H is a subgroup of the p-group (or p-local group) G, the coset valuation on G/H is defined by $||g + H|| = \sup\{|g + h| + 1 : h \in H\}$.

Main Theorem [**19**]. *Let H and H' be isotype subgroups of a simply presented p-group G. Then H and H' are equivalent in G if and only if*

(1) *H and H' have the same Ulm invariants*

and

(2) $G/H \cong G/H'$ *as valuated groups* (*endowed with the coset valuation*).

The time has come for us to discuss *how* an equivalence theorem is proved. After all, these automorphisms of G that map a subgroup H onto another subgroup H' do not just appear out of thin air. We will use the preceding theorem as a model on how to prove an equivalence theorem without really doing so. We might call the following discussion something like "the generic proof of an equivalence theorem."

In order to prove an equivalence theorem, we need first and foremost an Axiom 3 system for G. This is required in order that partial automorphisms can be extended. This is why the hypothesis that the containing group G is simply presented appears so often in the preceding theorems and, in particular, in the Main Theorem. Next we need a guide or a plan for the construction of the desired automorphism beginning at zero and building up to G. Often, and in the case of the Main Theorem, the guide is a chosen value-preserving isomorphism $\phi : G/H \rightarrowtail\!\!\!\!\twoheadrightarrow G/H'$. It is perhaps surprising that frequently this can be carried out regardless of the choice of ϕ. In this case, more is proved than is stated; in particular, when $H = H'$ we conclude that every automorphism of G/H lifts to an automorphism of G. After selecting the guide ϕ (or whatever), the rest of the proof consists only of the usually laborious details relevant to the case at hand (modeled after my proof of Ulm's theorem for totally projective p-groups).

One of the most important applications of the Main Theorem is that it yields necessary and sufficient conditions for two A-groups (in particular, for two S-groups) to be isomorphic. Therefore, this particular equivalence theorem provides a way for us to reach the pinnacle (at which point we can see when two groups are isomorphic) for a significantly broader class of p-groups than simply presented groups. For details, we refer to the original paper on A-groups [**15**], and we refer to [**16**] for an expository account of the classification of A-groups. It might be of sufficient interest to merit our pointing out that the isomorphism problem for A-groups was first solved without an equivalence theorem. But before publication, I realized that the proof could be shortened significantly by utilizing the Main Theorem, which at that time had just been proved (see [**15**]). Likewise, the most efficient way to prove the uniqueness theorem for S-groups is to use an equivalence theorem. These facts gave not only impetus to proving equivalence theorems, but also standing to the theory.

Conceptually, there is no mystery as how an equivalence theorem (like our Main Theorem) once it exists can be used to solve the isomorphism problem for a class of groups. Suppose that H and H' are groups, not subgroups, that belong to a certain class and have certain numerical invariants that agree. We try to embed both H and H' in a common Axiom 3 group G so that G/H and G/H' are isomorphic as valuated groups (or at least so that there is an isomorphism between G/H and G/H' that respects heights in some sense). As we already indicated, this can be carried out for A-groups and S-groups. Two remarks are appropriate here.

Remark 1. In the context of the isomorphism problem, the limitation of equivalence theorems to (isotype) subgroups of Axiom 3 groups is not as restrictive, at least at this time, as might first appear. There are many classes of isotype subgroups of Axiom 3 groups that have not yet been classified.

Remark 2. Whereas simply presented p-groups have an Axiom 3 characterization, this was not originally the case for simply presented mixed groups. However, this was achieved later not only for simply presented groups but also for their summands, which are called Warfield groups. In [**20**] and [**21**], respectively, Axiom 3 characterizations are given for p-local and global Warfield groups. As we tried to explain then and as I would like to say again now, these Axiom 3 characterizations seem to be essential in order to classify isotype subgroups of Warfield groups. We will soon encounter some of these groups that do not even have decomposition bases, but yet can be classified.

Although, in a sense, I feel that equivalence theorems hardly get any better than the Main Theorem stated above, it is true that M. Lane did a little better. Lane [**30**] proved the same theorem for p-local balanced projective groups, where of course the Warfield invariants are added to condition (1). Notice that [**29**] provided the crucial Axiom 3 system for balanced projectives; the rest follows the pattern of [**19**]. Lane's equivalence theorem is the following, which was generalized in [**24**].

