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BOUNDED ANALYTIC FAMILIES OF CAUCHY-STIELTJES INTEGRALS

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1. Introduction. Let $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and $\Lambda = \{z : |z| = 1\}$. Let $\mathcal{M}$ denote the set of (finite) complex-valued Borel measures on Λ. For $\alpha > 0$, let $\mathcal{F}_\alpha$ denote the family of functions f having the property that there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_\Lambda \frac{1}{(1 - xz)^\alpha} \, d\mu(x) \tag{1}$$

for $|z| < 1$. If $f \in \mathcal{F}_\alpha$, let $||f||_{\mathcal{F}_\alpha} = \inf ||\mu||$ where μ varies over all members of $\mathcal{M}$ for which (1) holds and where $||\mu||$ denotes the total variation of μ. Then $\mathcal{F}_\alpha$ is a Banach space with respect to this norm and the usual addition of functions and multiplication by complex numbers.

Properties of $\mathcal{F}_\alpha$ were studied in [**8**] and [**5**], where the related family denoted $\mathcal{F}_0$ was introduced. A function $f \in \mathcal{F}_0$ provided that there exists $\mu \in \mathcal{M}$ such that

$$f(z) = f(0) + \int_\Lambda \log\left(\frac{1}{1 - xz}\right) d\mu(x) \tag{2}$$

for $|z| < 1$. The family $\mathcal{F}_1$ has been studied extensively. The survey article [**1**] gives a number of references in this area.

A Banach space of analytic functions is defined in this paper for each real number α. It is shown that when $\alpha \geq 0$ the space is equivalent to $\mathcal{F}_\alpha$. This provides a natural extension of $\mathcal{F}_\alpha$ for $\alpha < 0$. The results obtained also clarify why $\mathcal{F}_0$ is an appropriate choice for $\mathcal{F}_\alpha$ when $\alpha = 0$.

Let α be a real number. Define the function G_α by

$$G_\alpha(z) = \sum_{n=1}^{\infty} n^{\alpha - 1} z^n \tag{3}$$

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for $|z| < 1$. Let $\mathcal{G}_\alpha$ denote the family of functions f having the property that there exists $\mu \in \mathcal{M}$ such that

$$f(z) = f(0) + \int_\Lambda G_\alpha(xz)\, d\mu(x) \tag{4}$$

for $|z| < 1$. Let $||f||_{\mathcal{G}_\alpha} = |f(0)| + \inf ||\mu||$ where μ varies over all members of $\mathcal{M}$ for which (4) holds. Then $\mathcal{G}_\alpha$ is a Banach space.

Theorem 1 shows that if $\alpha > 0$, then $f \in \mathcal{F}_\alpha$ if and only if $f \in \mathcal{G}_\alpha$. Moreover, the two norms are comparable. In the case $\alpha = 0$ this is evident since $G_0(z) = \sum_{n=1}^{\infty}(1/n)z^n = \log(1/(1-z))$, and hence (2) and (4) are the same. As a comparison of (1) and (4), note that (1) can be written

$$f(z) = \int_\Lambda F_\alpha(xz)\, d\mu(x) \tag{5}$$

where $F_\alpha(z) = 1/(1-z)^\alpha$.

If $F_\alpha(z) = \sum_{n=0}^{\infty} A_n(\alpha) z^n$, then $A_0(\alpha) = 1$ and $A_n(\alpha) = \alpha(\alpha+1)\cdots(\alpha+n-1)/n!$ for $n = 1, 2, \dots$. A function $f(z) = \sum_{n=0}^{\infty} a_n z^n (|z| < 1)$ belongs to $\mathcal{F}_\alpha$ if and only if there exists $\mu \in \mathcal{M}$ such that

$$\alpha_n = A_n(\alpha) \int_\Lambda x^n\, d\mu(x) \tag{6}$$

for $n = 0, 1, \dots$. Likewise, $f \in \mathcal{G}_\alpha$ if and only if there exists $\mu \in \mathcal{M}$ such that

$$a_n = n^{\alpha-1} \int_\Lambda x^n\, d\mu(x) \tag{7}$$

for $n = 1, 2, \dots$.

It follows from (7) that the Taylor coefficients of a function in $\mathcal{G}_\alpha$ satisfy

$$|a_n| \le n^{\alpha-1} ||\mu|| \tag{8}$$

for $n = 1, 2, \dots$. In particular, if $f \in \mathcal{G}_\alpha$ and $\alpha < 0$, then $\sum_{n=0}^{\infty} |a_n| < +\infty$. Hence f extends continuously to $\bar{\Delta}$. Later some additional facts are proved about the boundary behavior of such functions. One of these concerns the smoothness of the derivatives of the boundary function in terms of Lipschitz conditions. Another concerns weak L^p membership of these derivatives.

Several results about $\mathcal{F}_\alpha$ obtained earlier are extended to $\mathcal{G}_\alpha$ for all real α. This includes the fact that $\mathcal{G}_\alpha$ is closed under composition with analytic automorphisms of Δ. For easy reference, a few facts about $\mathcal{F}_\alpha$ are stated below. The proofs are contained in [**5, 8**].

Theorem A. 1. *$f \in \mathcal{F}_\alpha$ if and only if $f' \in \mathcal{F}_{\alpha+1}$.*

2. *If $0 \leq \alpha < \beta$, then $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$.*

3. *If φ is an analytic automorphism of Δ and $f \in \mathcal{F}_\alpha$, then the composition $f \circ \varphi \in \mathcal{F}_\alpha$.*

This paper also contains results on sequence multipliers of the space $\mathcal{G}_\alpha$ into $\mathcal{G}_\beta$ or into l^p for $0 < p \leq +\infty$.

