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FIXED POINTS OF PLANE CONTINUA

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Dedicated to Professor Sandra Barkdull

CONTENTS

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ABSTRACT. Suppose M is a plane continuum, $\mathcal{D}$ is a decomposition of M, and each element of $\mathcal{D}$ is a uniquely arcwise connected set. Our principal theorem states that every map of M that preserves the elements of $\mathcal{D}$ has a fixed point. It follows that every arc-component-preserving map of a plane continuum that does not contain a simple closed curve has a fixed point. This result generalizes the author's theorem [**17**] that every uniquely arcwise connected plane continuum has the fixed-point property. Our principal theorem also applies to planar dynamical systems. Suppose ψ is a continuous flow on the plane. Suppose M is an invariant continuum under ψ and $\mathcal{D}$ is the collection of orbits of ψ in M. Then, according to our principal theorem, some element of $\mathcal{D}$ is a point or a simple closed curve. Hence, every invariant continuum under ψ contains an equilibrium point or a closed orbit. This result implies the Poincare-Bendixson theorem, a compact limit set of an orbit of ψ that does not contain an equilibrium point is a closed orbit.

1. Introduction. Does every nonseparating plane continuum have the fixed-point property? This unsolved problem was first considered in the late 1920's [**11, 29**]. A variety of partial solutions indicates that a counterexample will have to be very pathological [**4, 12, 13, 21, 22, 42, 3, 5, 6, 41, 26, 15, 30, 34**]. For example, it is known that every arcwise connected nonseparating plane continuum has the fixed-point property [**14**]. Recently, the author [**19**] generalized this theorem by proving that every arc-component-preserving map of a nonseparating plane continuum has a fixed point. Note that the collection of arc components is a decomposition of a continuum. In this paper, we establish the fixed-point property for maps of a plane continuum that preserve the elements of another decomposition, a decomposition with elements that are uniquely arcwise connected.

In 1933, H. Whitney [**43**] defined a special type of decomposition to study the topological properties of solutions of differential equations. He called this decomposition a regular family of curves. The regularity condition is a convergence property that restricts the crookedness of arcs in the curves. Solutions of differential equations always have this property. Whitney proved the existence of cross-sections of these curves. Recall that the Poincare-Bendixson theorem is usually established with a cross-section argument [**24**, p. 248]. Whitney also defined a flow on the curves of the family. Each curve in Whitney's family is homeomorphic to a circle or an open interval. Recently, J.M. Aarts and L. Oversteegen [**2**] extended Whitney's results to regular curves that

FIGURE 1.

are one-to-one continuous images of an open interval.

The decomposition in our principal theorem is a generalization of Whitney's decomposition. Instead of ruling out crooked arcs, we only assume that the curves are uniquely arcwise connected. Instead of restricting our attention to flows, we consider all maps that preserve these curves. Although a cross-section may fail to exist, certain flow-box type arguments still work. Whitney's regular families of curves are in separable metric spaces. We must require that our curves lie in the plane.

A special case of our principal theorem, when $\mathcal{D}$ has a compact element, follows immediately from [**17**]. As in [**17**], our proof is based on Bing's dog-chases-rabbit principle [**10**, p. 123]. The present argument is more complicated because the dog may be forced to run down infinitely many rays (instead of just one ray) in pursuit of the rabbit (see Figure 1). Since our proof is rather long, it may be helpful to read the summary in the appendix at the end of this paper before considering the technical details.

2. More background. A collection $\mathcal{D}$ of sets is a *decomposition* of a space if $\cup\mathcal{D}$ is the space and the elements of $\mathcal{D}$ are pairwise disjoint.

A map f *preserves the elements* of a decomposition $\mathcal{D}$ if f sends each element of $\mathcal{D}$ into itself.

A set is *uniquely arcwise connected* if it is arcwise connected and does not contain a simple closed curve.

A *continuum* is a nondegenerate compact connected metric space.

A continuum M has *the fixed-point property* if for each map f of M into M, there exists a point p of M such that $f(p) = p$.

A map f of a continuum M is an *arc-component-preserving map* if f sends each arc component of M into itself.

A map f of a continuum M is a *deformation* if there exists a map h of $M \times [0, 1]$ onto M such that $h(p, 0) = p$ and $h(p, 1) = f(p)$ for each point p of M.

For each number t, $0 \leq t \leq 1$, let h_t be the map of M defined by $h_t(p) = h(p, t)$. The deformation f is called an *isotopic deformation* of M if each h_t is a homeomorphism.

Note that every deformation of a continuum is an arc-component-preserving map.

Every uniquely arcwise connected plane continuum has the fixed-point property [**17**]. Furthermore, every uniquely arcwise connected continuum has the fixed-point property for local homeomorphisms [**35**]. There are uniquely arcwise connected continua in Euclidean 3-space that admit fixed-point-free maps [**45, 10, 25, 35**]. Note that since these continua are arcwise connected, the fixed-point-free maps are obviously arc-component-preserving maps. However, they are not deformations. It is known that every deformation of a uniquely arcwise connected continuum has a fixed point [**18**].

Let $\mathbf{R}$ be the set of real numbers.

A *continuous flow* on a space S is a map ψ of $\mathbf{R} \times S$ onto S such that for each point p of S,

(2.1) $$\psi(0, p) = p \quad \text{and}$$

(2.2) $\psi(t_1 + t_2, p) = \psi(t_1, \psi(t_2, p))$ for all numbers t_1, t_2 in $\mathbf{R}$.

For each number t in $\mathbf{R}$, let ψ_t be the map of S defined by $\psi_t(p) = \psi(t, p)$. By (2.1) and (2.2), $\psi_{-t} = (\psi_t)^{-1}$. Hence, each ψ_t is a homeomorphism. In fact, each ψ_t is an isotopic deformation of S.

The *orbit* of ψ through a point p is $\{\psi_t(p) : t \in \mathbf{R}\}$.

The orbit of ψ through p is a *closed orbit* if there is a smallest positive number σ such that $\psi_\sigma(p) = p$. The number σ is called the *period* of the closed orbit.

A point q is an *ω-limit point* of p if there is an unbounded sequence of positive numbers $t_1, t_2, \ldots$ such that $\psi_{t_1}(p), \psi_{t_2}(p), \ldots$ converges to q. The set of all ω-limit points of p is the *ω-limit set* $L_\omega(p)$. A set of the form $L_\omega(p)$ is called a *limit set* of ψ.

An *equilibrium point* of ψ is a point p of S such that $\psi_t(p) = p$ for each number t in $\mathbf{R}$.

A *tree* is a finite graph that does not contain a simple closed curve.

A continuum M is *tree-like* if for each positive number δ, there is a cover of M with mesh less than δ whose nerve is a tree.

There are tree-like continua without the fixed-point property [**8, 38, 39**]. No such continuum is arcwise connected [**13**]. It is not known if every arc-component-preserving map of a tree-like continuum has a fixed point [**19**, Q. 4.23]. In fact, the question is still open for deformations [**33**, p. 369]. It is known that every isotopic deformation of a tree-like continuum has a fixed point [**20**, Theorem 4.1]. Hence, every flow on a tree-like continuum has an equilibrium point [**20**, Theorem 4.2, **1, 23**, Theorem 4.2].

3. The decomposition $\mathcal{D}$. Let $\mathbf{R}^2$ be the Cartesian plane with metric ρ. Let $\mathbf{S}^2$ denote the 2-sphere that is the one-point compactification $\mathbf{R}^2 \cup \{\omega\}$ of $\mathbf{R}^2$. We denote the boundary, closure, and interior of a given set Y relative to $\mathbf{S}^2$ by $\mathrm{Bd}\, Y$, $\mathrm{Cl}\, Y$, and $\mathrm{Int}\, Y$, respectively.

Let M be a continuum in $\mathbf{R}^2$.

Let $\mathcal{D}$ be a decomposition of M with the property that each element is uniquely arcwise connected.

For each point p of M, let $\mathbf{D}_p$ denote the element of $\mathcal{D}$ that contains p. If p and q are distinct points of M and $q \in \mathbf{D}_p$, then the arc, the half-open arc, and the arc segment (open arc) in $\mathbf{D}_p$ with end points p and q are denoted by $[p, q]$, $[p, q)$, and (p, q), respectively. We define $[p, p]$ to be $\{p\}$.

For each proper subcontinuum C of $\mathbf{S}^2$ and each point v of $\mathbf{S}^2\backslash C$, let $\mathbf{T}_v(C)$ denote the continuum in $\mathbf{S}^2$ that is the complement of the v-component of $\mathbf{S}^2\backslash C$.

Let $\mathcal{Y}$ be the collection of all continua Y in $\mathbf{S}^2$ such that

$$Y = \mathbf{T}_\omega(C) \quad \text{for some subcontinuum } C \text{ of } M \quad \text{and} \tag{3.1}$$

$$\mathbf{D}_p \subset Y \quad \text{for every point } p \text{ of } M \cap Y. \tag{3.2}$$

Note that $\mathbf{T}_\omega(M) \in \mathcal{Y}$.

Lemma 3.3. *There exists an element Y of $\mathcal{Y}$ such that no element of $\mathcal{Y}$ is a proper subcontinuum of Y.*

Proof. Let $\mathcal{Z}$ be a subcollection of $\mathcal{Y}$ that is linearly ordered by inclusion. By the Brouwer reduction theorem [**44**, Theorem 11.1, p. 17], it is sufficient to show that the continuum $\cap\mathcal{Z}$ is an element of $\mathcal{Y}$.

Since (3.1) holds for each element of $\mathcal{Z}$, (3.1) holds for $\cap\mathcal{Z}$ when C is defined to be the boundary of $\mathbf{T}_\omega(\cap\mathcal{Z})$ [**37**, Theorem 24, p. 176]. Since (3.2) holds for each element of $\mathcal{Z}$, it follows that $\mathbf{D}_p \subset \cap\mathcal{Z}$ for every point p of $M \cap (\cap\mathcal{Z})$. Hence, $\cap\mathcal{Z} \in \mathcal{Y}$. This completes the proof of Lemma 3.3. □

4. Rays. Let x be a point of M.

Let $\mathbf{P}_x$ be the image in $\mathbf{D}_x$ of the nonnegative real numbers $[0, +\infty)$ under a one-to-one continuous function φ with the property that $\varphi(0) = x$. The function φ determines a linear ordering $\ll$ of $\mathbf{P}_x$ with x as the first point. We call $\mathbf{P}_x$ a *ray*.

For points y and z of $\mathbf{P}_x$, the notation $[y, z], [y, z)$, and (y, z) will be used only when $y \ll z$.

An arc $[y, z]$ in $\mathbf{P}_x$ is *ordered* from a set Y to a set Z if $y \in Y$ and $z \in Z$.

For each point y of $\mathbf{P}_x$, let $\mathbf{P}_y$ denote the ray $\{z \in \mathbf{P}_x : y = z \text{ or } y \ll z\}$.

Let A_1 and A_2 be disjoint arcs in $\mathbf{R}^2\backslash\mathbf{P}_x$.

For each integer $i = 1, 2$ and $j = 1, 2, 3, 4$, let $a_{i,j}$ be a point of A_i with the following properties:

The end points of A_i are $a_{i,1}$ and $a_{i,4}$ and the order of A_i is such that $a_{i,j} < a_{i,j+1}$ for $j = 1, 2$, and 3.

Suppose there exist disjoint arc segments B_1, B_2, B_3, and B_4 in $\mathbf{R}^2 \backslash (A_1 \cup A_2)$ such that each B_j has $a_{1,j}$ and $a_{2,j}$ as end points and $B_2 \cup B_3$ is in one complementary domain of the simple closed curve $A_1 \cup B_1 \cup A_2 \cup B_4$.

Suppose

(4.1) for $j = 1$ and 2, every arc in $\mathbf{P}_x$ that is ordered from B_{j+1} to B_j intersects B_{j+2},

(4.2) every arc in $\mathbf{P}_x$ that is ordered from B_4 to B_3 intersects B_1, and

(4.3) every arc in $\mathbf{P}_x$ that is ordered from B_1 to B_4 intersects B_2.

Suppose there exist points y and u of $B_1 \cap \mathbf{P}_x$ such that (y, u) intersects B_4 and misses B_1 (see Figure 2). By (4.1) and (4.3), (y, u) intersects B_2 and B_3.

Suppose there exists a point w of $B_4 \cap \mathbf{P}_u$ such that $(u, w) \cap B_4 = \phi$. By (4.1) and (4.3), (u, w) intersects B_2 and B_3.

For $j = 1, 2, 3$, and 4, let C_j be an arc segment in B_j such that $\mathrm{Cl}\, C_j$ is irreducible between $[y, u)$ and $[u, w]$.

The point y is an end point of C_1. Let v be the end point of C_1 opposite y.

Let z be the first point of $[y, u]$ that belongs to B_4.

Note that

(4.4) the simple closed curve $C_1 \cup [y, v]$ separates A_1 from A_2 in $\mathbf{R}^2$.

To see this, for $i = 1$ and 2, let D_i be the arc segment in $\mathrm{Cl}\, B_2 \backslash [y, u]$ that has $a_{i,2}$ and a point of $[y, z]$ as end points. Let E be the disk bounded by $A_1 \cup B_1 \cup A_2 \cup B_4$ that contains $[y, z]$. The set $\{y, z\}$ separates $a_{1,2}$ from $a_{2,2}$ in $\mathrm{Bd}\, E$. Hence, $[y, z]$ separates $a_{1,2}$ from $a_{2,2}$ in E [**37**, Theorem 28, p. 156]. It follows that D_1 and D_2 abut $[y, z]$ from opposite sides [**37**, p. 180]. By (4.1) and (4.2), $B_2 \cap (C_1 \cup [z, v]) = \phi$. Thus, D_1 and D_2 abut $[y, z]$ from opposite sides with respect to

FIGURE 2.

$C_1 \cup [y,v]$ [**37**, Theorem 32, p. 181]. Therefore, $C_1 \cup [y,v]$ separates $a_{1,2}$ from $a_{2,2}$ in $\mathbf{R}^2$. Hence, (4.4) is true.

Next we prove that

$$C_2 \cup [y,w] \text{ separates } C_1 \text{ from } C_3 \text{ in } \mathbf{R}^2. \tag{4.5}$$

Let q be the end point of C_4 opposite w. Let G be the disk bounded by $C_1 \cup [y,q] \cup C_4 \cup [v,w]$ that contains C_2. Let A be an arc from C_1 to C_3 in $C_1 \cup \operatorname{Int} G$. By [**37**, Theorem 28, p. 156], $\operatorname{Cl} C_2$ separates C_1 from C_3 in G. Therefore, $C_2 \cup C_4 \cup [y,w]$ separates C_1 from C_3 in $\mathbf{R}^2$. Since $(C_2 \cup [y,w]) \cap (C_4 \cup [y,w]) = [y,w]$, either $C_2 \cup [y,w]$ or $C_4 \cup [y,w]$ separates C_1 from C_3 in $\mathbf{R}^2$ [**37**, Theorem 20, p. 173]. Since $(q,v) \cap G = \phi$, it follows that $A \cap (C_4 \cup [y,w]) = \phi$. Therefore, $C_4 \cup [y,w]$ does not separate C_1 from C_3 in $\mathbf{R}^2$. Hence, (4.5) is true.

Let Ω be the complementary domain of $C_1 \cup [y,v]$ in $\mathbf{R}^2$ that contains w. Since $(v,w) \subset \Omega$, $(v,w) \cap \operatorname{Cl} C_3 \neq \phi$, and $C_3 \cap (C_1 \cup [y,v]) = \phi$, it follows that $C_3 \subset \Omega$.

Note that

$$C_2 \cup [y,w) \text{ separates } B_4 \cap \Omega \text{ from } C_1 \text{ in } \mathbf{R}^2. \tag{4.6}$$

To see this, let c be a point of $B_4 \cap \Omega$. Let p be the end point of C_3 that belongs to $[y,z]$. Since $C_3 \subset \Omega$, there is an arc segment C from c to $[p,v]$ in $B_4 \cap \Omega$ such that C and C_3 abut $[p,v]$ from the same side with respect to $C_1 \cup [y,v]$. Since $C \cup C_3 \cup [p,v]$ and $C_2 \cup [y,p) \cup (v,w)$ are disjoint, by [**37**, Theorem 32, p. 181], C and C_3 abut $[p,v]$ from the same side with respect to the simple closed curve in $C_2 \cup [y,w)$. Thus, $C_2 \cup [y,w)$ does not separate c from C_3 in $\mathbf{R}^2$. Hence, by (4.5), $C_2 \cup [y,w)$ separates c from C_1 in $\mathbf{R}^2$. Therefore, (4.6) is true.