Theorem [**30**]. *Let H and H' be isotype subgroups of a p-local balanced projective group G. The H and H' are equivalent if and only if*

(i) *H and H' have the same Ulm invariants and the same Warfield invariants, and*

(ii) *$G/H \cong G/H'$ as valuated groups (endowed with the coset valuation).*

Since equivalence theorems are now considered a primary means for resolving the isomorphism problem and reaching the pinnacle defined in the introduction, it is not surprising that Lane's enhancement of the previously stated Main Theorem would lead even to a wider class of (mixed) groups for which we can determine when two are isomorphic by comparing numerical invariants. Indeed, this was accomplished in [**26**] for a large class of mixed groups, many of which are without decomposition bases. More precisely, it is shown that a certain set of numerical invariants (modeled after the invariants of an A-group) uniquely determine, up to isomorphism, a p-local B-group. Here is the definition of a B-group in two stages. First, a B-group is a p-local group that can be expressed as the coproduct (= direct sum) of a balanced projective group and μ-elementary B-groups for various limit ordinals μ not cofinal with ω. We say that H is a μ-elementary B-group if $H \subseteq G$ where G is a reduced p-local balanced projective group of length not exceeding μ and where the following conditions are satisfied:

(a) H is an isotype in G.

(b) $p^\alpha(G/H) = p^\alpha G + H/H$ for all $\alpha < \mu$.

(c) G/H is a balanced projective group.

We remark that an A-group is a B-group, so the above theorem extends the uniqueness theorem of [**15**]. We believe that we have made a pretty strong case already in favor of using equivalence theorems to determine structure. We will next consider torsion-free groups where some progress has been made dealing with structure via equivalence theorems but we are hoping to obtain better results as our investigations continue.

Currently, there is great interest in Butler groups both of finite rank and infinite rank. The "classical" case is the finite-rank case. A finite rank Butler group is simply a pure subgroup of a completely decomposable group of finite rank. Alternately, they can be described as torsion-free homomorphic images of completely decomposable groups of finite rank. A completely decomposable group is the traditional name and description of a torsion-free simply presented group. These groups have a natural Axiom 3 characterization, but nevertheless have offered the most resistance to satisfactory equivalence theorems for its isotype (= pure) subgroups. However, we do have a couple of results, and we are still working toward more and better results. The first of our theorems applies only to weakly $*$-pure subgroups. A pure subgroup

H of a torsion-free group G is weakly $*$-pure if

$$H \cap (G(s^*) + G(ps)) = H \cap G(s^*) + H \cap G(ps).$$

Theorem [**22**]. *If H and H' are weakly $*$-pure subgroups of a completely decomposable group G, then H and H' are equivalent if and only if*

(1) $H \cap G(\sigma)/H \cap G(\sigma^*) \otimes \mathbf{Q} \cong H' \cap G(\sigma)/H' \cap G(\sigma^*) \otimes \mathbf{Q}$ *and*

(2) *There is an isomorphism $\phi : G/H \twoheadrightarrow G/H'$ that respects heights.*

Recall that a Butler group G is a B_0-group if $G(\sigma^*)$ is always pure, where $G(\sigma^*) = \langle G(s^*) : s \in \sigma \rangle$ for any type σ. One application of this equivalence theorem is the following.

Theorem [**22**]. *There is a natural bijection between the isomorphism classes of B_0-groups G with prescribed Baer invariants manifested by $A_\sigma = G(\sigma)/G(\sigma^*)$ and the equivalence classes of balanced subgroups K of the fixed group $A = \oplus A_\sigma$ that satisfy $K \cap A(\sigma) = K \cap A(\sigma^*)$.*

The next theorem applies to arbitrary pure subgroups of completely decomposable groups, not just the weakly $*$-pure ones. Notice that the theorem is a kind of reduction theorem to the homogeneous case. Since the homogeneous case can probably be negotiated, the theorem should prove to be useful. However, that remains to be seen.