2. Comparison of $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$. Our comparison of $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ for $\alpha > 0$ is based on Lemma 2 which appears below. This lemma shows that the sequence $\{A_n(\alpha)/n^{\alpha-1}\}$ has an asymptotic expansion. This is more than needed to make the comparison which only uses the first two terms in the expansion. Two facts about the gamma function Γ are required, and they are stated first. The proofs are given in [**2**, pp. 209, 211, 223].

For all complex numbers z, except the nonpositive integers, $(z-1)\Gamma(z-1) = \Gamma(z)$, and hence

$$\Gamma(z) = \lim_{k\to\infty} \frac{k^{z-1}k!}{z(z+1)\cdots(z+k-1)}. \tag{9}$$

The following asymptotic expansion holds

$$(2\pi)^{-1/2} e^z z^{-z+1/2}\Gamma(z) \approx 1 + \sum_{k=1}^{\infty} a_k/z^k \tag{10}$$

as $|z| \to +\infty$ and $|\arg z| \leq \pi - \varepsilon$ for each ε $(0 < \varepsilon < \pi)$.

Lemma 1. *Let n be a positive integer and let α be a complex number such that $\alpha \neq 0, -1, -2, \dots$. Then*

$$(n+1)\Gamma(n+\alpha) = A_n(\alpha)\Gamma(\alpha)\Gamma(n+2). \tag{11}$$

Proof. Let n and α satisfy the stated conditions. In (9) let $z = n+\alpha$, $z = \alpha$ and $z = n+2$. Hence

$$\begin{aligned}
&\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+2)} \\
&= \lim_{k\to\infty}\left\{\frac{1}{k\cdot k!}\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)(n+2)(n+3)\cdots(n+k+1)}{(n+\alpha)(n+\alpha+1)\cdots(n+\alpha+k-1)}\right\} \\
&= \frac{A_n(\alpha)}{n+1}\lim_{k\to\infty}\left\{\frac{1}{k\cdot k!}\frac{(\alpha+n)(\alpha+n+1)\cdots(\alpha+k-1)}{(n+\alpha)(n+\alpha+1)\cdots(n+\alpha+k-1)}\cdot(n+k+1)!\right\} \\
&= \frac{A_n(\alpha)}{n+1}\lim_{k\to\infty}\left\{\frac{1}{k\cdot k!}\frac{(n+k+1)!}{(k+\alpha)(k+\alpha+1)\cdots(k+\alpha+n-1)}\right\} \\
&= \frac{A_n(\alpha)}{n+1}\lim_{k\to\infty}\left\{\frac{1}{k\cdot k!}\frac{(n+k+1)!}{k^n}\right\} \\
&= \frac{A_n(\alpha)}{n+1}\lim_{k\to\infty}\left\{\frac{(k+1)(k+2)\cdots(k+n+1)}{k^{n+1}}\right\} \\
&= \frac{A_n(\alpha)}{n+1}. \qquad \square
\end{aligned}$$

Lemma 2. *For $\alpha > 0$ the asymptotic expansion*

$$\frac{A_n(\alpha)}{n^{\alpha-1}} \approx \sum_{k=0}^{\infty}\frac{b_k}{n^k} \tag{12}$$

holds as $n \to \infty$ and $b_0 = 1/\Gamma(\alpha)$.

Proof. In (10), let $z = n+\alpha$ and $z = n+2$. Also note that for any β, then for large n, $1/(n+\beta)^k = 1/(n^k(1+\beta/n)^k)$ and $1/(1+\beta/n)^k$ can be expanded in a power series in $1/n$. Hence

$$(2\pi)^{-1/2}e^{n+\alpha}(n+\alpha)^{-n-\alpha+1/2}\Gamma(n+\alpha) \approx 1 + \sum_{k=1}^{\infty} c_k/n^k \tag{13}$$

and

$$(2\pi)^{-1/2}e^{n+2}(n+2)^{-n-3/2}\Gamma(n+2) \approx 1 + \sum_{k=1}^{\infty} d_k/n^k \tag{14}$$

as $n \to \infty$, for suitable sequences $\{c_k\}$ and $\{d_k\}$. Division of (13) by (14) yields

$$e^{\alpha-2}\frac{(n+2)^{n+3/2}}{(n+\alpha)^{n+\alpha-1/2}}\frac{\Gamma(n+\alpha)}{\Gamma(n+2)} \approx 1 + \sum_{k=1}^{\infty} e_k/n^k \tag{15}$$

as $n \to \infty$, for suitable $\{e_k\}$.

It will be shown that if $\gamma_n = (n+2)^{n+3/2}/(n+\alpha)^{n+\alpha-1/2}$, then

$$\gamma_n = n^{2-\alpha}e^{2-\alpha}\{1+p(1/n)\} \tag{16}$$

where p is a power series which converges in a neighborhood of 0 and vanishes at 0. The power series expansion of $\log(1+x)$ at $x=0$ implies that for sufficiently large n,

$$\begin{aligned}\log\gamma_n &= (n+3/2)[\log n + \log(1+2/n)] \\ &\quad - (n+\alpha-1/2)[\log n + \log(1+\alpha/n)] \\ &= (2-\alpha)\log n + 2 - \alpha + q(1/n)\end{aligned}$$

where q is a power series which vanishes at 0. Exponentiation of this relation gives (16).