Finally,

$$\mathbf{P}_w \subset \Omega. \tag{4.7}$$

To see this, assume the contrary. Then, since $\mathbf{P}_y$ does not contain a simple closed curve, $\mathbf{P}_w \cap C_1 \neq \phi$. Let s be the first point of $\mathbf{P}_w$ in C_1. Let r be the last point of $[w,s]$ in B_4. By (4.6), $C_2 \cap [r,s] \neq \phi$, and this contradicts (4.1) and (4.2). Hence, (4.7) is true.

5. Indecomposability in the limit. Let $\mathbf{L}_x$ denote the set $\cap\{\mathrm{Cl}\,\mathbf{P}_y : y \in \mathbf{P}_x\}$.

Hereafter, we assume $\mathbf{L}_x$ is not degenerate. Hence, $\mathbf{L}_x$ is a continuum. We call $\mathbf{L}_x$ the *limit* of the ray $\mathbf{P}_x$.

A continuum is *indecomposable* if it is not the union of two proper subcontinua.

A subset Ψ of a continuum Φ is a *composant* of Φ if there is a point p of Φ such that Ψ is the union of all proper subcontinua of Φ that contain p.

Let Φ be an indecomposable continuum in $\mathbf{S}^2$. J. Krasinkiewicz [**31**] defined a composant Ψ of Φ to be *internal* if every continuum in $\mathbf{S}^2$ that intersects Ψ and $\mathbf{S}^2 \backslash \Psi$ intersects every composant of Φ.

Lemma 5.1. *Suppose Φ is an indecomposable subcontinuum of $\mathbf{L}_x$. Suppose for some point y of $\mathbf{P}_x$, no internal composant of Φ intersects $\mathbf{P}_y$. Then Φ contains uncountably many elements of $\mathcal{D}$.*

Proof. Assume Φ contains only countably many elements of $\mathcal{D}$. Since each element of $\mathcal{D}$ is arcwise connected, no element of $\mathcal{D}$ that is contained in Φ intersects more than one composant of Φ. By [**31**, Theorem 2.3], Φ has uncountably many internal composants. Hence, there exists an internal composant Ψ of Φ and an arc A in an element of $\mathcal{D}$ such that $A \cap \Psi \neq \phi$ and $A \backslash \Psi \neq \phi$.

Since A intersects every internal composant of Φ more than once, there exist points r and s of $A \cap \Psi$ such that $(r, s) \not\subset \Psi$. Let B be a continuum in Ψ that contains $\{r, s\}$. Note that $A \cup B$ separates $\mathbf{S}^2$ [**37**, Theorem 22, p. 175]. Since Ψ is internal, each component of $\mathbf{S}^2 \backslash (A \cup B)$ intersects Φ. Since $\Phi \subset \mathbf{L}_x$ and $\mathbf{P}_y \cap \Psi = \phi$, it follows that $\mathbf{P}_y$ crosses A infinitely many times. Since $\mathbf{P}_y \subset \mathbf{D}_y$, the arc A is in $\mathbf{D}_y$. Hence, $\mathbf{D}_y$ contains a simple closed curve, and contradicts the assumption that each element of $\mathcal{D}$ is uniquely arcwise connected. This completes the proof of Lemma 5.1. $\square$

6. Folded rays. The ray $\mathbf{P}_x$ is *folded* on a subcontinuum L of $\mathbf{L}_x$ if there exist two points y and z of $\mathbf{P}_x$ and a complementary domain Δ of L in $\mathbf{S}^2$ such that $\{y, z\} \subset \Delta$ and $[y, z] \not\subset \mathrm{Cl}\, \Delta$.

Lemma 6.1. *Suppose L is a subcontinuum of $\mathbf{L}_x$ that contains only countably many elements of $\mathcal{D}$ and $\mathbf{P}_x$ is not folded on L. Then there exists a complementary domain Δ of L in $\mathbf{S}^2$ and a point y of $\mathbf{P}_x$ such that $\mathrm{Cl}\, \Delta$ contains $\mathbf{P}_y$.*

Proof. Assume the contrary.

Since $\mathbf{P}_x$ is not folded on L, it follows that

(6.2) for each complementary domain Δ of L, there is a point p of $\mathbf{P}_x$ such that $\Delta \cap \mathbf{P}_p = \phi$.

It follows from (6.2) that

$$L = \mathbf{L}_x. \tag{6.3}$$

Let $\mathcal{E}$ be the collection of elements of $\mathcal{D} \backslash \{\mathbf{D}_x\}$ that are contained in L. Since $\mathcal{E}$ is countable, there exists a countable subset Σ of L such that $\mathcal{E} = \{\mathbf{D}_\sigma : \sigma \in \Sigma\}$.

Let $F_1, F_2, \ldots$ be the elements of a countable base of $\mathbf{S}^2$ that intersect L.

For each positive integer α, let $\mathcal{G}_\alpha$ be the collection of complementary domains of $\mathrm{Cl}\,(\mathbf{P}_x \cup F_\alpha)$ in $\mathbf{S}^2$. Since $\mathbf{S}^2$ is a separable metric space, each $\mathcal{G}_\alpha$ is countable.

For each positive integer α, let

$$H_\alpha = \{p \in L \cap \mathbf{D}_x : [x,p] \cap \mathrm{Cl}\, F_\alpha = \phi\}.$$

For each positive integer α and each point σ of Σ, let

$$H_{\alpha,\sigma} = \{p \in L \cap \mathbf{D}_\sigma : [\sigma,p] \cap \mathrm{Cl}\, F_\alpha = \phi\}.$$

For each positive integer α and each element Γ of $\mathcal{G}_\alpha$, let

$$H_{\alpha,\Gamma} = \{p \in L \backslash \cup (\mathcal{E} \cup \{\mathbf{D}_x\}) : \text{an arc in } \mathbf{D}_p \backslash \mathrm{Cl}\, F_\alpha \text{ goes from } p \text{ into } \Gamma\}.$$

Note that

$$(6.4) \quad L = \cup\{H_\alpha \cup H_{\alpha,\sigma} \cup H_{\alpha,\Gamma} : \alpha = 1, 2, \dots, \sigma \in \Sigma, \text{ and } \Gamma \in \mathcal{G}_\alpha\}.$$

To see this, let p be a point of L. We consider three cases.

Suppose p is a point of $L \cap \mathbf{D}_x$. By (6.2), L is not an arc. Therefore, $[x,p]$ does not contain L. Thus, there is an integer α such that $[x,p] \cap \mathrm{Cl}\, F_\alpha = \phi$. Hence, $p \in H_\alpha$.

Suppose p is a point of $L \backslash \mathbf{D}_x$ and L contains $\mathbf{D}_p$. Let σ be a point of Σ such that $\mathbf{D}_p = \mathbf{D}_\sigma$. Since $[\sigma, p]$ does not contain L, there is an integer α such that $[\sigma,p] \cap \mathrm{Cl}\, F_\alpha = \phi$. Hence, $p \in H_{\alpha,\sigma}$.

Suppose p is a point of $L \backslash \mathbf{D}_x$ and L does not contain $\mathbf{D}_p$. Let q be a point of $\mathbf{D}_p \backslash L$. Since $[p,q]$ does not contain L, there is an integer α such that $[p,q] \cap \mathrm{Cl}\, F_\alpha = \phi$. Note that $q \notin \mathrm{Cl}\,(\mathbf{P}_x \cup F_\alpha)$. Let Γ be the q-component of $\mathbf{S}^2 \backslash \mathrm{Cl}\,(\mathbf{P}_x \cup F_\alpha)$. Then $p \in H_{\alpha,\Gamma}$. Hence, (6.4) is true.

By (6.4) and the Baire category theorem, there exist an integer α, a point σ of Σ, and a complementary domain Γ of $\mathrm{Cl}\,(\mathbf{P}_x \cup F_\alpha)$ such that either H_α, $H_{\alpha,\sigma}$, or $H_{\alpha,\Gamma}$ is somewhere dense in L.

If H_α is somewhere dense in L, define H to be a subset of H_α such that $\mathrm{Cl}\, H$ contains an open subset of L.

If H_α is nowhere dense and $H_{\alpha,\sigma}$ is somewhere dense in L, define H to be a subset of $H_{\alpha,\sigma}$ such that $\mathrm{Cl}\,H$ contains an open subset of L.

If $H_\alpha \cup H_{\alpha,\sigma}$ is nowhere dense and $H_{\alpha,\Gamma}$ is somewhere dense in L, define H to be a subset of $H_{\alpha,\Gamma}$ such that $\mathrm{Cl}\,H$ contains an open subset of L.

Let Y be a disk in F_α such that

$$L \cap \mathrm{Int}\,Y \neq \phi. \tag{6.5}$$

Since $L = \mathbf{L}_x$, there exists a point x' of $\mathbf{P}_x$ in Y.

Note that

(6.6)
each two points of H are the end points of an arc in $\mathbf{S}^2 \backslash (\mathbf{P}_{x'} \cup Y)$.

To see this, let a and b be distinct points of H. We consider three cases.

Suppose $H \subset H_\alpha$. Then Y and $[x,a] \cup [x,b]$ are disjoint. Since $x' \in Y$ and $\mathbf{D}_x$ does not contain a simple closed curve, $\mathbf{P}_{x'}$ and $[x,a] \cup [x,b]$ are disjoint. Hence, $[a,b] \subset \mathbf{S}^2 \backslash (\mathbf{P}_{x'} \cup Y)$.

Suppose $H \subset H_{\alpha,\sigma}$. Then Y and $[\sigma,a] \cup [\sigma,b]$ are disjoint. Since $\mathbf{D}_\sigma \cap \mathbf{P}_x = \phi$, it follows that $[a,b] \subset \mathbf{S}^2 \backslash (\mathbf{P}_{x'} \cup Y)$.

Suppose $H \subset H_{\alpha,\Gamma}$. Let I and J be two arcs in $M \backslash (\mathbf{D}_x \cup \mathrm{Cl}\,F_\alpha)$ from a into Γ and b into Γ, respectively. Since Γ is a connected open subset of $\mathbf{S}^2 \backslash (\mathbf{P}_{x'} \cup \mathrm{Cl}\,F_\alpha)$, there exists an arc K in $I \cup J \cup \Gamma$ from a to b that misses $\mathbf{P}_{x'} \cup \mathrm{Cl}\,F_\alpha$. Since $Y \subset F_\alpha$, the arc K is in $\mathbf{S}^2 \backslash (\mathbf{P}_{x'} \cup Y)$. Hence, (6.6) is true.

Let Λ be a nonempty open subset of L contained in $\mathrm{Cl}\,H$ (see Figure 3).

Let Z be a disk in $\mathbf{S}^2 \backslash Y$ such that

$$L \cap \mathrm{Int}\,Z \neq \phi \quad \text{and} \tag{6.7}$$

$$L \cap Z \subset \Lambda. \tag{6.8}$$

Let η be a point of H.

FIGURE 3.

For each component A of $\mathbf{P}_{x'} \backslash Y$, let $\mathbf{O}(A)$ denote the complementary domain of $A \cup Y$ that misses η.

Let $\mathcal{A}$ be the collection of all components A of $\mathbf{P}_{x'} \backslash Y$ such that $Z \cap \mathbf{O}(A) \neq \phi$. Note that each element of $\mathcal{A}$ is an arc segment with both end points in $\operatorname{Bd} Y$.

For each element A of $\mathcal{A}$,

$$\Lambda \cap \mathbf{O}(A) = \phi \quad \text{and} \tag{6.9}$$

$$A \cap \operatorname{Int} Z \neq \phi. \tag{6.10}$$

Statement (6.9) is true; for otherwise, since $\Lambda \subset \operatorname{Cl} H$, by (6.6), there is an arc in $\mathbf{S}^2 \backslash (A \cup Y)$ that runs from η to a point of $H \cap \mathbf{O}(A)$, and this contradicts the definition of $\mathbf{O}(A)$.

By (6.7) and (6.8), $L \cap \operatorname{Int} Z$ is a nonempty subset of Λ. Therefore, since $Z \cap \mathbf{O}(A) \neq \phi$, (6.10) follows from (6.9).

Since Y and Z are disjoint, it follows from (6.5) and (6.7) that $\mathcal{A}$ is infinite.

By (6.10), for each point p of $\mathbf{P}_{x'}$,

(6.11) $\mathbf{P}_p$ contains all but finitely many elements of $\mathcal{A}$.

For each element A of $\mathcal{A}$,

(6.12) $\mathbf{O}(A)$ contains at most finitely many elements of $\mathcal{A}$

and

(6.13) only finitely many elements of $\mathcal{A}$ separate η from A in $\mathbf{S}^2\backslash Y$.

To verify (6.12), assume there exists an infinite subcollection $\mathcal{B}$ of $\mathcal{A}$ such that $\cup\mathcal{B} \subset \mathbf{O}(A)$. By (6.10), each element of $\mathcal{B}$ intersects Z. Hence, by (6.11), there is a point t of L in $Z \cap \mathrm{Cl}\,(\cup\mathcal{B})$. By (6.8) and (6.9), $t \notin \mathbf{O}(A)$. Therefore, $t \in A$.

Since Λ is an open subset of L that contains t, there exists an arc segment I in $\mathbf{O}(A)$ with end points in $A\backslash\{t\}$ that has the following property. There is a complementary domain K of $A \cup I$ in $\mathbf{O}(A)$ such that $t \in \mathrm{Cl}\,K$ and $K \cap L \subset \Lambda$.

Since infinitely many elements of $\mathcal{B}$ intersect K, it follows from (6.2) and (6.11) that no complementary domain of L contains $K \cap (\cup\mathcal{B})$. Therefore, $K \cap L \neq \phi$. Since $K \subset \mathbf{O}(A)$, this contradicts (6.9). Hence, (6.12) is true.

To establish (6.13), assume the contrary. By (6.12), there exists a sequence $\{A_n : n = 1, 2, \dots\}$ of distinct elements of $\mathcal{A}$ such that $\mathbf{O}(A_n) \subset \mathbf{O}(A_{n+1})$ for each n. For each n, by (6.9) and (6.10), $Z \cap A_n$ misses L and is not empty. Since $\mathbf{P}_x$ is not folded on L, by (6.2) and (6.11), there is an integer n such that $L \cap Z$ separates $A_1 \cap Z$ from $A_n \cap Z$ in Z. Let I be an arc segment in Z such that $\mathrm{Cl}\,I$ is an arc irreducible between A_1 and A_n. Since $A_1 \subset \mathbf{O}(A_n)$, it follows that $I \subset \mathbf{O}(A_n)$. Since $I \cap L \neq \phi$, this contradicts (6.9). Hence, (6.13) is true.

Let $\mathcal{C} = \{A \in \mathcal{A} :$ no element of $\mathcal{A}$ separates η from A in $\mathbf{S}^2\backslash Y\}$. Since $\mathcal{A}$ is infinite, it follows from (6.12) and (6.13) that $\mathcal{C}$ is infinite.

Next we define a special pair p, q of points of $\mathbf{P}_{x'} \cap \mathrm{Int}\,Y$. We consider two cases.

Case 6.14a. Suppose $L \cap \mathbf{P}_{x'} \cap \mathrm{Int}\,Y \neq \phi$. Define p to be a point of $L \cap \mathbf{P}_{x'} \cap \mathrm{Int}\,Y$, and let q be a point of $(\mathbf{P}_p \cap \mathrm{Int}\,Y)\backslash\{p\}$.

Case 6.14b. Suppose $L \cap \mathbf{P}_{x'} \cap \operatorname{Int} Y = \phi$. Applying (6.2), we define p and q to be points of $\mathbf{P}_{x'} \cap \operatorname{Int} Y$ such that $\mathbf{P}_q$ misses the p-component of $\mathbf{S}^2 \backslash L$.