Theorem [**25**]. *Suppose that H and H' are pure subgroups of a completely decomposable group G. Then H and H' are equivalent if and only if: there exists a height-respecting isomorphism ϕ from G/H onto G/H' such that, for each type,*

$$\phi_\sigma : G_\sigma/H_\sigma \twoheadrightarrow G_\sigma/H'_\sigma$$

lifts to an automorphism $\pi_\sigma : G_\sigma \twoheadrightarrow G_\sigma$, where $G_\sigma = G(\sigma) + G(\sigma^*)/G(\sigma^*)$, $H_\sigma = H(\sigma) + G(\sigma^*)/G(\sigma^*)$, and ϕ_σ is induced by ϕ.

6. Other applications of equivalence theorems. Throughout, we have emphasized that the strong motivating force for equivalence theorems is structure, in particular, the isomorphism problem. And success has been obtained by solving Problem 3 in a number of instances. There are, however, other applications of equivalence theorems. We want to mention a couple of these here.

The simultaneous decompositions of a group and a subgroup can be related directly to an equivalence theorem just as Baer [**2**] related the decomposition of a countable p-group to Ulm's theorem. We cite here a simple result which is a direct consequence of the following equivalence theorem. Note that the subgroups do not have to be contained in the *same* group (as long as the containing groups are isomorphic).

Theorem. *Let C and C' be direct sums of cyclic groups (finite or infinite) with pure subgroups B and B' respectively. Then there exists an isomorphism $\pi : C \twoheadrightarrow C'$ that maps B onto B' if and only if $B \cong B'$ and $C/B \cong C'/B'$.*

Corollary 1 [**17**]. *Suppose that C is a direct sum of cyclic groups and that B is a pure subgroup of C. Then B and C have a common summand (in the set-theoretic sense) isomorphic to a given group K if and only if $B \cong K \oplus B'$, where*

$$B' \rightarrowtail C' \twoheadrightarrow A$$

is a pure revolution of $A = C/B$ and C' is a direct sum of cyclic groups.

The above is only intended to introduce the flavor of simultaneous decompositions and how they relate to equivalence theorems. For more substantial results, we refer to [**17**].

The simplest of all simply presented groups are the free groups. However, equivalence theorems here are not relevant to structure because a subgroup of a free group is again free, so the structure of the subgroup is already known. Nevertheless, equivalence theorems are still important and have another application in this (from the standpoint of structure) degenerate case. We want to discuss here an application of an equivalence theorem dealing with subgroups of free groups to an old and famous problem of Kaplansky: the stacked bases problem.

Recall that, years ago, Kaplansky [**28**] raised the following question.

If

$$K \mapsto F \twoheadrightarrow A$$

is an arbitrary free resolution of an arbitrary direct sum A of cyclic groups, must it be the case that F and K have stacked bases? This means that we can write $F = \oplus\langle x_i \rangle$ and $K = \oplus\langle n_i x_i \rangle$ for *some* basis $\{x_i\}$ of F and suitable integers n_i. Cohen and Gluck [**4**] provided an affirmative answer in 1970. We would like to show how the answer follows from the following equivalence theorem, which is a direct generalization of the first equivalence theorem due to Erdös that was discussed earlier.

Theorem [**23**]. *Suppose that H and H' are subgroups of a free group of G. Then H and H' are equivalent if and only if:*

(a) $\dim (H + pG/pG) = \dim (H' + G/pG)$ *for every prime p, and*

(b) $G/H \cong G/H'$.

To get a general idea of how the stacked bases theorem is a consequence of this equivalence theorem, suppose that F is a free group and that $F/K = A$ is a direct sum of cyclic groups. If $F = \oplus\langle x_i \rangle$, it is apparent that we can choose integers n_i so that $\oplus\langle x_i \rangle / \oplus \langle n, x_i \rangle \cong A$ because A is a direct sum of cyclics. Now, $F/K \cong A \cong F/H$ where $H = \oplus\langle n_i x_i \rangle$. With a little care (see [**23**] for details) we can arrange it so that $H + pF/pF \cong K + pF/pF$ for each prime p. The preceding equivalence theorem permits us to conclude that H and K are positioned in F in precisely the same way—up to an automorphism. Since F and H have stacked bases by construction, then F and K must also have stacked bases.