Lemma 1 gives

$$\frac{A_n(\alpha)}{n^{\alpha-1}} = \frac{n+1}{n^{\alpha-1}}\frac{1}{\Gamma(\alpha)}\frac{\Gamma(n+\alpha)}{\Gamma(n+2)}. \tag{17}$$

An asymptotic expansion for $\Gamma(n+\alpha)/\Gamma(n+2)$ follows from (15) and (16). Applied to (17) it yields (12) and $b_0 = 1/\Gamma(\alpha)$. □

Theorem 1. *If $\alpha > 0$, then $f \in \mathcal{F}_\alpha$ if and only if $f \in \mathcal{G}_\alpha$. There is a positive constant C depending only on α such that if $f \in \mathcal{F}_\alpha$, then $(1/C)||f||_{\mathcal{F}_\alpha} \leq ||f||_{\mathcal{G}_\alpha} \leq C||f||_{\mathcal{F}_\alpha}$.*

Proof. Suppose that $f \in \mathcal{F}_\alpha$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. There exists $\mu \in \mathcal{M}$ such that

$$a_n = A_n(\alpha) \int_\Lambda x^n \, d\mu(x) \tag{18}$$

for $n = 0, 1, \dots$. Lemma 2 implies that

$$A_n(\alpha) = n^{\alpha-1}(1/\Gamma(\alpha) + B_n(\alpha)) \tag{19}$$

for $n = 1, 2, \dots$ and there is a positive constant $B(\alpha)$ such that $|B_n(\alpha)| \le B(\alpha)/n$ for $n = 1, 2, \dots$. For $n = 1, 2, \dots$ let $C_n(\alpha) = B_n(\alpha) \int_\Lambda x^n \, d\mu(x)$ and define the function g by $g(z) = \sum_{n=1}^{\infty} C_n(\alpha) z^n$ for $|z| < 1$. Since $|C_n(\alpha)| \le |B_n(\alpha)| \, ||\mu|| \le B(\alpha)||\mu||/n$, $\sum_{n=1}^{\infty} |C_n(\alpha)|^2 \le B^2(\alpha)||\mu||^2 \sum_{n=1}^{\infty} 1/n^2 < +\infty$. Hence g belongs to the Hardy space H^2. Since $H^2 \subset H^1 \subset \mathcal{F}_1$, there exists $\nu \in \mathcal{M}$ such that

$$g(z) = \int_\Lambda \frac{1}{1 - xz} \, d\nu(x) \tag{20}$$

for $|z| < 1$. This implies that $C_n(\alpha) = \int_\Lambda x^n \, d\nu(x)$ for $n = 1, 2, \dots$. Thus, (18) and (19) yield

$$a_n = n^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)} \int_\Lambda x^n \, d\mu(x) + \int_\Lambda x^n \, d\nu(x)\right) = n^{\alpha-1} \int_\Lambda x^n \, d\lambda(x)$$

for $n = 1, 2, \dots$ where $\lambda = (1/\Gamma(\alpha))\mu + \nu$. Since $\lambda \in \mathcal{M}$ and $f(z) = f(0) + \int_\Lambda G_\alpha(xz) \, d\lambda(x)$, this proves that $f \in \mathcal{G}_\alpha$.

Conversely suppose that $f \in \mathcal{G}_\alpha$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. There exists $\mu \in \mathcal{M}$ such that

$$a_n = n^{\alpha-1} \int_\Lambda x^n \, d\mu(x) \tag{21}$$

for $n = 1, 2, \dots$. Lemma 2 implies that

$$n^{\alpha-1} = A_n(\alpha)[\Gamma(\alpha) + D_n(\alpha)] \tag{22}$$

for $n = 1, 2, \dots$ and $|D_n(\alpha)| \le D(\alpha)/n$ for some positive constant $D(\alpha)$. For $n = 1, 2, \dots$, let $E_n(\alpha) = D_n(\alpha) \int_\Lambda x^n \, d\mu(x)$ and define the function h by $h(z) = \sum_{n=1}^{\infty} E_n(\alpha) z^n$ for $|z| < 1$. Then $h \in H^2$ since $\sum_{n=1}^{\infty} |E_n(\alpha)|^2 \leq D^2(\alpha)||\mu||^2 \sum_{n=1}^{\infty} 1/n^2 < +\infty$. Hence there exists $\nu \in \mathcal{M}$ such that $h(z) = \int_\Lambda 1/(1-xz)\, d\nu(x)$ for $|z| < 1$. This gives $E_n(\alpha) = \int_\Lambda x^n\, d\nu(x)$, and thus (21) and (22) yield $a_n = A_n(\alpha) \int_\Lambda x^n\, d\lambda(x)$ for $n = 1, 2, \dots$ where $\lambda = \Gamma(\alpha)\mu + \nu$. Let $b = f(0) - \lambda(\Lambda)$ and let $\sigma = \lambda + b\tau$ where τ is normalized Lebesgue measure. Then $\sigma \in \mathcal{M}$ and $a_n = A_n(\alpha) \int_\Lambda x^n d\sigma(x)$ for $n = 0, 1, \dots$. Hence $f(z) = \int_\Lambda F_\alpha(xz)\, d\sigma(x)$ and therefore $f \in \mathcal{F}_\alpha$. This proves the first statement in the theorem.

It remains to verify that the norms are comparable. This will be done by examining the arguments given above. In the first half of the argument, the measure ν satisfies

$$||\nu|| \leq \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|\, d\theta \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^2\, d\theta \right)^{1/2}$$
$$\leq B(\alpha) \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} ||\mu||.$$

Hence $||\lambda|| \leq (1/\Gamma(\alpha))||\mu|| + ||\nu|| \leq B'(\alpha)||\nu||$ for some constant $B'(\alpha)$. Also $|f(0)| = |\int_\Lambda d\mu(x)| \leq ||\mu||$. Thus $|f(0)| + ||\lambda|| \leq (1 + B'(\alpha))||\mu||$, and therefore $||f||_{\mathcal{G}_\alpha} \leq (1 + B'(\alpha))||\mu||$. This inequality holds for every $\mu \in \mathcal{M}$ for which (1) holds. Hence $||f||_{\mathcal{G}_\alpha} \leq (1 + B'(\alpha))||f||_{\mathcal{F}_\alpha}$. The reverse inequality $||f||_{\mathcal{F}_\alpha} \leq C||f||_{\mathcal{G}_\alpha}$ can be proved in a similar way. □