There exist an element (y, u) of $\mathcal{C}$ in $\mathbf{P}_q$, a point z of $(y, u) \cap \operatorname{Int} Z$, and an arc segment I in $(\operatorname{Int} Z) \backslash \operatorname{Cl} \mathbf{O}((y, u))$ such that

(6.15) $$(\operatorname{Cl} I) \cap ([x', u] \cup \operatorname{Bd} Z) = \{z\},$$

(6.16) $$L \cap \operatorname{Cl} I \neq \phi, \quad \text{and}$$

(6.17) $$\text{either } z \in L \quad \text{or} \quad \mathbf{P}_{x'} \cap \operatorname{Cl} I = \{z\}.$$

To verify this, consider two cases.

Case 6.18a. Suppose $L \cap \mathbf{P}_q \cap \operatorname{Int} Z \neq \phi$. Let z be a point of $L \cap \mathbf{P}_q \cap \operatorname{Int} Z$. Let (y, u) be the element of $\mathcal{A}$ that contains z. It follows from (6.9) that $(y, u) \in \mathcal{C}$. Let I be an arc segment in $(\operatorname{Int} Z) \backslash \operatorname{Cl} \mathbf{O}((y, u))$ that satisfies (6.15). Since $z \in L$, (6.16) and (6.17) hold.

Case 6.18b. Suppose $L \cap \mathbf{P}_q \cap \operatorname{Int} Z = \phi$.

First we note that

(6.19) $$(L \backslash \mathbf{P}_x) \cap \operatorname{Int} Z \neq \phi.$$

To see this, assume otherwise. Since $L \cap \operatorname{Int} Z \neq \phi$ and $L \cap \mathbf{P}_q \cap \operatorname{Int} Z = \phi$, there exists a point t of $[x, q]$ in $L \cap \operatorname{Int} Z$. Let T be an open disk in Z containing t such that $T \backslash [x, q]$ has exactly two components. Since $t \in L$, infinitely many elements of $\mathcal{A}$ intersect $T \backslash [x, q]$. Since both components of $T \backslash [x, q]$ are in $\mathbf{S}^2 \backslash L$, this contradicts (6.2) and (6.11). Hence, (6.19) is true.

By (6.19), there is a point t of $L \backslash \mathbf{P}_x$ in $\operatorname{Int} Z$. Let T be an open disk in Z that contains t and misses $[x, q]$. By (6.2), there exist an element (y', u') of $\mathcal{A}$ in $\mathbf{P}_{x'}$ and a point r of $T \cap (y', u')$ such that $T \cap \mathbf{P}_{u'}$ misses the r-component R of $\mathbf{S}^2 \backslash L$.

Let S be an arc in T that runs from r to t. Define s to be the first point of S that belongs to L.

Let I be the arc segment in S that precedes s with the property that $\operatorname{Cl} I$ is irreducible between s and $[q, u']$. Define z to be the end point of

FIGURE 4.

I opposite s. Let (y, u) be the element of $\mathcal{A}$ that contains z. By (6.8) and (6.9), $s \notin \mathrm{Cl}\, \mathbf{O}((y,u))$. Since $z \in R$, it follows that $\mathbf{P}_x \cap \mathrm{Cl}\, I = \{z\}$. Hence, by (6.9), $(y, u) \in \mathcal{C}$. Clearly, (6.15), (6.16) and (6.17) hold.

Let J be a polygonal arc segment in $Z \backslash ([x,u] \cup \mathbf{O}((y,u)) \cup \mathrm{Cl}\, I)$ with end points b and b_0 in (y, u) such that $z \in (b, b_0) \subset Z$ (see Figure 4).

Define K_0 to be the component of $\mathbf{S}^2 \backslash (J \cup [b, b_0])$ that contains I. Note that $K_0 \subset Z$.

The following statements and definitions (6.20_n)–(6.29_n) will be used inductively.

By (6.12), (6.13), and (6.16), for $n = 1$, there exists an element (y_n, u_n) of $\mathcal{C}$ such that

(6.20_n) $$K_{n-1} \cap (y_n, u_n) \neq \phi$$

and

(6.21_n) no element of $\mathcal{C}$ in (b_{n-1}, y_n) intersects K_{n-1}.

For $n = 1$,

(6.22_n) let $P_n = (b_{n-1}, u_n) \cap \mathrm{Cl}\, K_{n-1}$,

(6.23_n)
let I_n be the component of $I\backslash(p, u_n)$ whose closure contains z,

(6.24_n) let J_n be the arc in $\mathrm{Cl}\, J$ that is irreducible between b and (y_n, u_n),

and

(6.25_n) let b_n be the end point of J_n that belongs to (y_n, u_n).

Let J' be the arc in $\mathrm{Cl}\, J$ that is irreducible between b_0 and (y_1, u_1). Since $Y \cup \mathbf{O}((y, u))$ misses $(y_1, u_1) \cup \mathrm{Cl}\, K_0$, there exists an arc segment Ω in $Y \cup \mathbf{O}((y, u))$ from p to z such that $\mathrm{Cl}\, I_1$ and $\mathrm{Cl}\, \Omega$ abut on $[b, b_0]$ from opposite sides with respect to the simple closed curve Υ in $J_1 \cup J' \cup (b, b_0) \cup (y_1, u_1)$ [**37**, Theorem 32, p. 181]. Since $\Upsilon \cap (I_1 \cup \Omega \cup \{p\}) = \phi$, it follows that $J_1 \cup J' \cup (y, u_1)$ separates p from I_1 in $\mathbf{S}^2$. Since $J_1 \cap J' = \phi$, either $J_1 \cup (y, u_1)$ or $J' \cup (y, u_1)$ separates p from I_1 in $\mathbf{S}^2$ [**37**, Theorem 20, p. 173].

We assume without loss of generality that $J_1 \cup (y, u_1)$ separates p from I_1 in $\mathbf{S}^2$.

For $n = 1$,
(6.26_n)
let K_n be the complementary domain of $J_n \cup (b, b_n)$ that contains I_n.

This situation is complicated because $J_n \cup (b, b_n)$ may fail to be a simple closed curve (see Figure 5).

Clearly, for $n = 1$,

(6.27_n) $$p \notin K_n.$$

By (6.16) and (6.17), for $n = 1$,

(6.28_n) $$L \cap (K_n \cup \{z\}) \neq \phi.$$

Next we show that for $n = 1$,

(6.29_n) $$K_n \cap Y \cap \mathbf{P}_{b_n} = \phi.$$

FIGURE 5.

Let A be an arc in J_1 such that $A \cap (p, u_1) = \{b\}$. Since (y, u) and (y_1, u_1) belong to $\mathcal{C}$, it follows that (y, u) and (y_1, u_1) are not separated in $\mathbf{S}^2 \backslash Y$ by an element of $\mathcal{A}$. Hence, there is a polygonal arc segment B in $\mathbf{S}^2 \backslash (Y \cup (p, u_1))$ such that $A \cup \{b_1\} \subset \mathrm{Cl}\, B$. Let I' be an arc in $\mathrm{Cl}\, I_1 \backslash B$ that contains z. Note that $B \cup [b, b_1]$ is a simple closed curve. Let K be the complementary domain of $B \cup [b, b_1]$ that intersects I'.

Observe that

$$K_1 \cap Y \subset K. \tag{6.30}$$

To see this, let c be a point of $K_1 \cap Y$. We must show that $c \in K$. Let C be a polygonal arc in K_1 from c to I' such that $B \cap C$ is finite and C crosses B at each point of $B \cap C$. Clearly, $c \in K$ if $B \cap C = \phi$, so we assume $B \cap C \neq \phi$. Since $I' \cup Y \cup (z, u)$ misses $B \cup J_1$, one complementary domain of $B \cup J_1$ contains $I' \cup \{c\}$. Since $C \cap J_1 = \phi$, it follows that C crosses B an even number of times. Therefore, C crosses $B \cup [b, b_1]$ an even number of times. Thus, $c \in K$. Hence, (6.30) is true.

It follows from (6.30) that (6.29_1) can be established by proving

$$K \cap Y \cap \mathbf{P}_{b_1} = \phi. \tag{6.31}$$

To verify (6.31), first note that since $p \notin K_1$ and $J_1 \cup (b, b_1)$ misses $[p, b) \cup (I' \backslash \{z\})$, the arcs I' and $[p, b]$ abut on $A \cup [b, u]$ from opposite sides with respect to a simple closed curve in $J_1 \cup (b, b_1)$. Hence, I' and $[p, b]$ abut on $A \cup [b, u]$ from opposite sides with respect to $B \cup [b, b_1]$ [**37**, Theorem 32, p. 181]. Since $[p, b) \cap (B \cup [b, b_1]) = \phi$, it follows that $p \notin K$.

Let E be an arc from p to q in $\operatorname{Int} Y$. Either $p \in L$ (Case 6.14a) or $L \cap \mathbf{P}_{x'} \cap \operatorname{Int} Y = \phi$ and $\mathbf{P}_q$ misses the p-component of $\mathbf{S}^2 \backslash L$ (Case 6.14b). Hence, the p-component of $E \backslash \operatorname{Cl} K$ contains a point e of L. Let D be an open set in $Y \backslash \operatorname{Cl} K$ that contains e.

Now suppose that (6.31) is false. Since $e \in L$, there exists an arc $[r, s]$ in $\mathbf{P}_{b_1}$ such that $r \in K \cap Y$ and $s \in D$.

Let (r', s') be an arc segment in $[r, s] \backslash Y$ such that $r' \in K \cap \operatorname{Bd} Y$ and $s' \in \operatorname{Bd} Y \backslash K$. A component of $(B \cup [b, b_1]) \backslash \operatorname{Int} Y$ separates r' from s' in $\mathbf{S}^2 \backslash \operatorname{Int} Y$ [**37**, Theorem 27, p. 177]. Since $B \cap (b, b_1) = \phi$ and $(r', s') \cap (Y \cup (b, b_1)) = \phi$, it follows that $B \cup [b, u] \cup [y_1, b_1]$ separates r' from s' in $\mathbf{S}^2 \backslash \operatorname{Int} Y$. Hence, $\{u, y_1\}$ separates r' from s' in $\operatorname{Bd} Y$. Thus, (r', s') is an element of $\mathcal{A}$ that separates (y, u) from (y_1, u_1) in $\mathbf{S}^2 \backslash Y$ [**37**, Theorem 30, p. 158], and this contradicts the fact that (y, u) and (y_1, u_1) belong to $\mathcal{C}$. Hence, (6.31) is true. Consequently, (6.29_1) is true.

Proceeding inductively, for each integer $n > 1$, we define (y_n, u_n), P_n, I_n, J_n, b_n, and K_n satisfying (6.20_n)–(6.29_n). For $n > 1$, (6.20_n) and (6.21_n) follow from (6.12), (6.13) and (6.28_{n-1}). To verify (6.27_n) for $n > 1$, note that by (6.27_{n-1}), there exists an A in J_n such that $[p, b]$ and $\operatorname{Cl} I_n$ abut on $A \cup [b, u]$ from opposite sides with respect to a simple closed curve in $J_n \cup (b, b_n)$ [**37**, Theorem 32, p. 181]. The arguments given for (6.28_n) and (6.29_n) when $n = 1$ hold when $n > 1$.

Since $Y \cap Z = \phi$, for each point v of $\mathbf{P}_x$,

(6.32) there exists an integer n such that $v \notin \mathbf{P}_{u_n}$.

Let X be the limit superior of $P_1, P_2, \ldots$. By (6.32), $X \subset \mathbf{L}_x$. Thus, by (6.3), $X \subset L$.

Since K_0 misses Y, so does P_1.

It follows from (6.29_n) that

(6.33) $$Y \cap \cup \{P_n : n = 1, 2, \ldots\} = \phi.$$

Since $J_1, J_2, \dots$ is a nested sequence of arcs, $b_1, b_2, \dots$ converges to a point c of $X \cap J_1$. For each positive integer n, let B_n be the polygonal arc in J_n with end points c and b_n.

Since $J_1 \cap [x,y] = \phi$, it follows from (6.21_n) and (6.24_n) that $(J_1 \backslash B_1) \cap \cup\{P_n : n = 1,2,\dots\} = \phi$. Thus, for each positive integer n, every component of P_{n+1} intersects B_n. Therefore, $B_1 \cup P_2$, $B_2 \cup P_3, \dots$ is a sequence of continua whose limit superior is X. Hence, X is connected.

For each component A of $\mathbf{P}_y \backslash Y$,

$$X \cap \operatorname{Int}(Y \cup \mathbf{O}(A)) = \phi. \tag{6.34}$$

To see this, let v be the last point of $\operatorname{Cl} A$ with respect to the ordering of $\mathbf{P}_x$. By (6.21_n) and (6.24_n), $c \notin (b,v)$. Let i be a positive integer such that $B_i \cap (b,v) = \phi$. By (6.8), $c \in \Lambda$. Thus, by (6.9), $B_i \cap \mathbf{O}(A) = \phi$. Since $\mathbf{P}_v$ contains $\cup\{P_n : n = i+1, i+2, \dots\}$ and $\mathbf{P}_x$ does not contain a simple closed curve, $A \cap \cup\{P_n : n = i+1, i+2, \dots\} = \phi$. Since B_i intersects each component of $\cup\{P_n : n = i+1, i+2, \dots\}$, it follows from (6.33) that $\cup\{P_n : n = i+1, i+2, \dots\}$ misses $Y \cup \mathbf{O}(A)$. Hence, (6.34) is established.

Next we prove that Knaster's chainable indecomposable continuum with one end point [**32**, Example 1, p. 204] is a continuous image of X. We use a result [**16**, Theorem 1] that was derived from an argument of Bellamy [**7**].

According to Theorem 1 of [**16**], X can be mapped onto Knaster's continuum if there exists a sequence $G_1, G_2, \dots$ of nonempty open sets in X such that $\operatorname{Cl} G_1 \cap \operatorname{Cl} G_2 = \phi$ and, for each n,

$$G_{2n+1} \cup G_{2n+2} \subset G_{2n-1} \quad \text{and} \tag{6.35_n}$$

there exists a separation of $V_n \cup W_n$ of $X \backslash G_{2n}$ such that

$$G_{2n+1} \subset V_n \text{ and } G_{2n+2} \subset W_n. \tag{6.36_n}$$

To establish the existence of the sequence $G_1, G_2, \dots$, order B_1 so that b_1 is its first point. Let c_1 be the first point of B_1 that belongs to L. By (6.2) and (6.32), $c_1 \neq c$.

If $b_1 \neq c_1$, define C_1 to be the arc in B_1 from b_1 to c_1. If $b_1 = c_1$, let $C_1 = \{c_1\}$.

FIGURE 6.

Note that

$$c_1 \in X. \tag{6.37}$$

To see this, consider two cases.

Case 6.38a. Suppose $c_1 \in \mathbf{P}_z$. Let m be an integer such that $J_m \cap [z, c_1] = \phi$. Then $[z, c_1] \subset \mathrm{Cl}\, K_n$ for each integer $n \geq m$. It follows from (6.12) that $c_1 \in X$.

Case 6.38b. Suppose $c_1 \notin \mathbf{P}_z$. There is a point v of $\mathbf{P}_x$ such that $C_1 \cap \mathbf{P}_v = \phi$; otherwise, by (6.2), $L \cap (C_1 \backslash \{c_1\}) \neq \phi$, and this contradicts the definition of c_1. Let d be the last point of $C_1 \cap \mathbf{P}_x$ that precedes c_1 with respect to the ordering of B_1. Since $C_1 \cap [x, z] = \phi$, it follows that $d \in \mathbf{P}_z$.

Let Π be the arc in C_1 from d to c_1. Let (y', u') be the d-component of $\mathbf{P}_x \backslash Y$. Since $c_1 \in \Lambda$ and $\Pi \cap \mathbf{P}_x = \{d\}$, it follows from (6.9) that $(y', u') \in \mathcal{C}$.

Let A be an arc in $J_1 \backslash \Pi$ such that $A \cap (p, u') = \{b\}$ (see Figure 6).

FIGURE 7.

Observe that

(6.39) A and Π abut on $[y, u']$ from the same side.

To verify (6.39), first note that, by (6.9), $\Pi \cap \mathbf{O}(y', u') = \phi$. Hence, there exists a polygonal arc segment B in $\mathbf{S}^2 \backslash (Y \cup (p, u_1))$ such that $A \cup \Pi \subset \mathrm{Cl}\, B$. Let C be an arc in $\mathbf{S}^2 \backslash (p, u')$ from p to u' such that $B \cap C$ is finite and C crosses B at each point of $B \cap C$.