Remark . We feel compelled to make a final comment for the sake of clarity. From the introduction onward, beginning with the quotation by Hungerford, we have identified the isomorphism problem with the classification problem, but there is a distinction at least for abelian groups. The classification problem requires a solution to the isomorphism problem (uniqueness theorem) *and* a companion existence theorem. In this discussion we have dealt exclusively with the isomorphism problem, not the classification problem.

REFERENCES

1. D. Arnold, *Finite rank Butler groups*, preprint.

2. R. Baer, *The decomposition of enumerable, primary groups into direct summands*, Quart. J. Math. Oxford **6** (1935), 217–221.

3. K. Benabdallah, D. Cutler, and A. Mader, *Extensions of torsion-complete p-groups*, Contemp. Math. **87** (1989), 11–21.

4. J. Cohen and H. Gluck, *Stacked bases for modules over principal ideal domains*, J. Algebra **14** (1970), 493–505.

5. D. Cutler, *Primary abelian groups having all high subgroups isomorphic*, Proc. Amer. Math. Soc. **83** (1981), 467–470.

6. L. Fuchs, *Abelian groups*, Pergamon Press, New York, 1960.

7. ———, *Infinite abelian groups*, Vol. 1, Academic Press, New York, 1973.

8. ———, *Infinite abelian groups*, Vol. 2, Academic Press, New York, 1973.

9. ———, *Extensions of isomorphisms between subgroups*, in *Abelian group theory*, Lecture Notes in Math. #874, Springer-Verlag, Berlin, Heidelberg, New York, 1981.

10. P. Griffith, *Infinite abelian groups*, Univ. Chicago Press, Chicago, 1970.

11. D. Harrison, *Infinite abelian groups and homological methods*, Ann. Math. **69** (1959), 366–391.

12. P. Hill, *Automorphisms of countable primary abelian groups*, Proc. Amer. Math. Soc. **25** (1970), 135–140.

13. ———, *The equivalence of high subgroups*, Proc. Amer. Math. Soc. **88** (1983), 207–211.

14. ———, *Balanced subgroup of totally projective groups*, J. Algebra **88** (1984), 346–349.

15. ———, *On the structure of abelian p-groups*, Trans. Amer. Math. Soc. **288** (1985), 505–525.

16. ———, *The classification problem*, in *Abelian groups and modules*, CISM Courses and Lectures, Springer-Verlag, Wien and New York, 1984.

17. ———, *The simultaneous decomposition of an abelian group and a subgroup*, Oxford J. Math. **36** (1985), 425–433.

18. P. Hill and C. Megibben, *On the congruence of subgroups in totally projective groups*, Lecture Notes in Math. **1006**, 513–518.

19. ———, *On the theory and classification of abelian p-groups*, Math. Z. **190** (1985), 17–38.

20. ———, *Axiom* 3 *modules*, Trans. Amer. Math. Soc. **295** (1986), 715–734.

21. ———, *Mixed groups*, Trans. Amer. Math. Soc., to appear.

22. ———, *The classification of certain Butler groups*, Journal of Algebra, to appear.

23. ———, *Generalizations of the stacked bases theorem*, Trans. Amer. Math. Soc. **312** (1989), 377–402.

24. ———, *The local equivalence theorem*, Contemp. Math. **87** (1989), 201–219.

25. ———, *Equivalence theorems for torsion free groups*, to appear.

26. P. Hill, M. Lane, and C. Megibben, *On the structure of p-local abelian groups*, Journal of Algebra, to appear.

27. R. Hunter, F. Richman and E. Walker, *Subgroups of bounded abelian groups*, in *Abelian groups and modules*, CISM Courses and Lectures #287, Springer-Verlag, Wien and New York, 1984.

28. I. Kaplansky, *Infinite abelian groups*, Univ. Michigan Press, Ann Arbor, 1954 and 1969.

29. M. Lane, *A new characterization of p-local balanced projective groups*, Proc. Amer. Math. Soc. **91** (1986), 379–386.

30. ———, *Isotype submodules of p-local balanced projective groups*, Trans. Amer. Math. Soc. **301** (1987), 313–325.

31. D. Tarwater and E. Walker, *Decompositions of direct sums of cyclic p-groups*, Rocky Mountain J. Math. **2** (1972), 275–282.

32. R. Warfield, *A classification theorem for abelian p-groups*, Trans. Amer. Math. Soc. **210** (1975), 149–168.

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