3. Properties of the family $\mathcal{G}_\alpha$.

Theorem 2. *For all real numbers α, $f \in \mathcal{G}_\alpha$ if and only if $f' \in \mathcal{G}_{\alpha+1}$.*

Proof. Suppose that $f \in \mathcal{G}_\alpha$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. There exists $\mu \in \mathcal{M}$ such that $a_n = n^{\alpha-1} \int_\Lambda x^n\, d\mu(x)$ for $n = 1, 2, \dots$. If $f'(z) = \sum_{n=0}^{\infty} b_n z^n$ for $|z| < 1$, then $b_n/n^\alpha = (n+1)a_{n+1}/n^\alpha = ((n+1)^\alpha/n^\alpha) \int_\Lambda x^{n+1}\, d\mu(x)$ for $n = 1, 2, \dots$. This can be written

$$b_n/n^\alpha = (1 + B_n(\alpha)) \int_\Lambda x^n\, d\nu(x) \tag{23}$$

where $d\nu(x) = x\, d\mu(x)$ and $|B_n(\alpha)| \leq B(\alpha)/n$ for $n = 1, 2, \dots$ and $B(\alpha)$ is a positive constant. Let $C_n(\alpha) = B_n(\alpha) \int_\Lambda x^n\, d\nu(x)$ for

$n = 1, 2, \ldots$ and let $g(z) = \sum_{n=1}^{\infty} C_n(\alpha) z^n$ for $|z| < 1$. Then $g \in H^2$ and there exists $\lambda \in \mathcal{M}$ such that $C_n(\alpha) = \int_\Lambda x^n \, d\lambda(x)$. Thus (23) yields $b_n = n^\alpha \int_\Lambda x^n \, d\sigma(x)$ for $n = 1, 2, \ldots$, where $\sigma = \nu + \lambda$. Therefore $f' \in \mathcal{G}_{\alpha+1}$.

Conversely, suppose that $f' \in \mathcal{G}_{\alpha+1}$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f'(z) = \sum_{n=0}^{\infty} b_n z^n$ for $|z| < 1$. There exists $\mu \in \mathcal{M}$ such that $b_n = n^\alpha \int_\Lambda x^n \, d\mu(x)$ for $n = 1, 2, \ldots$, and hence $a_n = ((n-1)^\alpha / n) \int_\Lambda x^{n-1} \, d\mu(x)$ for $n = 2, 3, \ldots$. This implies that

$$a_n / n^{\alpha - 1} = (1 + D_n(\alpha)) \int_\Lambda x^n \, d\nu(x) \tag{24}$$

for $n = 2, 3, \ldots$, where $d\nu(x) = \bar{x} \, d\mu(x)$ and $|D_n(\alpha)| \leq D(\alpha)/n$ for some positive constant $D(\alpha)$. Let $E_n(\alpha) = D_n(\alpha) \int_\Lambda x^n \, d\nu(x)$ for $n = 2, 3, \ldots$, and let $g(z) = \sum_{n=2}^{\infty} E_n(\alpha) z^n$. Then $g \in H^2$ and hence $E_n(\alpha) = \int_\Lambda x^n \, d\lambda(x)$ for some $\lambda \in \mathcal{M}$. Hence (24) yields $a_n = n^{\alpha-1} \int_\Lambda x^n \, d\sigma(x)$ for $n = 2, 3, \ldots$, where $\sigma = \nu + \lambda$. Let $b = \int_\Lambda x \, d\sigma(x)$, $c = a_1 - b$ and $\omega = \sigma + c\tau$ where $d\tau(x) = \bar{x} d\psi(x)$ and ψ is normalized Lebesgue measure. Then $\omega \in \mathcal{M}$, $\int_\Lambda x^n \, d\omega(x) = \int_\Lambda x^n \, d\sigma(x)$ for $n = 2, 3, \ldots$, and $\int_\Lambda x d\omega(x) = a_1$. Hence $a_n = n^{\alpha-1} \int_\Lambda x^n d\omega(x)$ for $n = 1, 2, \ldots$, and therefore $f \in \mathcal{G}_\alpha$. □

The argument given for Theorem 2 also shows that there is a constant C depending only on α such that if $f \in \mathcal{G}_\alpha$, then $(1/C) ||f||_{\mathcal{G}_\alpha} \leq ||f'||_{\mathcal{G}_{\alpha+1}} \leq C ||f||_{\mathcal{G}_\alpha}$.

Theorem 3. *If $\alpha < \beta$, then $\mathcal{G}_\alpha \subset \mathcal{G}_\beta$ and $\mathcal{G}_\alpha \neq \mathcal{G}_\beta$.*

Proof. Suppose that $\alpha < \beta$ and $f \in \mathcal{G}_\alpha$. Let n be a positive integer such that $\alpha + n > 0$. Theorem 2 implies that $f^{(n)} \in \mathcal{G}_{\alpha+n}$. Since $\alpha + n > 0$, Theorem 1 implies that $f^{(n)} \in \mathcal{F}_{\alpha+n}$. Statement 2 of Theorem A yields $f^{(n)} \in \mathcal{F}_{\beta+n}$. By Theorem 1 this is the same as $f^{(n)} \in \mathcal{G}_{\beta+n}$ and Theorem 2 implies $f \in \mathcal{G}_\beta$. Therefore $\mathcal{G}_\alpha \subset \mathcal{G}_\beta$.