Suppose (6.39) is false. Then C crosses B an odd number of times (see Figure 7). It follows that p and u' are separated in $\mathbf{S}^2$ by $B \cup [b, d]$. By the argument for (6.31), there is a component of $\mathbf{P}_z \backslash Y$ that separates (y, u) from (y', u') in $\mathbf{S}^2 \backslash Y$, and this contradicts the fact that (y, u) and (y', u') belong to $\mathcal{C}$. Hence, (6.39) is true.

Let i be a positive integer such that $J_i \cap (b, u') = \phi$. Let E be an arc in $\mathrm{Cl}\, I_i$ such that $E \cap [y, u'] = \{z\}$. Since E and A abut on $[y, u']$ from the same side, by (6.39), E and Π abut on $[b, u']$ from the same side. Since $\Pi \cap \mathbf{P}_x = \{d\}$, it follows that $c_1 \in K_j$ for each integer $j \geq i$.

Let F be a disk in K_i such that $c_1 \in \mathrm{Int}\, F$. Since $c_1 \in \mathbf{L}_x$, it follows that $F \cap \mathbf{P}_{b_i} \neq \phi$. For each integer $j > i$, since $c_1 \in K_j$, if $F \cap (b_i, b_j) = \phi$, then $F \subset K_j$. Hence, for some $j > i$, the set P_j contains the first point of $F \cap \mathbf{P}_{b_i}$ with respect to the ordering of $\mathbf{P}_x$ (recall (6.22_n)). It follows that $c_1 \in X$. Thus, (6.37) is established.

FIGURE 8.

Let D_1 and D_2 be open disks in $\mathbf{S}^2$ such that $B_1 \subset D_1$, $\operatorname{Cl} I_1 \subset D_2$, and $\operatorname{Cl} D_1 \cap \operatorname{Cl} D_2 = \phi$ (see Figure 8).

Let $i(1) = 1$.

Let $j(1)$ be an integer greater than 1 such that

$$C_1 \cap \mathbf{P}_{y_{j(1)}} = \phi \quad \text{and} \quad J_{j(1)} \cap [z, b_1] = \phi. \tag{6.40}$$

By (6.12), (6.13), (6.16) and (6.17), there exists an integer $i(2) > j(1)$ such that $D_2 \cap K_1 \cap (y_{i(2)}, u_{i(2)}) \neq \phi$.

Let Θ be an arc segment in $(\operatorname{Bd} Y \cap \operatorname{Cl} \mathbf{O}((y_{i(2)}, u_{i(2)})))\backslash[p, u_{i(2)}]$ that has $u_{i(2)}$ as an end point. Let Θ_1 be a polygonal arc segment in $\mathbf{O}((y_{i(2)}, u_{i(2)}))\backslash[b, b_{i(2)}]$ from a point e of Θ to $D_2 \cap K_1$ such that $B_1 \cap \Theta_1$ is finite and Θ_1 crosses B_1 at each point of $B_1 \cap \Theta_1$.

By (6.29_1), $u_{i(2)} \notin K_1$. Since $\Theta \cap (J_1 \cup [b, b_1]) = \phi$, it follows that $e \notin K_1$.

Let Π_1 be the arc in B_1 from c_1 to c. Since Θ_1 misses $[b, b_1] \cup (J_1 \backslash \Pi_1)$, it follows that Θ_1 crosses Π_1 an odd number of times.

By (6.34), there exists a simple closed curve Σ_1 in $Y \cup \Theta_1 \cup D_2 \cup \mathbf{O}((y, u))$ such that $\Theta_1 \subset \Sigma_1$, $X \cap \Sigma_1 \subset D_2$, and $\Pi_1 \cap \Sigma_1 \subset \Theta_1$. Since Π_1 crosses Σ_1 an odd number of times, Σ_1 separates c from c_1 in $\mathbf{S}^2$.

FIGURE 9.

Define Ω_1 to be the c-component of $\mathbf{S}^2\backslash\Sigma_1$. Note that $B_{i(2)} \subset \Omega_1 \subset \mathbf{S}^2\backslash C_1$.

Let $V_1 = \Omega_1 \cap (X\backslash D_2)$ and $W_1 = X\backslash(D_2 \cup V_1)$.

For $i = 1$ and 2, let $G_i = D_i \cap X$. Note that $V_1 \cup W_1$ is a separation of $X\backslash G_2$.

Let c_2 be the first point of $L \cap B_{i(2)}$ with respect to the ordering of B_1. By (6.2) and (6.32), $c_2 \neq c$.

If $b_{i(2)} \neq c_2$, define C_2 to be the arc in B_1 from $b_{i(2)}$ to $c_{i(2)}$. If $b_{i(2)} = c_2$, let $C_2 = \{c_2\}$. By the argument for (6.37), $c_2 \in X$.

Let D_3 and D_4 be open disks such that $B_{i(2)} \subset D_3 \subset D_1 \cap \Omega_1$ and $C_1 \subset D_4 \subset D_1\backslash\Omega_1$. Note that $D_3 \cap X \subset V_1$ and $D_4 \cap X \subset W_1$ (see Figure 9).

It follows from (6.40) and the arguments for Cases 6.38a and 6.38b that $c_1 \in \mathrm{Cl}\, K_i$ for each integer $i \geq i(2)$.

Proceeding inductively, we let n be an integer greater than 1.

We assume that, for each integer m, $1 < m \leq n$, an integer $i(m)$, a point c_m of $X\backslash\{c\}$, a subset C_m of B_1, and disjoint open disks D_{2m-1},

D_{2m} have been defined such that
(6.41_n)
c_m is the first point of $L \cap B_{i(m)}$ with respect to the ordering of B_1,

$$(6.42_n) \qquad C_m \text{ is a minimal connected set containing } \{b_{i(m)}, c_m\},$$

$$(6.43_n) \qquad B_{i(m)} \subset D_{2m-1},$$

$$(6.44_n) \qquad C_{m-1} \subset D_{2m}, \quad \text{and}$$

$$(6.45_n) \qquad c_{m-1} \in \operatorname{Cl} K_i \qquad \text{for each integer } i \geq i(m).$$

For each integer i, $2 < i \leq 2n$, let $G_i = D_i \cap X$. We assume that for each positive integer i less than n, (6.35_i) and (6.36_i) are satisfied.

Let $j(n)$ be an integer greater than $i(n)$ such that $C_n \cap \mathbf{P}_{y_{j(n)}} = \phi$ and $J_{j(n)} \cap [z, b_{i(n)}] = \phi$. Define $i(n+1)$ to be an integer greater than $j(n)$ such that $D_{2n} \cap K_{i(n)} \cap (y_{i(n+1)}, u_{i(n+1)}) \neq \phi$.

Let Π_n be the arc in B_1 from c_n to c. Define Θ_n to be a polygonal arc segment in $\mathbf{O}((y_{i(n+1)}, u_{i(n+1)}))$ from $(\operatorname{Bd} Y)\backslash X$ to $D_{2n} \cap K_{i(n)}$ that crosses Π_n an odd number of times.

Let Σ_n be a simple closed curve in

$$Y \cup \Theta_n \cup D_{2n} \cup \mathbf{O}((y_{i(n-1)}, u_{i(n-1)}))$$

such that $\Theta_n \subset \Sigma_n$, $X \cap \Sigma_n \subset D_{2n}$, and $\Pi_n \cap \Sigma_n \subset \Theta_n$. Since Π_n crosses Σ_n an odd number of times, Σ_n separates c from c_n in $\mathbf{S}^2$.

Define Ω_n to be the c-component of $\mathbf{S}^2 \backslash \Sigma_n$. Let $V_n = \Omega_n \cap (X \backslash D_{2n})$ and $W_n = X \backslash (D_{2n} \cup V_n)$.

To complete the inductive step, define $c_{n+1}, C_{n+1}, D_{2n+1}$, and D_{2n+2} satisfying (6.41_{n+1})–(6.45_{n+1}), (6.35_n) and (6.36_n) when $G_{2n+1} = X \cap D_{2n+1}$ and $G_{2n+2} = X \cap D_{2n+2}$. It follows from the existence of $G_1, G_2, \ldots$ that Knaster's continuum is a continuous image of X [**16**, Theorem 1].

Since Knaster's continuum is indecomposable, X contains an indecomposable continuum Φ [**32**, Theorem 4, p. 208].

For each internal composant Ψ of Φ,

$$\Psi \cap \mathbf{P}_y = \phi. \tag{6.46}$$

To see this, assume the contrary. Let v be a point of $\Psi \cap \mathbf{P}_y$. Since Ψ is internal and $\Psi \subset X \subset M \backslash \operatorname{Int} Y$, it follows that $v \notin Y$. Let A be the v-component of $\mathbf{P}_y \backslash Y$. Since Ψ is internal, $\Psi \cap \mathbf{O}(A) \neq \phi$, and this contradicts (6.34). Hence, (6.46) is true.

By (6.46) and Lemma 5.1, Φ contains uncountably many elements of $\mathcal{D}$. Since L contains Φ, this contradicts the assumption that L contains only countably many elements of $\mathcal{D}$. This completes the proof of Lemma 6.1. □

Lemma 6.47. *Suppose L is a subcontinuum of $\mathbf{L}_x$ that contains only countably many elements of $\mathcal{D}$ and $\mathbf{P}_x$ is not folded on L. Then for each point v of $\mathbf{P}_x$, the ray $\mathbf{P}_v$ intersects $M \backslash L$.*

Proof. Suppose $\mathbf{P}_v \subset L$ for some point v of $\mathbf{P}_x$. Assume without loss of generality that $\mathbf{P}_x \subset L$. The proof of Lemma 6.1 with Cases 6.14b, 6.18b, and 6.38b deleted shows that this assumption involves a contradiction. □

7. Frames. Suppose $\mathbf{L}_x$ contains only countably many elements of $\mathcal{D}$ and $\mathbf{P}_x$ is not folded on $\mathbf{L}_x$. By Lemma 6.1, there exists a complementary domain Δ of $\mathbf{L}_x$ in $\mathbf{S}^2$ and a point τ of $\mathbf{P}_x$ such that $\operatorname{Cl} \Delta$ contains $\mathbf{P}_\tau$.

We assume without loss of generality that $\mathbf{P}_x \subset \operatorname{Cl} \Delta$. By Lemma 6.47, we can also assume without loss of generality that $x \in \Delta$.

Change (if necessary) the embedding of M in $\mathbf{S}^2$ so that $\omega \in \Delta$.

Suppose L is a nondegenerate subcontinuum of $\mathbf{L}_x$. Since $\mathbf{P}_x \subset \operatorname{Cl} \Delta$ and $\Delta \subset \mathbf{S}^2 \backslash \mathbf{T}_x(L)$, it follows that $\operatorname{Bd} \mathbf{T}_x(L) = L$.

Note that $\mathbf{T}_x(L)$ does not separate $\mathbf{R}^2$.

K. Sieklucki [**41**, Lemma 5.5] proved that $\mathbf{T}_x(L)$ has the following properties:

FIGURE 10.

There exists a sequence $Q_1, Q_2, \dots$ of disks in $\mathbf{R}^2$ such that $\mathbf{T}_x(L) = \cap\{Q_n : n = 1, 2, \dots\}$ and for each n,

$$Q_{n+1} \subset \operatorname{Int} Q_n, \tag{7.1}$$

(7.2)
$$\begin{gathered}\text{the boundary } B_n \text{ of } Q_n \text{ is a polygonal simple closed curve}\\ \text{with consecutive vertices } b_n(1), b_n(2), \dots, b_n(\mu_n), b_n(\mu_{n+1}) = b_n(1),\end{gathered}$$

and

$$\begin{gathered}\text{for } j = 1, 2, \dots, \mu_n, \text{ the interval in } B_n \text{ from } b_n(j) \text{ to } b_n(j+1)\\ \text{has diameter less than } 2^{-n} \text{ (see Figure 10).}\end{gathered} \tag{7.3}$$

For every $b_n(j)$, $n = 1, 2, \dots$, and $j = 1, 2, \dots, \mu_n$, there exists a vertex $b_{n+1}(\nu(j))$ such that

$$\begin{gathered}\text{the interval } N_n(j) \text{ in } \mathbf{R}^2 \text{ from } b_n(j) \text{ to } b_{n+1}(\nu(j))\\ \text{has diameter less than } 2^{-n},\end{gathered} \tag{7.4}$$

(7.5) $$N_n(j)\backslash\{b_n(j), b_{n+1}(\nu(j))\} \subset \operatorname{Int} Q_n\backslash Q_{n+1},$$

and

(7.6) $$N_n(j)\cap N_n(k)=\phi \qquad \text{for each integer } k\neq j,\ 1\leq k\leq \mu_n.$$

For each n, let $\Sigma_n = \cup\{N_n(j) : j = 1, 2, \dots, \mu_n\}$.

Let $N = \cup\{\Sigma_n : n = 1, 2, \dots\}$. Note that each component of N is a half-open arc in $\mathbf{R}^2\backslash\mathbf{T}_x(L)$ with an endpoint in L.

Let m be a given positive integer. Define N_m to be the union of all components of N that intersect B_m. Let O_m be a subset of N_m that is maximal with respect to the property that each component of O_m is a component of N_m and each pair of components of O_m with a common endpoint is separated in $Q_m\backslash L$ by another pair of components of O_m.

Let $c_m(1), c_m(2), \dots, c_m(\xi_m), c_m(\xi_{m+1}) = c_m(1)$ denote the consecutive vertices of B_m that belong to O_m. Since L is not degenerate, we can assume without loss of generality that $\xi_m > 3$.

Let n be an integer greater than m. Define E to be the closure of a component of $Q_n\backslash(O_m \cup \mathbf{T}_x(L))$. We call E an (m,n)-*section* on $\mathbf{T}_x(L)$. The polygonal arc $B_n\cap E$ is called the *bottom* of E. The two components of $E\cap O_m$ are called the *sides* of E. Note that the sides of E are half-open arcs in $Q_n\backslash\mathbf{T}_x(L)$ with distinct end points in L. The diameter of the union of the sides of E is less than 2^{3-m}.

For $j = 1, 2, \dots, \xi_m$, let E_j be the (m,n)-section whose sides are contained in the components of O_m that intersect $\{c_{m(j)}, c_{m(j+1)}\}$.

Assume there is a point σ of L and an arc segment Σ in $\mathbf{S}^2\backslash\mathbf{T}_x(L)$ such that $\sigma \in \operatorname{Cl}\Sigma \subset \mathbf{D}_\sigma \subset M\backslash\mathbf{D}_x$.

Assume without loss of generality that $B_m\cap\Sigma \neq \phi$.

Suppose there exist two (m,n)-sections E and F that have a common side such that $Q_n\cap\Sigma \subset E\cup F$ and $\operatorname{Cl}\Sigma$ misses the closure of each uncommon side of E and F. Change the indexing of the (m,n)-sections (if necessary) so that $E = E_1$, $F = E_{\xi_m}$, and each pair of consecutive sections has a common side.

Define F_1 to be the closure of the component of $(E_1\cup E_{\xi_m})\backslash(\Sigma\cup L)$ that contains a side of E_2. Let $F_j = E_j$ for $1 < j < \xi_m$. Define F_{ξ_m} to be the closure of the component of $(E_1\cup E_{\xi_m})\backslash(\Sigma\cup L)$ that contains a side of $E_{\xi_m - 1}$. We call $\mathcal{F} = \{F_j : 1 \leq j \leq \xi_m\}$ an *m-frame* on $(\mathbf{T}_x(L), \Sigma)$. We call F_1 and F_{ξ_m} the *end sections* of $\mathcal{F}$. Each F_j, $1 < j < \xi_m$, is called an *interior section* of $\mathcal{F}$. Let A be the arc in $\mathrm{Cl}\,\Sigma$ that is irreducible between σ and B_n. The half-open arc $A \backslash \{\sigma\}$ is the *common side* of F_1 and F_{ξ_m}.

Since Σ is an arc segment, for each positive integer m, there exists an m-frame on $(\mathbf{T}_x(L), \Sigma)$.

F.B. Jones [**27**] defined a continuum C to be *nonaposyndetic* at a point p of C with respect to a point q of $C \backslash \{p\}$ if every subcontinuum of C that contains p in its interior also contains q.

Lemma 7.7. *For each positive integer i, there exists an m-frame $\mathcal{F}$ on $(\mathbf{T}_x(L), \Sigma)$ such that $m > i$ and no pair of consecutive sections of $\mathcal{F}$ contains L in its union.*

Proof. Let m and m' be integers such that $0 < m \leq m'$. Suppose $\mathcal{F}$ is an m-frame on $(\mathbf{T}_x(L), \Sigma)$ and U is the union of a pair of consecutive sections of an m'-frame on $(\mathbf{T}_x(L), \Sigma)$. Then the union of some pair of consecutive sections of $\mathcal{F}$ contains $U \cap L$. Hence, it is sufficient to show that there exists an m-frame $\mathcal{F}$ on $(\mathbf{T}_x(L), \Sigma)$ such that no pair of consecutive sections of $\mathcal{F}$ contains L in its union.

Assume that, for each positive integer m, every m-frame on $(\mathbf{T}_x(L), \Sigma)$ has a pair of consecutive sections whose union contains L. Then, for each m, there exist a positive number δ_m, a pair of consecutive sections E_m, F_m of an m-frame on $(\mathbf{T}_x(L), \Sigma)$ and an arc segment A_m in $B_m \cup O_m \cup \Sigma$ such that $\delta_1, \delta_2, \ldots$ converges to zero, $L \subset E_m \cup F_m$, the arc A_m has diameter less than δ_m and contains the uncommon sides of E_m and F_m, and $A_m \cup \mathbf{T}_x(L)$ separates $(\mathrm{Int}\,E_m \cup \mathrm{Int}\,F_m) \backslash L$ from $\mathbf{S}^2 \backslash Q_m$ in $\mathbf{S}^2$.

For each positive integer m, let x_m and y_m be the end points of A_m. Note that $\{x_m, y_m\} \subset L$ for each m. For each m, let W_m be the complementary domain of $A_m \cup \mathbf{T}_x(L)$ in $\mathbf{S}^2$ whose closure contains $E_m \cup F_m$. Let y be a limit point of $y_1, y_2, \ldots$.

The continuum L is nonaposyndetic at y with respect to each point of $L \backslash \{y\}$. For assume otherwise. Then there exists a continuum Y, an open disk G, and a point z of $L \backslash \mathrm{Cl}\,G$ such that $y \in G \cap L \subset Y \subset L \backslash \{z\}$.

FIGURE 11.

Let Z be an open disk such that $z \in Z \subset \mathbf{S}^2 \backslash (G \cup Y)$.

Let i be an integer such that $B_i \cap Z \neq \phi$. Define m to be an integer greater than i such that $A_m \subset G$. Let p be a point of $Z \cap (Q_i \backslash Q_m)$. Since $z \in L \cap Z$ and $L \subset E_m \cup F_m \subset \operatorname{Cl} W_m$, there is a point q of W_m in Z (see Figure 11).

There exists a polygonal arc I in $Q_i \backslash \mathbf{T}_x(L)$ from p to q such that $A_m \cap I$ is finite and I crosses A_m at each point of $A_m \cap I$. Since A_m separates p from q in $Q_i \backslash \mathbf{T}_x(L)$, the arc I crosses A_m an odd number of times. It follows that $I \cup Z$ contains a simple closed curve that separates x_m from y_m in $\mathbf{S}^2$. Since $\{x_m, y_m\} \subset Y \subset \mathbf{S}^2 \backslash (I \cup Z)$, this violates the connectivity of Y. Hence, L is nonaposyndetic at y with respect to each point of $L \backslash \{y\}$.

According to a theorem of H.E. Schlais [**40**, Theorem 9, **16**, Theorem 4], L contains an indecomposable continuum Φ.

For each internal composant Ψ of Φ, note that $\Psi \cap \mathbf{P}_x = \phi$. To see this, assume there is a point y of $\mathbf{P}_x$ in Ψ. By Lemma 6.47, there is a point z of $\mathbf{P}_y$ in Δ. Let J be an arc in Δ that is irreducible between $[x, y]$ and $[y, z]$. Since Ψ is internal, each complementary domain of

$J \cup [x, z]$ intersects Φ. Since $\Phi \subset \mathbf{L}_x$ and $[x, z) \cap \mathbf{P}_z = \phi$, it follows that $J \cap \mathbf{P}_u \neq \phi$ for each point u of $\mathbf{P}_x$. Thus, $J \cap \mathbf{L}_x \neq \phi$, and this contradicts the fact that J is in Δ. Hence, $\Psi \cap \mathbf{P}_x = \phi$.

According to Lemma 5.1, Φ contains uncountably many elements of $\mathcal{D}$. Since $\Phi \subset L \subset \mathbf{L}_x$, this contradicts the assumption that $\mathbf{L}_x$ contains only countably many elements of $\mathcal{D}$. This completes the proof of Lemma 7.7. □

For each real number ζ, let $I(\zeta)$ denote the interval $\{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq 1 \text{ and } y = \zeta\}$.

Suppose A is an arc, B is a continuum, and $A \cup B \subset \mathbf{S}^2$. Then B *straddles* A if, for each homeomorphism h of $\cup\{I(\zeta) : -1 \leq \zeta \leq 1\}$ into $\mathbf{S}^2$ with $h(I(0)) = A$, there exists a positive real number δ such that $B \cap h(I(\zeta)) \neq \phi$ when $|\zeta| < \delta$.

Lemma 7.8. *Suppose F is a section of an m-frame $\mathcal{F}$ on $(\mathbf{T}_x(L), \Sigma)$, y is a point of $(\mathbf{P}_x \cap \operatorname{Int} F)\backslash L$, z is a point of $\mathbf{P}_y\backslash F$, and $\cup\mathcal{F}$ contains $[y, z]$. Then $[y, z]$ intersects the closure of a side of F.*

Proof. Assume $[y, z]$ misses the closure of each side of F. By Lemma 6.47, there is a point u of $\mathbf{P}_z$ in Δ. Let p be a point of $L \cap [y, z] \cap \operatorname{Bd} F$ such that every arc in $[y, p]$ from $\mathbf{P}_x\backslash\mathbf{T}_x(L)$ to p intersects $\operatorname{Int} F$, and every arc in $[p, u]$ from p to $\mathbf{P}_x\backslash\mathbf{T}_x(L)$ intersects $\mathbf{P}_x\backslash F$. Let J be an arc in Δ that is irreducible between $[x, p]$ and $[p, u]$.

The continuum $L \cap \operatorname{Bd} F$ straddles every arc in $\mathbf{P}_x$ that contains p and has both end points in Δ. Consequently, each complementary domain of $J \cup [x, u]$ intersects $L \cap \operatorname{Bd} F$. Since $[x, u) \cap \mathbf{P}_u = \phi$, it follows that $J \cap L \neq \phi$, and this contradicts the fact that J is in Δ. Hence, $[y, z]$ intersects the closure of a side of F. □

8. Framing over a sequence of rays. Suppose there exists a sequence $\mathbf{P}_{x_1}, \mathbf{P}_{x_2}, \dots$ of pairwise disjoint rays in M such that $\mathbf{L}_{x_n} \supset \operatorname{Cl} \mathbf{P}_{x_{n+1}}$ and $\mathbf{L}_{x_n} \neq \mathbf{L}_{x_{n+1}}$ for each positive integer n. Note that $\mathbf{L}_{x_1}, \mathbf{L}_{x_2}, \dots$ is a strictly decreasing sequence of continua.

Let $L = \cap\{\mathbf{L}_{x_n} : n = 1, 2, \dots\}$. Assume L is not degenerate.

Let $\mathbf{Q}_{x_1}$ denote the connected set $\cup\{\mathbf{P}_{x_n} : n = 1, 2, \dots\}$.

Define a linear order $\lll$ on $\mathbf{Q}_{x_1}$ as follows:

Define $y \lll z$ if $\{y, z\} \subset \mathbf{P}_{x_n}$ and $y \ll z$ or if $y \in \mathbf{P}_{x_m}$, $z \in \mathbf{P}_{x_n}$, and $m < n$.

For each point y of $\mathbf{Q}_{x_1}$, let $\mathbf{Q}_y$ denote $\{z \in \mathbf{Q}_{x_1} : y = z \text{ or } y \lll z\}$.

Suppose there exists a point y of $\mathbf{Q}_{x_1}$ in a complementary domain Δ of L such that $\mathbf{Q}_y \subset \operatorname{Cl}\Delta$. Since $\Delta = \mathbf{S}^2 \backslash \mathbf{T}_y(L)$, it follows that $\operatorname{Bd}\mathbf{T}_y(L) = L$.

Assume there is a point σ of L and an arc segment Σ in Δ such that $\sigma \in \operatorname{Cl}\Sigma \subset \mathbf{D}_\sigma \subset M \backslash \mathbf{Q}_y$.

Let $\mathcal{F} = \{F_j : 1 \leq j \leq \xi_m\}$ be an m-frame on $(\mathbf{T}_y(L), \Sigma)$.

Let $\mathcal{G} = \{F_j \in \mathcal{F} : \mathbf{P}_z \cap (\operatorname{Int} F_j \backslash L) \neq \phi$ for each point z of $\mathbf{Q}_y\}$.

Assume $\mathcal{G}$ has at least two elements. Let F_i and F_k be the first and last sections, respectively, of $\mathcal{F}$ that belong to $\mathcal{G}$.

Lemma 8.1. *Suppose A is an arc segment in $F_j \backslash L$, $i < j < k$, that has an end point in L. Then there exists a point z of $\mathbf{Q}_y$ such that each arc in $\mathbf{Q}_z$ that intersects both* $\operatorname{Int} F_i \backslash L$ *and* $\operatorname{Int} F_k \backslash L$ *also intersects A.*

Proof. Let $\mathcal{H}$ be an m-frame on $(\mathbf{T}_y(L), \Sigma)$ such that A intersects the bottom of a section of $\mathcal{H}$. Since $L = \cap\{\mathbf{L}_{x_n} : n = 1, 2, \dots\}$ and each $\mathbf{L}_{x_n}$ contains $\operatorname{Cl}\mathbf{P}_{x_{n+1}}$, there is a point z of $\mathbf{Q}_y$ such that $\mathbf{Q}_z \subset (\cup\mathcal{H}) \backslash (\operatorname{Cl} A \backslash A)$.

Suppose there is an arc B in $\mathbf{Q}_z \backslash A$ that intersects both $\operatorname{Int} F_i \backslash L$ and $\operatorname{Int} F_k \backslash L$. Since the elements of $\mathbf{P}_{x_1}, \mathbf{P}_{x_2}, \dots$ are pairwise disjoint, there is a point u of $\mathbf{Q}_z$ such that $\mathbf{P}_u$ contains B. Assume without loss of generality that u is an end point of B that belongs to $\operatorname{Int} F_i \backslash L$ and the other end point v of B belongs to $\operatorname{Int} F_k \backslash L$.

Let U be the u-component of $(\cup\mathcal{H}) \backslash (A \cup \Sigma \cup L)$. Note that U misses F_k and contains $(\cup\mathcal{H}) \cap (\operatorname{Int} F_i \backslash L)$. Since $B \cap \operatorname{Cl}(A \cup \Sigma) = \phi$, the continuum $L \cap \operatorname{Bd} U$ straddles each subarc of B that has one end point in U and the other end point in $M \backslash \operatorname{Cl} U$.

Since $F_i \in \mathcal{G}$, there is a point w of $\mathbf{P}_v$ in $\operatorname{Int} F_i \backslash L$. Let C be an arc in U that is irreducible between $[u, v]$ and $[v, w]$. Each complementary domain of $C \cup [u, w]$ intersects $L \cap \operatorname{Bd} U$. Note that $[u, w) \cap \mathbf{Q}_w = \phi$.

For each positive integer n, since $\mathbf{L}_{x_n} \supset L$, it follows that $C \cap \mathbf{L}_{x_n} \neq \phi$. Thus, $C \cap L \neq \phi$, and this contradicts the fact that C is in U. Hence, each arc in $\mathbf{Q}_z$ that intersects $\operatorname{Int} F_i \backslash L$ and $\operatorname{Int} F_k \backslash L$ intersects A. □

A *subframe* of $\mathcal{F}$ is a collection of consecutive sections of $\mathcal{F}$.

By Lemma 8.1, for each F_j, $i < j < k$, there is a point z of $\mathbf{Q}_y$ such that each arc in $\mathbf{Q}_z$ that intersects both $\operatorname{Int} F_i \backslash L$ and $\operatorname{Int} F_k \backslash L$ also intersects $\operatorname{Int} F_j \backslash L$. Since each $\mathbf{L}_{x_n}$ contains $\operatorname{Cl} \mathbf{P}_{x_{n+1}}$, it follows that $F_j \in \mathcal{G}$. Hence, $\mathcal{G}$ is the subframe $\{F_j : i \leq j \leq k\}$ of $\mathcal{F}$.

A connected subset K of $(\cup\mathcal{G}) \backslash \operatorname{Bd}(\mathbf{T}_y(L) \cup (\cup\mathcal{F}))$ that intersects $\operatorname{Int} F_i$ and $\operatorname{Int} F_k$ (the interiors of the end sections of $\mathcal{G}$) is a *trace* of $\mathcal{G}$ if for each arc A in K, there exists a function g of A into $\mathcal{G}$ such that (1) $a \in g(a)$ for each point a of A and (2) if a and b are points of A and $g(a) \neq g(b)$, then the arc in A from a to b intersects a side of $g(a)$ and the interior of each section of $\mathcal{G}$ between $g(a)$ and $g(b)$ (with respect to the index ordering of $\mathcal{G}$).

A set K *agrees* with $\mathcal{G}$ if K is a trace of $\mathcal{G}$, $\mathcal{G}\backslash\{F_i\}$, $\mathcal{G}\backslash\{F_k\}$, or $\mathcal{G}\backslash\{F_i, F_k\}$.

Lemma 8.2. *There is a point z of $\mathbf{Q}_y \cap \operatorname{Int} F_i$ such that $\mathbf{Q}_z$ is a trace of $\mathcal{G}$.*

Proof. Let $T = \{t \in L$: t is an end point of a side of a section of $\mathcal{G}\}$.

Define z to be a point of $\mathbf{Q}_y \cap (\operatorname{Int} F_i \backslash L)$ such that $\mathbf{Q}_z$ is contained in $\cup\mathcal{F}$ and misses $T \cup \operatorname{Bd}(\mathbf{T}_y(L) \cup (\cup\mathcal{F}))$ and $\cup\{\operatorname{Int} F_j \backslash L : 1 \leq j < i$ or $k < j \leq \xi_m\}$.

Using Lemma 7.8, we define a function g^* of $\mathbf{Q}_z$ onto $\mathcal{G}$ such that (1) $u \in g^*(u)$ for each point u of $\mathbf{Q}_z$ and (2) if v and w are points of $\mathbf{P}_u$ for some point u of $\mathbf{Q}_z$ and $g^*(v) \neq g^*(w)$, then $[v, w]$ intersects a side of $g^*(v)$ and the interior of each section of $\mathcal{G}$ between $g^*(v)$ and $g^*(w)$.

By considering the restriction of g^* to each arc in $\mathbf{Q}_z$, we see that $\mathbf{Q}_z$ is a trace of $\mathcal{G}$. □

9. Framing over one ray. Suppose L is a nondegenerate subcontinuum of L_x. Suppose there exists a point y of $\mathbf{P}_x$ in a complementary domain Δ of L such that $\mathbf{P}_y \subset \mathrm{Cl}\,\Delta$. Assume there exist a point σ of L and an arc segment Σ in Δ such that $\sigma \in \mathrm{Cl}\,\Sigma \subset \mathbf{D}_\sigma \subset M \backslash \mathbf{D}_x$.

As in Lemma 8.1, let $\mathcal{F} = \{F_j : 1 \le j \le \xi_m\}$ be an m-frame on $(\mathbf{T}_y(L), S)$. Let $\mathcal{G} = \{F_j \in \mathcal{F} : \mathbf{P}_z \cap (\mathrm{Int}\, F_j \backslash L) \neq \phi$ for each point z of $\mathbf{P}_y\}$.

Assume $\mathcal{G}$ has at least two elements. Let F_i and F_k be the first and last sections, respectively, of $\mathcal{F}$ that belong to $\mathcal{G}$.