When $0 < \alpha < \beta$, it is clear that $\mathcal{F}_\alpha \neq \mathcal{F}_\beta$. For example, $F_\beta \in \mathcal{F}_\beta$ and $F_\beta \notin \mathcal{F}_\alpha$. Hence the argument given above yields $\mathcal{G}_\alpha \neq \mathcal{G}_\beta$. (In fact, $G_\beta \in \mathcal{G}_\beta$ and $G_\beta \notin \mathcal{G}_\alpha$.) □

Lemma 3. *If g is analytic in $\bar{\Delta}$ and $f \in \mathcal{G}_\alpha$, then $gf \in \mathcal{G}_\alpha$.*

Proof. The lemma holds when $\alpha \geq 0$ due to [**5**] and Theorem 1. An inductive argument is given to treat the case $\alpha < 0$. For $n = 1, 2, \ldots$, consider the proposition: if g is analytic in $\bar{\Delta}$, $-n+1 \leq \alpha < -n+2$ and $f \in \mathcal{G}_\alpha$, then $gf \in \mathcal{G}_\alpha$. The first sentence asserts that the proposition holds when $n = 1$. Assume that it holds for the positive integer n. Suppose that g is analytic in $\bar{\Delta}$, $-n \leq \alpha < -n+1$ and $f \in \mathcal{G}_\alpha$. Theorem 2 implies that $f' \in \mathcal{G}_{\alpha+1}$. Since $-n+1 \leq \alpha+1 < -n+2$, the inductive assumption yields $gf' \in \mathcal{G}_{\alpha+1}$. Theorem 3 implies that $f \in \mathcal{G}_{\alpha+1}$, and because g' is analytic in $\bar{\Delta}$, the inductive assumption gives $g'f \in \mathcal{G}_{\alpha+1}$. Thus, $gf'+g'f \in \mathcal{G}_{\alpha+1}$. This is the same as $h' \in \mathcal{G}_{\alpha+1}$ where $h = gf$. Theorem 2 yields $h \in \mathcal{G}_\alpha$. This completes the induction. □

A function g is called a multiplier of $\mathcal{G}_\alpha$ provided that g is analytic in Δ and $f \in \mathcal{G}_\alpha$ implies $gf \in \mathcal{G}_\alpha$. By Lemma 3, every function analytic in $\bar{\Delta}$ is a multiplier of $\mathcal{G}_\alpha$. For $\alpha > 0$, the following stronger result is proved in [**6**]: If $\alpha > 0$ and $g' \in H^1$, then g is a multiplier of $\mathcal{F}_\alpha$.

Suppose that g is a multiplier of $\mathcal{G}_\alpha$. Since the constant function $1 \in \mathcal{G}_\alpha$ for every α, it follows that $g = g \cdot 1 \in \mathcal{G}_\alpha$. In particular, the Taylor series of a multiplier of $\mathcal{G}_\alpha$ is absolutely convergent in $\bar{\Delta}$ when $\alpha < 0$. For $\alpha \geq 0$, multipliers of $\mathcal{F}_\alpha$ are bounded in Δ [**4, 6**].

Theorem 4. *If $f \in \mathcal{G}_\alpha$ and φ is an analytic automorphism of Δ, then $f \circ \varphi \in \mathcal{G}_\alpha$.*

Proof. Assertion 3 of Theorem A and Theorem 1 imply that this theorem holds when $\alpha \geq 0$. An inductive argument will be given for $\alpha < 0$. It is required to show that for $n = 1, 2, \ldots$, the following holds: if $-n+1 \leq \alpha < -n+2$, $f \in \mathcal{G}_\alpha$ and φ is an analytic automorphism of Δ, then $f \circ \varphi \in \mathcal{G}_\alpha$. This assertion holds when $n = 1$ by the first remark. Assume that it holds for the positive integer n. Let $-n \leq \alpha < -n+1$, $f \in \mathcal{G}_\alpha$, and let φ be an analytic automorphism of Δ. Theorem 2 implies that $f' \in \mathcal{G}_{\alpha+1}$, and hence the inductive assumption gives $f' \circ \varphi \in \mathcal{G}_{\alpha+1}$. The function φ has the form $\varphi(z) = w(z+\zeta)/(1+\bar{\zeta}z)$ where $|w| = 1$ and $|\zeta| < 1$. In particular, φ' is analytic in $\bar{\Delta}$. Hence Lemma 3 and $f' \circ \varphi \in \mathcal{G}_{\alpha+1}$ imply that $\varphi' \cdot (f' \circ \varphi) \in \mathcal{G}_{\alpha+1}$. This is the same as $g' \in \mathcal{G}_{\alpha+1}$ where $g = f \circ \varphi$. Thus Theorem 2 yields $g \in \mathcal{G}_\alpha$. □

The next two theorems concern the boundary behavior of a function $f \in \mathcal{G}_\alpha$. The first result concerns the smoothness behavior of $f(e^{i\theta})$ when $\alpha < 0$.

For $0 < \beta \le 1$ let Λ_β denote the set of functions F defined on $[0, 2\pi]$ which satisfy a Lipschitz condition

$$|F(\theta_1) - F(\theta_2)| \le A|\theta_1 - \theta_2|^\beta. \tag{25}$$

Suppose that f is analytic in Δ. Then f is continuous in $\bar{\Delta}$ and $f(e^{i\theta}) \in \Lambda_\beta$ if and only if there is a positive constant B such that

$$|f'(z)| \le B/(1 - |z|)^{1-\beta} \tag{26}$$

for $|z| < 1$ [**3**, p. 74]. Let Λ_* denote the set of functions F that are continuous on $[0, 2\pi]$ and for some $A > 0$ satisfy

$$|F(\theta + h) - 2F(\theta) + F(\theta - h)| \le Ah \tag{27}$$

for all θ in $[0, 2\pi]$ and all $h > 0$. Suppose that f is analytic in Δ. Then f is continuous in $\bar{\Delta}$ and $f(e^{i\theta}) \in \Lambda_*$ if and only if there is a constant $B > 0$ such that

$$|f''(z)| \le B/(1 - |z|) \tag{28}$$

for $|z| < 1$ [**3**, p. 76].