The argument given for Lemma 8.1 can be modified to establish the following:

Lemma 9.1. *Suppose A is an arc segment in $F_j \backslash L$, $i < j < k$, that has an end point in L. Then there exists a point z of $\mathbf{P}_y$ such that each arc in $\mathbf{P}_z$ that intersects both* $\mathrm{Int}\, F_i \backslash L$ *and* $\mathrm{Int}\, F_k \backslash L$ *also intersects A.*

It follows from Lemma 9.1 that $\mathcal{G}$ is the subframe $\{F_j : i \le j \le k\}$ of $\mathcal{F}$.

The proof of Lemma 8.2 can be modified to establish the following:

Lemma 9.2. *There is a point z of $\mathbf{P}_y \cap \mathrm{Int}\, F_i$ such that $\mathbf{P}_z$ is a trace of $\mathcal{G}$.*

10. Borsuk rays. Let f be a map of M that preserves the elements of $\mathcal{D}$.

Assume f moves each point of M.

By the compactness of M and the continuity of f, there is a positive number δ such that for every point x of M,

$$\rho(x, f(x)) > \delta. \tag{10.1}$$

It follows from (10.1) and an argument of K. Borsuk [**13**] that for each point x of M, there exists a unique sequence $a_x(1), a_x(2), \dots$ of points of $\mathbf{D}_x$ such that $a_x(1) = x$ and, for each positive integer n,

$$\rho(a_x(n), a_x(n+1)) = \delta/3 \qquad [\mathbf{13},\ (4_n)], \tag{10.2}$$

$$\text{if } y \in (a_x(n), a_x(n+1)), \quad \text{then } \rho(a_x(n), y) < \delta/3 \qquad [\mathbf{13}, (5_n)], \tag{10.3}$$

$$[x, a_x(n)] \cap [a_x(n), a_x(n+1)] = \{a_x(n)\} \quad [\mathbf{13}, (11)], \tag{10.4}$$

and

$$\{a_x(n), a_x(n+1)\} \subset [x, f(a_x(n))] \qquad [\mathbf{13}, (7_n)\ (13)]. \tag{10.5}$$

For each positive integer n, let φ_n be a homeomorphism of the half-open real line interval $[n-1, n)$ onto $[a_x(n), a_x(n+1))$. Let φ be the map defined by $\varphi(r) = \varphi_n(r)$ if $n - 1 \leq r < n$.

Define $\mathbf{P}_x$ to be $\cup\{[x, a_x(n)) : n = 2, 3, \dots\}$. By (10.4), φ is a one-to-one map of the nonnegative real line $[0, +\infty)$ onto $\mathbf{P}_x$. The map φ determines a linear ordering $\ll$ of $\mathbf{P}_x$ with x as the first point.

We call $\mathbf{P}_x$ a *Borsuk ray*.

By [**18**, (3.6)],

$$\mathbf{P}_x = \{z \in \mathbf{D}_x : [x, z] \cap [z, f(z)] = \{z\}\}. \tag{10.6}$$

Let y be a point of $\mathbf{P}_x \backslash \{x\}$. By (10.6), $\mathbf{P}_y = \mathbf{P}_x \backslash [x, y)$. Therefore, $\mathbf{P}_y = \{z \in \mathbf{P}_x : y = z \text{ or } y \ll z\}$.

As in Section 5, define $\mathbf{L}_x$ to be $\cap\{\mathrm{Cl}\,\mathbf{P}_y : y \in \mathbf{P}_x\}$. By (10.2), $\mathbf{L}_x$ is not degenerate. Hence, $\mathbf{L}_x$ is a subcontinuum of $\mathrm{Cl}\,\mathbf{P}_x$.

Note that, for each point y of $\mathbf{P}_x$,

$$\mathbf{L}_y = \mathbf{L}_x. \tag{10.7}$$

By [**18**, (3.7)], for each point y of $\mathbf{D}_x$,

$$\mathbf{P}_x \cap \mathbf{P}_y \neq \phi. \tag{10.8}$$

By (10.2), $\mathbf{P}_x \not\subset [x, f(x)]$. Since $\mathbf{D}_x$ does not contain a simple closed curve, there exists a point ζ of $\mathbf{P}_x$ such that $\mathbf{P}_\zeta \cap [x, f(x)] = \{\zeta\}$.

By [**17**, (7), p. 98],

(10.9) $$\zeta \in f([x,\zeta]).$$

By [**17**, (8), p. 99], for each point y of $\mathbf{P}_\zeta$,

(10.10) $$y \in f([x,y]).$$

By [**17**, (9), p. 99], for each point y of $\mathbf{P}_x$,

(10.11) there exists a point z of $\mathbf{P}_y$ such that $\mathbf{P}_z \cap f([x,y]) = \phi$.

11. Nested limits. As in Section 3, assume M is embedded in $\mathbf{S}^2\backslash\{\omega\}$.

By Lemma 3.3, there exists a continuum Y in $\mathbf{S}^2$ such that

(11.1) $Y = \mathbf{T}_\omega(C)$ for some subcontinuum C of M,

(11.2) $\mathbf{D}_p \subset Y$ for every point p of $M \cap Y$, and

(11.3)
no proper subcontinuum of Y satisfies conditions (11.1) and (11.2).

Let Λ be an arc segment in $\mathbf{S}^2\backslash Y$ that has an end point x_1 in C. By (11.2), $\mathbf{P}_{x_1} \subset Y$. Therefore, $\mathbf{L}_{x_1} \subset Y$.

Let $\mathcal{U} = \{U$: there is a point x of $\mathbf{L}_{x_1}$ such that $\mathbf{L}_x = U$ and $\mathbf{P}_x \subset \mathbf{L}_{x_1}\}$.

Case 11.4a. Suppose $\mathcal{U}$ is empty. Then define L to be $\mathbf{L}_{x_1}$.

Case 11.4b. Suppose $\mathcal{U}$ is not empty.

Define a binary relation $\rightarrow$ on $\mathcal{U}$ as follows:

For each pair of elements U, V of $\mathcal{U}$, define $U \rightarrow V$ if there is a point x of U such that $\mathbf{P}_x \subset U$ and $\mathbf{L}_x = V \neq U$.

An $\to$ *nest* in $\mathcal{U}$ is a nonempty subcollection of $\mathcal{U}$ that is linearly ordered by $\to$.

Let $\mathcal{W} = \{W\colon$ there is an $\to$ nest $\mathcal{N}$ in $\mathcal{U}$ such that $W = \cap\mathcal{N}\}$.

By (10.2), each element of $\mathcal{W}$ is nondegenerate. Thus, each element of $\mathcal{W}$ is a continuum. Since each singleton of $\mathcal{U}$ is an $\to$ nest, $\mathcal{U} \subset \mathcal{W}$.

Define a binary relation $\to\!\to$ on $\mathcal{W}$ as follows:

Let V and W be elements of $\mathcal{W}$.

Suppose $\{V, W\} \subset \mathcal{U}$. Then $V \to\!\to W$ if $V \to W$.

Suppose $V \in \mathcal{U}$ and $W \in \mathcal{W}\backslash\mathcal{U}$. Then $V \to\!\to W$ if there is an $\to$ nest $\mathcal{N}$ in $\mathcal{U}$ that has V as its first element such that $W = \cap\mathcal{U}$; and $W \to\!\to V$ if there is a point x in W such that $\mathbf{P}_x \subset W$ and $\mathbf{L}_x = V \neq W$.

Suppose $\{V, W\} \subset \mathcal{W}\backslash\mathcal{U}$. Then $V \to\!\to W$ if there is an $\to$ nest $\mathcal{N}$ in $\mathcal{U}$ that has a first element U such that $V \to\!\to U$ and $W = \cap\mathcal{N}$.

Note that $\to\!\to$ is transitive.

Furthermore,

(11.5) each $\to\!\to$ nest in $\mathcal{W}$ has an upper bound in $\mathcal{W}$.

To see this, let $\mathcal{N}$ be an $\to\!\to$ nest in $\mathcal{W}$. If $\mathcal{N}$ is finite, then clearly $\mathcal{N}$ has an upper bound. Thus, we assume that $\mathcal{N}$ is infinite. For each pair of elements V, W of $\mathcal{N}$, note that $V \to\!\to W$ only if W is a proper subcontinuum of V. Since M is a compact metric space, it follows that there is a sequence $V_1, V_2, \dots,$ of elements of $\mathcal{N}$ such that $V_n \to\!\to V_{n+1}$ for each n and $\cap\mathcal{N} = \cap\{V_n : n = 1, 2, \dots\}$. For each n, if $V_n \in \mathcal{U}$, define $U_n = V_n$; and if $V_n \notin \mathcal{U}$, define U_n to be an element of $\mathcal{U}$ such that $V_n \to\!\to U_n \to\!\to V_{n+1}$. Then $\cap\mathcal{N} = \cap\{U_n : n = 1, 2, \dots\}$. Thus, $\cap\mathcal{N}$ is an element of $\mathcal{W}$ and an upper bound for $\mathcal{N}$. Hence, (11.5) is true.

Using (11.5) and Zorn's lemma [**28**, p. 33], define L to be a maximal element of $\mathcal{W}$ with respect to $\to\!\to$.

In Cases 11.4a and 11.4b, for each point x of L

(11.6) either $\mathbf{P}_x \not\subset L$ or $\mathbf{L}_x = L$.

12. An uncountable subcollection of $\mathcal{D}$. Let $\mathcal{E}$ be the collection of elements of $\mathcal{D}$ that are contained in the continuum L.

In this section, we prove that

$$\mathcal{E} \text{ is uncountable.} \tag{12.1}$$

Either $L \in \mathcal{W}\backslash\mathcal{U}$ or $L \in \mathcal{U} \cup \{\mathbf{L}_{x_1}\}$.

Case 12.2a. Suppose $L \in \mathcal{W}\backslash\mathcal{U}$. Then, by the argument for (11.5), there exists a sequence $U_1, U_2, \dots$ of elements of $\mathcal{U}$ such that $U_1 \to U_2 \to \cdots$ and $L = \cap\{U_n : n = 1, 2, \dots\}$.

For $n = 2, 3, \dots$, there is a point x_n of $\mathbf{L}_{x_{n-1}}$ such that $\mathrm{Cl}\,\mathbf{P}_{x_n} \subset \mathbf{L}_{x_{n-1}}$ and $\mathbf{L}_{x_n} = U_{n-1}$.

Note that, for each positive integer n,

$$\mathrm{Cl}\,\mathbf{P}_{x_n} \supset \mathbf{L}_{x_n} \supset \mathrm{Cl}\,\mathbf{P}_{x_{n+1}}. \tag{12.3}$$

For each pair of distinct positive integers m and n,

$$\mathbf{D}_{x_m} \cap \mathbf{D}_{x_n} = \phi. \tag{12.4}$$

To see this, assume the contrary. Then $\mathbf{P}_{x_m} \cup \mathbf{P}_{x_n} \subset \mathbf{D}_{x_m}$. Since $U_1 \to U_2 \to \cdots$, it follows that $\mathbf{L}_{x_m} \neq \mathbf{L}_{x_n}$. Thus, by (10.7), $\mathbf{P}_{x_m} \cap \mathbf{P}_{x_n} = \phi$, and this contradicts (10.8). Hence, (12.4) is true.

Note that

$$\mathbf{P}_{x_1} \text{ is not folded on } L. \tag{12.5}$$

To see this, assume the contrary. Then there exist a complementary domain Δ of L and three points y, z, and w of $\mathbf{P}_{x_1}$ such that $\{y, w\} \subset \Delta$ and $z \in [y, w]\backslash\mathrm{Cl}\,\Delta$. Let J be an arc in Δ that is irreducible between $[y, z]$ and $[z, w]$. Let n be a positive integer such that $U_n \cap J = \phi$. Every arc in M that intersects Δ and $\mathbf{S}^2\backslash\mathrm{Cl}\,\Delta$ is straddled by $\mathrm{Bd}\,\Delta$. Hence, $\mathrm{Bd}\,\Delta$ intersects each complementary domain of $J \cup [y, w]$. Since $\mathrm{Bd}\,\Delta \subset L \subset \mathrm{Cl}\,\mathbf{P}_{x_{n+2}} \subset U_n$, it follows that $\mathbf{P}_{x_{n+2}} \cap [y, w] \neq \phi$, and this contradicts (12.4). Hence, (12.5) is true.

As in Section 8, let $\mathbf{Q}_{x_1}$ denote the set $\cup\{\mathbf{P}_{x_n} : n = 1, 2, \dots\}$. Define a linear order $\lll$ on $\mathbf{Q}_{x_1}$ as follows:

Let $y \lll z$ if $\{y, z\} \subset \mathbf{P}_{x_n}$ and $y \ll z$ or if $y \in \mathbf{P}_{x_m}$, $z \in \mathbf{P}_{x_n}$, and $m < n$.

For each point y of $\mathbf{Q}_{x_1}$, let $\mathbf{Q}_y$ denote the set $\{z \in \mathbf{Q}_{x_1} : y = z$ or $y \lll z\}$.

Assume $\mathcal{E}$ is countable ((12.1) is false).

By (12.5) and Lemmas 6.1 and 6.47, there exist a complementary domain Δ of L and a point y of $\mathbf{P}_{x_1}$ such that $y \in \Delta$ and $\mathbf{P}_y \subset \mathrm{Cl}\,\Delta$. Since $\Delta = \mathbf{S}^2 \backslash \mathbf{T}_y(L)$, it follows that $\mathrm{Bd}\,\mathbf{T}_y(L) = L$.

Since $\mathbf{P}_y \subset \mathrm{Cl}\,\Delta$, it follows that $\mathbf{L}_{x_1} \subset \mathrm{Cl}\,\Delta$. By (12.3), $\mathbf{P}_{x_n} \subset \mathrm{Cl}\,\Delta$ for each $n = 2, 3, \dots$. Hence, $\mathbf{Q}_y \subset \mathrm{Cl}\,\Delta$.

There exist a point σ of L and an arc segment Σ in Δ such that

$$\sigma \in \mathrm{Cl}\,\Sigma \subset \mathbf{D}_\sigma. \tag{12.6}$$

To see this, we consider two cases.

Case 12.7a. Suppose $\Delta = \mathbf{S}^2 \backslash \mathbf{T}_\omega(L)$. By (11.3), there is a point p of $M \cap \mathbf{T}_\omega(L)$ such that $\mathbf{D}_p \not\subset \mathbf{T}_\omega(L)$. Hence, there exist a point σ of L and an arc segment Σ in $\mathbf{D}_p$ that satisfy (12.6).

Case 12.7b. Suppose $\Delta \neq \mathbf{S}^2 \backslash \mathbf{T}_\omega(L)$. The arc segment Λ is in $\mathbf{S}^2 \backslash \mathbf{T}_\omega(Y)$. Thus, $\Lambda \subset \mathbf{S}^2 \backslash \mathbf{T}_\omega(L)$. Since $\mathbf{P}_y \subset \mathrm{Cl}\,\Delta$, it follows that $[x_1, y] \cap L \neq \phi$. There exist a point σ of L and an arc segment Σ in $[x_1, y]$ that satisfy (12.6).

By (12.4), $\mathbf{D}_\sigma$ intersects at most one element of $\{\mathbf{P}_{x_n} : n = 1, 2, \dots\}$. Hence, by Lemma 6.47, we can assume without loss of generality that $\mathbf{Q}_y \cap \mathrm{Cl}\,\Sigma = \phi$.

Using Sieklucki's nested sequence of polygonal disks (described in Section 7), define a sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ with the property that each $\mathcal{F}_m$ is an m-frame on $(\mathbf{T}_y(L), \Sigma)$ refined by $\mathcal{F}_{m+1}$.

For each positive integer m, let $\mathcal{G}_m = \{F \in \mathcal{F}_m : \mathbf{P}_z \cap (\mathrm{Int}\, F \backslash L) \neq \phi$ for each point z of $\mathbf{Q}_y\}$. By Lemma 8.1, each $\mathcal{G}_m$ is a subframe of $\mathcal{F}_m$. Note that if m and n are integers and $0 < m < n$, then $\mathcal{G}_n$ refines $\mathcal{G}_m$ and each end section of $\mathcal{G}_m$ contains an end section of $\mathcal{G}_n$.

For each positive integer, m, let $G_m(1), G_m(2), \dots, G_m(\lambda_m)$ be the consecutive sections of $\mathcal{G}_m$.

By Lemma 8.2, for each positive integer m, there exists a point z_m of $\mathbf{Q}_y \cap \operatorname{Int} G_m(1)$ such that $\mathbf{Q}_{z_m}$ is a trace of $\mathcal{G}_m$. Hence, $L \subset \cup\mathcal{G}_m$ for each m. By (12.3), for each $G_m(i)$, $1 \leq i \leq \lambda_m$, and for each point u of $\mathbf{P}_{z_m}$, the Borsuk ray $\mathbf{P}_u$ intersects $\operatorname{Int} G_m(i) \backslash L$.

It follows from (12.4) and the proof of Lemma 8.1 that, for each positive integer m,

$$\text{no arc segment in } (\mathbf{Q}_y \backslash L) \cap \cup\{G_m(i) : 1 < i < \lambda_m\} \text{ has an end point in } L. \tag{12.8}$$

For each positive integer m, there exist arcs A_m and B_m in $\mathbf{Q}_{z_m}$ such that A_m is a trace of $\mathcal{G}_m$, $B_m \subset A_m$, $f(B_m) \subset (\cup\mathcal{G}_m)\backslash L$, and $f(B_m)$ is a trace of $\mathcal{G}_m$. To see this, let u be a point of $\mathbf{P}_{z_m} \cap \operatorname{Int} G_m(\lambda_m)$. By (12.8), we can assume without loss of generality that $[z_m, u] \cap L = \phi$. Let n be an integer greater than m such that $[z_m, u] \cap \cup\mathcal{G}_n = \phi$. By (10.10), (10.11) and (12.8), there exist points v and w of $\mathbf{Q}_u \backslash \mathbf{P}_u$ such that $w \in \mathbf{P}_v \subset \cup\mathcal{G}_n$, $f(v) \in \mathbf{P}_v \cap \operatorname{Int} G_m(1)$, $f(w) \in \mathbf{P}_w \cap \operatorname{Int} G_m(\lambda_m)$, and $[z_m, u]$ separates $[f(v), f(w)] \cap \cup\{G_m(i) : 1 < i < \lambda_m\}$ from the bottom of each $G_m(i)$, $1 < i < \lambda_m$, in $\cup\{G_m(i) : 1 < i < \lambda_m\}$. Let A_m be an arc in $\mathbf{P}_v$ that contains $[v, w]$ and is a trace of $\mathcal{G}_m$. By (12.4), $[z_m, u] \cap f([v, w]) = \phi$. It follows from (12.4) and (12.8) that there exists a subarc B_m of $[v, w]$ such that $f(B_m) \subset (\cup\mathcal{G}_m)\backslash L$ and $f(B_m)$ is a trace of $\mathcal{G}_m$. Note that $f(B_m)$ may fail to be in $\mathbf{P}_v$ (see Figure 12).

We have shown that $\mathbf{T}_y(L)$ has the following property:

Property 12.9. The continuum $\mathbf{T}_y(L)$ does not separate $\mathbf{S}^2$ and there exists an arc segment Σ in $M \backslash \mathbf{T}_y(L)$ with an end point in $\mathbf{T}_y(L)$, a sequence $A_1, A_2, \ldots$ of arcs in $\mathbf{Q}_y$ converging to $\operatorname{Bd} \mathbf{T}_y(L)$, and a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ such that for each positive integer m,

(a) $\mathcal{G}_m$ is a subframe of an m-frame on $(\mathbf{T}_y(L), \Sigma)$,

(b) $\mathcal{G}_{m+1}$ refines $\mathcal{G}_m$,

(c) each end section of $\mathcal{G}_m$ contains an end section of $\mathcal{G}_{m+1}$,

(d) A_m agrees with $\mathcal{G}_m$, and

(e) either $f(A_m) \subset (\cup\mathcal{G}_m)\backslash \mathbf{T}_y(L)$ or there exists a subarc B_m of A_m such that $f(B_m) \subset (\cup\mathcal{G}_m)\backslash \mathbf{T}_y(L)$ and $f(B_m)$ agrees with $\mathcal{G}_m$.

Next we prove that $\mathbf{T}_y(L)$ contains a continuum that is irreducible with respect to Property 12.9.

FIGURE 12.

Assume $X_1, X_2, \dots$ is a decreasing sequence of continua in $\mathbf{T}_y(L)$ that do not separate $\mathbf{S}^2$.

For each positive integer n, assume there exist an arc segment Σ_n in $M \backslash X_n$ that has an end point in X_n, a sequence $A_1(n), A_2(n), \dots$ of arcs in $\mathbf{Q}_y$ converging to $\mathrm{Bd}\, X_n$, and a sequence $\mathcal{G}_1(n), \mathcal{G}_2(n), \dots$ with the following property:

For each m,

$$\mathcal{G}_m(n) \text{ is a subframe of an } m\text{-frame on } (X_n, \Sigma_n), \tag{12.10}$$

$$\mathcal{G}_{m+1}(n) \text{ refines } \mathcal{G}_m(n), \tag{12.11}$$

(12.12)

$$\text{each end section of } \mathcal{G}_m(n) \text{ contains an end section of } \mathcal{G}_{m+1}(n),$$

$$A_m(n) \text{ agrees with } \mathcal{G}_m(n), \quad \text{and} \tag{12.13}$$

$$\begin{array}{c} \text{either } f(A_m(n)) \subset (\cup \mathcal{G}_m(n)) \backslash X_n \text{ or there exists a subarc} \\ B_m(n) \text{ of } A_m(n) \text{ such that } f(B_m(n)) \subset (\cup \mathcal{G}_m(n)) \backslash X_n \\ \text{and } f(B_m(n)) \text{ agrees with } \mathcal{G}_m(n). \end{array} \tag{12.14}$$

Let $X = \cap\{X_n : n = 1, 2, \dots\}$. By the Brouwer reduction theorem [**44**, p. 17], it is sufficient to prove that X is a continuum with Property 12.9.

For each positive integer n, since $A_1(n), A_2(n), \dots$ converges to $\mathrm{Bd}\, X_n$, it follows from (12.14) and the continuity of f that either $f(\mathrm{Bd}\, X_n) \subset \mathrm{Bd}\, X_n$ or $\mathrm{Bd}\, X_n \subset f(\mathrm{Bd}\, X_n)$. The sequence $\mathrm{Bd}\, X_1, \mathrm{Bd}\, X_2, \dots$ converges to $\mathrm{Bd}\, X$. Thus, $f(\mathrm{Bd}\, X) \subset \mathrm{Bd}\, X$ or $\mathrm{Bd}\, X \subset f(\mathrm{Bd}\, X)$. Since f is fixed-point free, $\mathrm{Bd}\, X$ is not degenerate. Hence, X is a continuum.

Since $\mathbf{S}^2 \backslash X = \cup\{\mathbf{S}^2 \backslash X_n : n = 1, 2, \dots\}$ and each $\mathbf{S}^2 \backslash X_n$ is connected,

(12.15) $\mathbf{S}^2 \backslash X$ is connected.

By the argument for (12.6) there is an arc segment Σ in $M \backslash X$ that lies in one element of $\mathcal{D}$ and has an end point in X. By (12.4), we can assume without loss of generality that $\mathbf{Q}_y \cap \mathrm{Cl}\, \Sigma = \phi$.

Define a sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ with the property that each $\mathcal{F}_m$ is an m-frame on (X, Σ) refined by $\mathcal{F}_{m+1}$.

There exists a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ such that for each m,

(12.16) $\mathcal{G}_m$ is a subframe of $\mathcal{F}_m$,

(12.17) $\mathcal{G}_{m+1}$ refines $\mathcal{G}_m$,

(12.18) each end section of $\mathcal{G}_m$ contains an end section of $\mathcal{G}_{m+1}$, and

(12.19) there exist integers i_m and j_m, $j_m > m$, such that

(a_m) $\cup\mathcal{G}_{i_m}(j_m) \subset (\cup\mathcal{G}_m) \backslash \mathrm{Bd}\,(X \cup \cup\mathcal{F}_m)$,

(b_m) the interior of each interior section of $\mathcal{G}_m$ contains the sides of two consecutive sections of $\mathcal{G}_{i_m}\ (j_m)$,

(c_m) no end point of a side of a section of $\mathcal{G}_m$ belongs to $A_{i_m}\ (j_m) \cup f(A_{i_m}(j_m))$, and

FIGURE 13.

(d_m) the Hausdorff distance [**32**, p. 47] from $A_{i_m}(j_m)$ to $\mathrm{Bd}\, X_{j_m}$ is less than m^{-1}.

Use the following procedure to define the sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$. If the sections of a subframe $\mathcal{G}_m(n)$ are contained in $X \cup \cup\mathcal{F}_1$, then, since $X \subset X_n$, the collection of sections of $\mathcal{F}_1$ whose interiors intersect $\cup\mathcal{G}_m(n)$ is a subframe of $\mathcal{F}_1$. Hence, there exist a subframe $\mathcal{G}_1$ of $\mathcal{F}_1$ and a sequence of subframes $\mathcal{G}_{i_1}(j_1), \mathcal{G}_{i_2}(j_2), \dots$ such that each $\cup\mathcal{G}_{i_m}(j_m)$ is in $(\cup\mathcal{G}_1)\backslash \mathrm{Bd}\,(X \cup \cup\mathcal{F}_1)$ and intersects the interior of each section of $\mathcal{G}_1$, the interior of each interior section of $\mathcal{G}_1$ contains the sides of two consecutive sections of each $\mathcal{G}_{i_m}(j_m)$, each $A_{i_m}(j_m) \cup f(A_{i_m}(j_m))$ misses the set of end points of the sides of the sections of $\mathcal{G}_1$ (recall (10.6)), and the Hausdorff distance from $A_{i_m}(j_m)$ to $\mathrm{Bd}\, X_{j_m}$ is less than m^{-1}. The subframes $\mathcal{G}_1$ and $\mathcal{G}_{i_1}(j_1)$ are drawn in Figure 13.

Next, define a subframe $\mathcal{G}_2$ of $\mathcal{F}_2$ and a subsequence $\mathcal{S}$ of $\mathcal{G}_{i_1}(j_1)$, $\mathcal{G}_{i_2}(j_2), \dots$ such that $\mathcal{G}_2$ refines $\mathcal{G}_1$, each end section of $\mathcal{G}_2$ contains an end section of $\mathcal{G}_1$, and the conditions described in preceding paragraph for $\mathcal{G}_1$ and $\mathcal{G}_{i_1}(j_1), \mathcal{G}_{i_2}(j_2), \dots$ hold for $\mathcal{G}_2$ and $\mathcal{S}$. Assume without loss of generality that $\mathcal{S} = \mathcal{G}_{i_2}(j_2), \mathcal{G}_{i_3}(j_3), \dots$. Repeat this procedure to define $\mathcal{G}_3, \mathcal{G}_4, \dots$.

For each positive integer m, let $A_m = A_{i_m}(j_m)$.

Since the sequence $\operatorname{Bd} X_1, \operatorname{Bd} X_2, \ldots$ converges to $\operatorname{Bd} X$, it follows from (12.19) (d_m) that

$$A_1, A_2, \ldots \text{ converges to } \operatorname{Bd} X. \tag{12.20}$$

By (12.13), (12.14), (12.19) (a_m)–(c_m) and Lemma 7.8, for each positive integer m,

$$A_m \text{ agrees with } \mathcal{G}_m \tag{12.21}$$

and either

$$f(A_m) \subset (\cup \mathcal{G}_m) \backslash X \tag{12.22}$$

or

$$\begin{gathered} \text{there exists a subarc } B_m \text{ of } A_m,\ B_m = B_{i_m}(j_m), \\ \text{such that } f(B_m) \subset (\cup \mathcal{G}_m) \backslash X \text{ and } f(B_m) \text{ agrees with } \mathcal{G}_m. \end{gathered} \tag{12.23}$$

It follows from (12.15)–(12.18) and (12.20)–(12.23) that X has Property 12.9. Hence, there exists a subcontinuum of $\mathbf{T}_y(L)$ that is irreducible with respect to Property 12.9.

For convenience, we assume that

$$\text{no proper subcontinuum of } X \text{ has Property 12.9.} \tag{12.24}$$

By Lemma 7.7, there exists a positive integer α such that $\delta > 2^{1-\alpha}$ and no pair of consecutive sections of $\mathcal{G}_\alpha$ contains $\operatorname{Bd} X$ in its union.

Since Σ is in one element of $\mathcal{D}$ and f preserves the elements of $\mathcal{D}$, by (10.6), $\Sigma \cap f(\mathbf{Q}_z) = \phi$ for some point z of $\mathbf{Q}_y$.

Assume without loss of generality that for each integer $m \geq \alpha$,

$$\Sigma \cap f(A_m) = \phi. \tag{12.25}$$

Let $G_1, G_2, \ldots, G_\beta$ be the consecutive sections of $\mathcal{G}_\alpha$. For $j = 1, 2, \ldots, \beta - 1$, let $H_j = \cup\{G_i : 1 \leq i \leq j\}$, and let C_j be the common side of G_j and G_{j+1}.

Let $\mathcal{C} = \{C_j : 2 \leq j \leq \beta - 2\}$. Note that each element of $\mathcal{C}$ has diameter less than δ.

For $m = \alpha, \alpha + 1, \dots$ and $j = 2, 3, \dots, \beta - 2$, let $C_j(m) = A_m \cap C_j$.

A point p of $C_j(m)$ is *sent back* by f if $f(p) \in H_j$; otherwise, p is *sent forward* by f.

The arc A_m has the *switch property* if a component of $A_m \setminus \cup \mathcal{C}$ has end points in $\cup\mathcal{C}$ that are sent in opposite directions by f.

Statement (12.22) is true for only finitely many integers $m \geq \alpha$. To see this, assume the contrary. Suppose without loss of generality that (12.22) is true for each integer $m \geq \alpha$.

For each integer $m \geq \alpha$, if f sends two points of $\cup\{C_j(m) : 2 \leq j \leq \beta - 2\}$ in opposite directions, then, by (10.1), (12.22), and (12.25), A_m has the switch property.

Suppose for infinitely many integer $m \geq \alpha$, two points of $\cup\{C_j(m) : 2 \leq j \leq \beta - 2\}$ are sent in opposite directions by f. Then infinitely many elements of $\{A_m : m = \alpha, \alpha + 1, \dots\}$ have the switch property.

Assume without loss of generality that there exists a component G of $(\cup\mathcal{G}_\alpha)\setminus(\Sigma \cup X \cup \cup\mathcal{C})$ such that for each integer $m \geq \alpha$, the arc A_m has the switch property on a component Υ_m of $A_m \setminus \cup \mathcal{C}$ that is in $\mathrm{Cl}\, G$.

For each integer $m \geq \alpha$, we have three cases.

Case 12.26a. Suppose $f(\mathrm{Cl}\,\Upsilon_m) \subset G$ (see Figure 14(1)). Then $\mathrm{Cl}\, G$ is a section of $\mathcal{G}_m$ and Υ_m has an end point in each side of $\mathrm{Cl}\, G$.

Case 12.26b. Suppose $f(\mathrm{Cl}\,\Upsilon_m)$ intersects two components of $(\cup\mathcal{G}_\alpha)\setminus(\Sigma \cup X \cup G)$ (see Figures 14(2) and 14(3)). Then $\mathrm{Cl}\, G$ is a section of $\mathcal{G}_m$ and there exists an arc A in $\mathrm{Cl}\,\Upsilon_m$ such that $f(A) \subset \mathrm{Cl}\, G$ and $f(A)$ intersects each side of $\mathrm{Cl}\, G$.

Case 12.26c. Suppose $f(\mathrm{Cl}\,\Upsilon_m)$ intersects only one component of $(\cup\mathcal{G}_\alpha)\setminus(\Sigma \cup X \cup G)$ (see Figure 14(4)). Then there exist an element C_j of $\mathcal{C}$ in $\mathrm{Cl}\, G$ and an arc A in $\mathrm{Cl}\,\Upsilon_m$ with an end point in C_j such that $f(A) \subset \mathrm{Cl}\, G$ and $C_j \cap f(A) \neq \phi$. In this case $\mathrm{Cl}\, G$ is either a section of $\mathcal{G}_\alpha$ or the union of the first two sections or the last two sections of $\mathcal{G}_\alpha$.