Theorem 5. *Suppose that $f \in \mathcal{G}_\alpha$ and $\alpha < 0$. If α is not an integer and $n = -[\alpha]$, then $f^{(n-1)}$ is continuous in $\bar{\Delta}$ and $f^{(n-1)}(e^{i\theta})$ satisfies a Lipschitz condition of order $1-\alpha+[\alpha]$. If α is an integer and $\alpha = -n$, then $f^{(n-1)}$ is continuous in $\bar{\Delta}$ and $F(\theta) \equiv f^{(n-1)}(e^{i\theta})$ satisfies* (27).

Proof. Suppose that $f \in \mathcal{G}_\alpha$ and $\alpha < 0$. Assume that α is not an integer and let $n = -[\alpha]$. Let $\gamma = \alpha + n$ and $g = f^{(n-1)}$. Theorem 2 implies that $g \in \mathcal{G}_{\gamma-1}$ and $g' \in \mathcal{G}_\gamma$. Since $0 < \gamma < 1$, Theorem 1 implies that there exists $\mu \in \mathcal{M}$ such that

$$g'(z) = \int_\Lambda 1/(1 - xz)^\gamma \, d\mu(x) \tag{29}$$

for $|z| < 1$. Hence $|g'(z)| \le ||\mu||/(1-|z|)^\gamma$ for $|z| < 1$. From the result quoted about (26) this shows that $f^{(n-1)}$ is continuous in $\bar{\Delta}$ and $f^{(n-1)}(e^{i\theta})$ satisfies a Lipschitz condition (25) where $\beta = 1-\gamma = 1-\alpha+[\alpha]$.

Next assume that α is an integer and let $\alpha = -n$ and $g = f^{(n-1)}$. Then $g \in \mathcal{G}_{-1}$ and $g'' \in \mathcal{G}_1$. Hence by Theorem 1

$$g''(z) = \int_\Lambda 1/(1-xz)\, d\mu(x) \tag{30}$$

for $|z| < 1$ and $\mu \in \mathcal{M}$. This implies that $|g''(z)| \le ||\mu||/(1-|z|)$ for $|z| < 1$. The result quoted about (28) yields $f^{(n-1)}$ is continuous in $\bar{\Delta}$ and $F(\theta) \equiv f^{(n-1)}(e^{i\theta})$ satisfies (27). □

The next theorem concerns weak L^p membership of functions in $\mathcal{G}_\alpha$ when $\alpha \le 1$. A measurable function F defined on $[0, 2\pi]$ is called weak L^p provided that there is a constant $A > 0$ such that $|\{\theta : |F(\theta)| > s\}| < A/s^p$ for all $s > 0$. Here $|E|$ denotes the Lebesgue measure of the set $E \subset [0, 2\pi]$.

Theorem 6. *Suppose that $f \in \mathcal{G}_\alpha$ and $\alpha \le 1$, and let $n = [-\alpha]$. Then $f^{(n+1)}(e^{i\theta})$ is weak L^p where $p = 1/(\alpha + [-\alpha] + 1)$.*

Proof. Suppose that $f \in \mathcal{G}_\alpha$ and $\alpha \le 1$. Let $n = [-\alpha]$, $\gamma = \alpha + n + 1$ and $g = f^{(n+1)}$ where $f^{(0)} = f$. Then $g \in \mathcal{G}_\gamma$ and $0 < \gamma \le 1$. Hence $g \in \mathcal{F}_\gamma$ and therefore $g \in H^p$ for $p < 1/\gamma$ [**8**]. Thus $g(e^{i\theta}) \equiv \lim_{r\to 1-} g(re^{i\theta})$ exists for almost all θ and defines a measurable function.

First consider the case $\alpha = 1$, that is, suppose that $f \in \mathcal{F}_1$. The fact that $f(e^{i\theta})$ is weak L^1 follows from a theorem of Kolmogorov about conjugate functions [**7**, p. 66]. Indeed, the Jordan decomposition theorem implies that $f = a_1 f_1 - a_2 f_2 + i a_3 f_3 - i a_4 f_4$ where $a_k \ge 0$ and $\operatorname{Re} f_k(z) > 1/2$ for $|z| < 1$ and $f_k(0) = 1$. Kolmogorov's theorem implies that $\operatorname{Im} f_k$ is weak L^1. Since $\operatorname{Re} f_k$ and $\operatorname{Im} f_k$ are weak L^1 for $k = 1, 2, 3, 4$, so is f.

Next suppose that $f \in \mathcal{F}_\alpha$ and $0 < \alpha < 1$. By the Jordan decomposition theorem it suffices to assume that

$$f(z) = \int_\Lambda 1/(1-xz)^\alpha\, d\mu(x) \tag{31}$$

for $|z| < 1$ where μ is a probability measure. The function F_α is a convex univalent mapping when $0 < \alpha \leq 1$. Because μ is a probability measure, the convexity and (31) imply that $f(\Delta) \subset F_\alpha(\Delta)$. Indeed, each probability measure is a weak $*$ limit of convex combinations of point masses. Hence for each $z\,(|z| < 1)$, $f(z)$ is a limit of numbers having the form $\sum_{j=1}^{n} t_j F_\alpha(x_j z)$ where $t_j \geq 0$, $|x_j| = 1$ and $\sum_{j=1}^{n} t_j = 1$ $(n = 1, 2, \ldots)$, and such numbers belong to $\{w : w = F_\alpha(\zeta), |\zeta| \leq |z|\}$. Since $f(\Delta) \subset F_\alpha(\Delta)$ and $F_\alpha(z) \neq 0$ for $|z| < 1$, we have $f(z) \neq 0$ for $|z| < 1$. Hence $h = f^{1/a}$ is analytic in Δ and satisfies $\operatorname{Re} h(z) > 1/2$ for $|z| < 1$. Also $h(0) = 1$ and hence the Riesz-Herglotz formula gives

$$h(z) = \int_\Lambda 1/(1 - xz)\, d\nu(x) \tag{32}$$

for $|z| < 1$ where ν is a probability measure. In particular, $h \in \mathcal{F}_1$. By the result proved in the case $\alpha = 1$, it follows that there is a constant $A > 0$ such that $|\{\theta : |h(e^{i\theta})| > s\}| < A/s$ for every $s > 0$. This is the same as $|\{\theta : |f(e^{i\theta})| > s^\alpha\}| < A/s$, which implies that $f(e^{i\theta})$ is weak $L^{1/\alpha}$.

Finally consider the case $\alpha \leq 0$. Since $n = [-\alpha]$, we have $g = f^{(n+1)} \in \mathcal{F}_\gamma$ where $\gamma = \alpha + n + 1$. Also $0 < \gamma \leq 1$ and hence the results above apply to g. Therefore $f^{(n+1)}(e^{i\theta})$ is weak L^p where $p = 1/\gamma = 1/(\alpha + [-\alpha] + 1)$. □

If $f \in \mathcal{F}_\alpha$ and $0 < \alpha \leq 1$ then $f(e^{i\theta})$ is weak $L^{1/\alpha}$ and hence $f(e^{i\theta})$ belongs to L^p for every $p < 1/\alpha$ [**7**, p. 65]. Thus Theorem 6 is an improvement of the result in [**8**] that $f \in H^p$ for $p < 1/\alpha$. Also note that if $f \in H^p$ then $|f(z)| \leq A/(1 - |z|)^{1/p}$ for some $A > 0$ [**3**, p. 36]. In the opposite direction, there are functions f that are analytic in Δ and satisfy $|f(z)| \leq A/(1 - |z|)^\gamma$ for $|z| < 1$ where $\gamma > 0$ and $A > 0$ but belong to no H^p class. Hence such a function cannot be weak L^p for any $p > 0$. Theorems 5 and 6 give independent information about the boundary behavior of functions in $\mathcal{G}_\alpha$.

4. Sequence multipliers. Sequence multipliers of $\mathcal{G}_\alpha$ into $\mathcal{G}_\beta$ and of $\mathcal{G}_\alpha$ into l^p will be discussed. The arguments depend on the fact that members of $\mathcal{G}_\alpha$ are so directly related to the sequence $\{n^{\alpha-1}\}$. The following lemma concerns the case of multipliers of $\mathcal{G}_1$ into $\mathcal{G}_1$. It is stated in the more convenient form relating to $\mathcal{F}_1$.

Lemma 4. *Suppose that* $f,\, g \in \mathcal{F}_1$ *and let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *for* $|z| < 1$. *If* $h(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ *for* $|z| < 1$, *then* $h \in \mathcal{F}_1$.

Proof. The Jordan decomposition theorem implies that it suffices to assume that

$$f(z) = \int_\Lambda 1/(1 - xz)\, d\mu(x) \tag{33}$$

and

$$g(z) = \int_\Lambda 1/(1 - xz)\, d\nu(x) \tag{34}$$

where μ and ν are probability measures in $\mathcal{M}$. Hence $\operatorname{Re} f(z) > 1/2$ and $\operatorname{Re} g(z) > 1/2$ for $|z| < 1$ and $f(0) = g(0) = 1$.

The conditions on f and g imply that h also satisfies $\operatorname{Re} h(z) > 1/2$ for $|z| < 1$ and $h(0) = 1$. It is clear that $h(0) = 1$. That $\operatorname{Re} h(z) > 1/2$ is a known fact, and the simple argument is included here. Suppose that $0 < r < 1$ and $0 \le \theta \le 2\pi$. Then

$$\begin{aligned} h(r^2 e^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} f(re^{i(\theta-\varphi)}) g(re^{i\varphi})\, d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \int_\Lambda \frac{1}{1 - xre^{i(\theta-\varphi)}}\, d\mu(x) \right\} g(re^{i\varphi})\, d\varphi \\ &= \int_\Lambda \left\{ \frac{1}{2\pi i} \int_{|w|=r} \frac{g(w)}{w - xr^2 e^{i\theta}}\, dw \right\} d\mu(x) \\ &= \int_\Lambda g(xr^2 e^{i\theta})\, d\mu(x). \end{aligned}$$

The last equality follows from Cauchy's formula. Since $\operatorname{Re} g(z) > 1/2$ for $|z| < 1$ and μ is a probability measure, $\operatorname{Re} h(r^2 e^{i\theta}) = \int_\Lambda \operatorname{Re} g(xr^2 e^{i\theta})\, d\mu(x) > \int_\Lambda (1/2)\, d\mu(x) = 1/2$. Therefore $\operatorname{Re} h(z) > 1/2$ for $|z| < 1$.

Since $h(0) = 1$ and $\operatorname{Re} h(z) > 1/2$ for $|z| < 1$, the Riesz-Herglotz formula implies there exists a probability measure $\lambda \in \mathcal{M}$ such that $h(z) = \int_\Lambda 1/(1-xz)\, d\lambda(x)$ for $|z| < 1$. Therefore $h \in \mathcal{F}_1$. □

In terms of the moments of measures, Lemma 4 can be reexpressed in the following way.

Lemma 5. *Suppose that $\mu, \nu \in \mathcal{M}$ and for $n = 0, 1, \dots,$ define the sequence $\{c_n\}$ by $c_n = \{\int_\Lambda x^n\, d\mu(x)\}\{\int_\Lambda x^n\, d\nu(x)\}$. There exists $\lambda \in \mathcal{M}$ such that $c_n = \int_\Lambda x^n\, d\lambda(x)$ for $n = 0, 1, \dots$.*

Theorem 7. *A sequence $\{\lambda_n\}$ $(n = 0, 1, \dots)$ multiplies G_α into $\mathcal{G}_\beta$ if and only if there is a measure $\lambda \in \mathcal{M}$ such that*

$$\lambda_n = n^{\beta-\alpha} \int_\Lambda x^n\, d\lambda(x) \tag{35}$$

for $n = 1, 2, \dots$.

Proof. Suppose that $\{\lambda_n\}$ multiplies $\mathcal{G}_\alpha$ into $\mathcal{G}_\beta$. In other words, if $f \in \mathcal{G}_\alpha$ and $f(z) = \sum_{n=0}^\infty a_n z^n (|z| < 1)$ then $g \in \mathcal{G}_\beta$ where $g(z) = \sum_{n=0}^\infty \lambda_n a_n z^n$. In particular, if $f = G_\alpha$ this requires that $h \in \mathcal{G}_\beta$ where $h(z) = \sum_{n=0}^\infty \lambda_n n^{\alpha-1} z^n$. Hence there exists $\lambda \in \mathcal{M}$ such that $\lambda_n n^{\alpha-1} = n^{\beta-1} \int_\Lambda x^n\, d\lambda(x)$ for $n = 1, 2, \dots$. Therefore (35) holds.

Conversely, suppose that $\lambda \in \mathcal{M}$ and let the sequence $\{\lambda_n\}$ be defined by (35) for $n = 1, 2, \dots$. Assume that $f \in \mathcal{G}_\alpha$ and let $f(z) = \sum_{n=0}^\infty a_n z^n (|z| < 1)$. There exists $\mu \in \mathcal{M}$ such that $a_n = n^{\alpha-1} \int_\Lambda x^n\, d\mu(x)$ for $n = 1, 2, \dots$. Lemma 5 implies that there exists $\nu \in \mathcal{M}$ such that

$$\lambda_n a_n = n^{\beta-1} \int_\Lambda x^n\, d\lambda(x) \cdot \int_\Lambda x^n\, d\mu(x) = n^{\beta-1} \int_\Lambda x^n\, d\nu(x)$$

for $n = 1, 2, \dots$. Hence $k \in \mathcal{G}_\beta$ where $k(z) = \sum_{n=0}^\infty \lambda_n a_n z^n$. Therefore $\{\lambda_n\}$ multiplies $\mathcal{G}_\alpha$ into $\mathcal{G}_\beta$. □

Theorem 7, of course, does not give an intrinsic characterization of the multipliers of $\mathcal{G}_\alpha$ into $\mathcal{G}_\beta$. It shows that such a problem is equivalent to the characterization of moment sequences of complex Borel measures on Λ.

Theorem 8. *For $0 < p \leq +\infty$, a sequence $\{\lambda_n\}$ $(n = 0, 1, \dots)$ is a multiplier of $\mathcal{G}_\alpha$ into l^p if and only if the sequence $\{n^{\alpha-1}\lambda_n\}$ $(n = 1, 2, \dots)$ belongs to l^p.*

Proof. Suppose that $\{\lambda_n\}$ is a multiplier of $\mathcal{G}_\alpha$ into l^∞. Since $G_\alpha \in \mathcal{G}_\alpha$ this requires that $\{\lambda_n n^{\alpha-1}\}$ $(n = 1, 2, \dots)$ belongs to l^∞. Conversely, assume that $\{n^{\alpha-1}\lambda_n\}$ $(n = 1, 2, \dots)$ belongs to l^∞. Suppose that $f \in \mathcal{G}_\alpha$ and $f(z) = \sum_{n=0}^\infty a_n z^n (|z| < 1)$. Then $a_n = n^{\alpha-1} \int_\Lambda x^n \, d\mu(x)$ for $n = 1, 2, \dots$ and for some $\mu \in \mathcal{M}$. Since $|a_n| \leq n^{\alpha-1} ||\mu||$ and $\{n^{\alpha-1}\lambda_n\}$ is bounded, the sequence $\{\lambda_n a_n\}$ is bounded. Therefore $\{\lambda_n\}$ multiplies $\mathcal{G}_\alpha$ into l^∞.

Let $0 < p < +\infty$. Suppose that $\{\lambda_n\}$ multiplies $\mathcal{G}_\alpha$ into l^p. Since $G_\alpha \in \mathcal{G}_\alpha$ this implies that $\{\lambda_n n^{\alpha-1}\}$ $(n = 1, 2, \dots)$ belongs to l^p. Conversely, suppose that $\{n^{\alpha-1}\lambda_n\}$ $(n = 1, 2, \dots)$ belongs to l^p. Assume that $f \in \mathcal{G}_\alpha$ and let $f(z) = \sum_{n=0}^\infty a_n z^n (|z| < 1)$. There exists $\mu \in \mathcal{M}$ such that $a_n = n^{\alpha-1} \int_\Lambda x^n \, d\mu(x)$ for $n = 1, 2, \dots$. Since $\{n^{\alpha-1}\lambda_n\}$ belongs to l^p, this implies that $\sum_{n=1}^\infty |\lambda_n a_n|^p \leq ||\mu||^p \sum_{n=1}^\infty (n^{\alpha-1}|\lambda_n|)^p < +\infty$. Therefore $\{\lambda_n\}$ multiplies $\mathcal{G}_\alpha$ into l^p. □

The sequences $\{\lambda_n\}$ described in Theorems 7 and 8 induce bounded mappings between the spaces. This is seen from the arguments. It also follows from the closed graph theorem.

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