Since one of these three cases holds for infinitely many elements of

FIGURE 14.

$\{\Upsilon_m : m = \alpha, \alpha+1, \dots\}$, there is a continuum L' in $X \cap \mathrm{Cl}\, G$ with the following properties:

There exist a sequence $\Phi_1, \Phi_2, \dots$ of arcs in $\mathbf{Q}_y$ converging to L' and a sequence $\mathcal{H}_1, \mathcal{H}_2, \dots$ such that for each positive integer m,

$$\mathcal{H}_m \text{ is a subframe of } \mathcal{F}_{m+\alpha}, \tag{12.27}$$

$$\cup\mathcal{H}_m \subset \mathrm{Cl}\, G, \tag{12.28}$$

$$\mathcal{H}_{m+1} \text{ refines } \mathcal{H}_m, \tag{12.29}$$

$$\text{each end section of } \mathcal{H}_m \text{ contains an end section of } \mathcal{H}_{m+1}, \tag{12.30}$$

$$\Phi_m \text{ agrees with } \mathcal{H}_m, \tag{12.31}$$

and

$$\begin{gathered} \text{either } f(\Phi_m) \subset (\cup\mathcal{H}_m)\backslash L' \text{ or there exists a subarc} \\ \Psi_m \text{ of } \Phi_m \text{ such that } f(\Psi_m) \subset (\cup\mathcal{H}_m)\backslash L' \\ \text{and } f(\Psi_m) \text{ agrees with } \mathcal{H}_m. \end{gathered} \tag{12.32}$$

Since $\Delta \subset \mathbf{S}^2 \backslash \mathbf{T}_y(L')$, it follows that $L' = \mathrm{Bd}\, \mathbf{T}_y(L')$.

By the argument for (12.6), there exists an arc segment Σ' in $M \backslash \mathbf{T}_y(L')$ that lies in one element of $\mathcal{D}$ and has an end point in L'. By (12.4), we can assume without loss of generality that $\mathbf{Q}_y \cap \mathrm{Cl}\, \Sigma' = \phi$.

Define a sequence $\mathcal{J}_1, \mathcal{J}_2, \dots$ with the property that each $\mathcal{J}_m$ is an m-frame on $(\mathbf{T}_y(L'), \Sigma')$ refined by $\mathcal{J}_{m+1}$.

There exists a sequence $\mathcal{K}_1, \mathcal{K}_2, \dots$ such that for each m,

(12.33) $$\mathcal{K}_m \text{ is a subframe of } \mathcal{J}_m,$$

(12.34) $$\mathcal{K}_{m+1} \text{ refines } \mathcal{K}_m,$$

(12.35)
$$\text{each end section of } \mathcal{K}_m \text{ contains an end section of } \mathcal{K}_{m+1}, \quad \text{and}$$

(12.36) there exists an integer i_m such that

(a) $\cup \mathcal{H}_{i_m} \subset (\cup \mathcal{K}_m) \backslash \mathrm{Bd}\, (\mathbf{T}_y(L') \cup \cup \mathcal{J}_m)$,

(b) the interior of each interior section of $\mathcal{K}_m$ contains the sides of two consecutive sections of $\mathcal{H}_{i_m}$, and

(c) no end point of a side of a section of $\mathcal{K}_m$ belongs to $\Phi_{i_m} \cup f(\Phi_{i_m})$.

It follows from (12.31)–(12.36) that $\mathbf{T}_y(L')$ has Property 12.9. Since $\mathrm{Cl}\, G$ is either a section or the union of two consecutive sections of $\mathcal{G}_\alpha$, it follows that $\mathrm{Bd}\, X \not\subset L'$. But $\mathbf{T}_y(L')$ and X are continua that do not separate $\mathbf{S}^2$ and $L' \subset \mathrm{Bd}\, X$. Consequently, $\mathbf{T}_y(L')$ is a proper subcontinuum of X, and this contradicts (12.24). Hence, for all but finitely many integers $m \geq \alpha$, the map f sends each point of $\cup\{C_j(m) : 2 \leq j \leq \beta - 2\}$ in the same direction.

Assume without loss of generality that for each integer $m \geq \alpha$, every point of $\cup\{C_j(m) : 2 \leq j \leq \beta - 2\}$ is sent back by f.

The set $\{m : f(A_m \cap (G_1 \cup G_2)) \not\subset G_1 \cup G_2\}$ is finite; for otherwise, a case similar to Case 12.26c (with $\mathrm{Cl}\, G = G_1 \cup G_2$ and $C_j = C_2$) holds for infinitely many elements of $\{A_m : m = \alpha, \alpha + 1, \dots\}$. It can be shown that this implies the existence of a continuum L' in

$X \cap (G_1 \cup G_2)$, a sequence $\Phi_1, \Phi_2, \dots$ of arcs in $\mathbf{Q}_y$ converging to L', and a sequence $\mathcal{H}_1, \mathcal{H}_2, \dots$ that satisfy (12.27)–(12.32). The argument following (12.32) shows that this is impossible.

Hence, we can assume without loss of generality that for each integer $m \geq \alpha$,

$$f(A_m \cap (G_1 \cup G_2)) \subset G_1 \cup G_2. \tag{12.37}$$

For each positive integer m, let $\mathcal{H}_m$ be the collection consisting of all sections of $\mathcal{G}_{m+\alpha}$ that intersect $\mathrm{Int}\,(G_1 \cup G_2)$, and let Φ_m be an arc in $A_{m+\alpha} \cap (G_1 \cup G_2)$ that intersects C_2 and agrees with $\mathcal{H}_m$.

The sequence $\Phi_1, \Phi_2, \dots$ converges to a continuum in $X \cap (G_1 \cup G_2)$. For each positive integer m, conditions (12.27)–(12.31) (with $\mathrm{Cl}\, G = G_1 \cup G_2$) are satisfied. By (12.22) and (12.37), $f(\Phi_m) \subset (\cup \mathcal{H}_m) \backslash X$ for each positive integer m. According to the argument following (12.32), a proper subcontinuum of X has Property 12.9, and this contradicts (12.24). Hence, (12.22) is true for at most finitely many integers.

Assume without loss of generality that (12.23) holds for each integer $m \geq \alpha$.

By the preceding argument, for infinitely many integers $m \geq \alpha$, the arc A_m does not have the switch property on a component of $A_m \backslash \cup \mathcal{C}$ that is contained in B_m. Hence, we can assume without loss of generality that for each integer $m \geq \alpha$, every point of $B_m \cap \cup \{C_j(m) : 2 \leq j \leq \beta - 2\}$ is sent forward by f.

It follows from a similar argument that, for infinitely many integers $m \geq \alpha$,

$$f(p) \notin H_j \text{ for each point } p \text{ of } B_m \backslash H_j, \qquad 2 \leq j \leq \beta - 2. \tag{12.38}$$

We assume without loss of generality that (12.38) holds for each integer $m \geq \alpha$.

Since for each positive integer m, the arc $B_{m+\alpha}$ has the properties given in (12.23) and (12.38), there exist a sequence $\Phi_1, \Phi_2, \dots$ of arcs in $\mathbf{Q}_y$ converging to a continuum in $X \cap (G_1 \cup G_2)$ and a sequence $\mathcal{H}_1, \mathcal{H}_2, \dots$ such that for each positive integer m,

$$\mathcal{H}_m \text{ is a subframe of } \mathcal{F}_{m+\alpha}, \tag{12.39}$$

(12.40) $$\cup\mathcal{H}_m \subset G_1 \cup G_2,$$

(12.41) $$\mathcal{H}_{m+1} \text{ refines } \mathcal{H}_m,$$

(12.42) $$\text{each end section of } \mathcal{H}_m \text{ contains an end section of } \mathcal{H}_{m+1},$$

(12.43) $$\Phi_m \text{ agrees with } \mathcal{H}_m,$$

(12.44) $$\Phi_m \cap C_2 \neq \phi,$$

and

(12.45) $$\text{there exists a subarc } \Psi_m \text{ of } \Phi_m \text{ such that } f(\Psi_m) \subset (\cup\mathcal{H}_m)\backslash X \text{ and } f(\Psi_m) \text{ agrees with } \mathcal{H}_m.$$

By the argument following (12.32), a proper subcontinuum of X has Property 12.9, and this contradicts (12.24). Hence, Case 12.2a is impossible if $\mathcal{E}$ is countable.

Case 12.2b. Suppose $L \in \mathcal{U}$. Then there is a point x of $\mathbf{L}_{x_1}$ such that $\mathbf{P}_x \subset \mathbf{L}_{x_1}$ and $\mathbf{L}_x = L$. Assume $\mathcal{E}$ is countable. The argument given in Case 12.2a can be modified by replacing the subsets of $\mathbf{Q}_{x_1}$ with subsets of $\mathbf{P}_x$ and applying Lemmas 9.1 and 9.2 instead of Lemmas 8.1 and 8.2 to get a contradiction. Hence, (12.1) is true.

13. Triods and bridges. A continuum T is a *triod* if there is a point v of T such that $T\backslash\{v\}$ is the union of three disjoint half-open arcs.

Since $\mathbf{S}^2$ does not contain uncountably many disjoint triods [**36**],

(13.1) $$\text{only countably many elements of } \mathcal{E} \text{ contain triods.}$$

An element E of $\mathcal{E}$ is *bridged* if there exists a sequence $p_1, p_2, \dots$ of points of L converging to a point p of E such that

(13.2) $$\text{the sequence of arcs } [p_1, f(p_1)], [p_2, f(p_2)], \dots \text{ converges to a continuum } S \text{ that does not contain } [p, f(p)].$$

Since f is continuous, $\{p, f(p)\} \subset S$. It follows from (13.2) that $S \cap [p, f(p)]$ is not connected. Hence, $S \cup [p, f(p)]$ separates $\mathbf{S}^2$ [**32**, Theorem 2, p. 506].

Note that

(13.3) L intersects only one complementary domain of $S \cup [p, f(p)]$.

To see this, assume L intersects two complementary domains A and B of $S \cup [p, f(p)]$. Let C and D be disks such that $C \subset A$, $D \subset B$, $L \cap \operatorname{Int} C \neq \phi$, and $L \cap \operatorname{Int} D \neq \phi$. For each point x of $\cup\mathcal{E}$, by (11.6), $\mathbf{P}_x \cap C \neq \phi$ and $\mathbf{P}_x \cap D \neq \phi$. Hence, by (12.1), there exist four elements E_1, E_2, E_3, and E_4 of $\mathcal{E}\backslash(\{\mathbf{D}_p\} \cup \{\mathbf{D}_{p_n} : n = 1, 2, \dots\})$ such that $E_i \cap C \neq \phi$ and $E_i \cap D \neq \phi$ for $i = 1, 2, 3$, and 4.

For $i = 1, 2, 3$, and 4, let F_i be an arc in E_i that is irreducible between C and D. Since $C \subset A$, $D \subset B$, and $\mathbf{D}_p \neq E_1, E_2, E_3$, or E_4, for each i, there is a point s_i of F_i in S. Assume without loss of generality that $C \cup D \cup F_1 \cup F_3$ separates s_2 from s_4 in $\mathbf{S}^2$.

Since $C \cup D \subset \mathbf{S}^2\backslash S$, there is an integer k such that $[p_n, f(p_n)] \cap (C \cup D) = \phi$ for each $n > k$. Since $\{s_2, s_4\} \subset S$, there is an integer $n > k$ such that $[p_n, f(p_n)]$ intersects both the s_2-component and the s_4-component of $\mathbf{S}^2\backslash(C \cup D \cup F_1 \cup F_3)$. It follows that $[p_n, f(p_n)] \cap (F_1 \cup F_3) \neq \phi$, and this contradicts the fact that $F_1 \subset E_1$ and $F_3 \subset E_3$. Hence, (13.3) is true.

Observe that

(13.4) only countably many elements of $\mathcal{E}$ are bridged.

To see this, assume the contrary. By [**9**, Theorem 3′], there exist a bridged element E of $\mathcal{E}$, a point p of E, and a continuum S satisfying (13.2) with the following property. For each point q of $[p, f(p)]\backslash S$ and each arc A that crosses $[p, f(p)]$ at q, each component of $A\backslash\{q\}$ intersects a bridged element of $\mathcal{E}$. Since each element of $\mathcal{E}$ is contained in L, this contradicts (13.3). Hence, (13.4) is true.

14. Our principal theorem.

Theorem 14.1. *Let M be a continuum in the plane $\mathbf{R}^2$. Suppose $\mathcal{D}$ is a decomposition of M whose elements are uniquely arcwise connected* *and f is a map of M that preserves the elements of $\mathcal{D}$. Then f has a fixed point.*

Proof. Assume f moves each point of M.

As in Section 11, define the collection $\mathcal{W}$ and let L be a subcontinuum of M that is a maximal element of $\mathcal{W}$ with respect to $\rightarrow\rightarrow$.

Let $G_1, G_2, \ldots$ be the elements of a countable base for $\mathbf{R}^2$ that intersect L. For each positive integer n, let $H_n = \{p \in L : [p, f(p)] \cap \mathrm{Cl}\, G_n = \phi\}$. Since f is fixed-point free and preserves the elements of $\mathcal{D}$, no element of $\mathcal{D}$ is contained in an arc. Hence, by (12.1), L is not an arc. Therefore, $L = \cup\{H_n : n = 1, 2, \ldots\}$.

By the Baire category theorem, there is an integer n such that $\mathrm{Cl}\, H_n$ contains a nonempty open subset H of L. Since $H_n \cap \mathrm{Cl}\, G_n = \phi$, it follows that $H \cap (L \backslash \mathrm{Cl}\, G_n) \neq \phi$. Let J be an open subset of $\mathbf{R}^2 \backslash G_n$ such that $J \cap L$ is a nonempty subset of H.

As in Section 12, let $\mathcal{E}$ be the collection of elements of $\mathcal{D}$ that lie in L.

Let $\mathcal{E}' = \{E \in \mathcal{E} : E$ is not bridged and does not contain a triod$\}$. By (12.1), (13.1) and (13.4), $\mathcal{E}'$ is uncountable.

By (11.6), for each point $p \in \cup\mathcal{E}'$,

$$\mathbf{L}_p = L. \tag{14.2}$$

Let p be a point of an element E of $\mathcal{E}$.

For each point q of $\mathbf{P}_p$, by (10.6), $q \in [p, f(q)]$.

Since E does not contain a triod, it follows that

$$f(q) \in \mathbf{P}_q \text{ for each point } q \text{ of } \mathbf{P}_p. \tag{14.3}$$

For each point q of $J \cap E$,

$$[q, f(q)] \cap G_n = \phi. \tag{14.4}$$

To see this, let $q_1, q_2, \ldots$ be a sequence of points of H_n that converges to q. For each positive integer i, the arc $[q_i, f(q_i)]$ misses G_n. Thus

FIGURE 15.

the limit set K of $[q_1, f(q_1)], [q_2, f(q_2)], \ldots$ misses G_n. Since E is not bridged, K contains $[q, f(q)]$. Hence, (14.4) is true.

For each element E of $\mathcal{E}'$, let $p(E)$ be a point of E.

By (14.2), there is an arc $A(E)$ in $\mathbf{P}_{p(E)}$ that contains a set $\{q_j(E) : 1 \leq j \leq 12\}$ such that $q_1(E) \ll q_2(E) \ll \cdots \ll q_{12}(E)$, the end points of $A(E)$ are $q_1(E)$ and $q_{12}(E)$, and for each odd integer k, $1 \leq k \leq 11$, $q_k(E) \in J$ and $q_{k+1}(E) \in G_n$.

For each element E of $\mathcal{E}'$, let g_E be a homeomorphism of $[0,1]$ onto $A(E)$ such that $g_E(0) = q_1(E)$. Let $\mathcal{G}$ denote the function space $\{g_E : E \in \mathcal{E}'\}$ with the topology of uniform convergence [**32**, p. 89]. Let μ be the metric on $\mathcal{G}$ defined by $\mu(g_E, g_F) = \text{maximum}\, \{\rho(g_E(t), g_F(t)) : t \in [0,1]\}$.

Since $\mathcal{G}$ is an uncountable separable metric space, $\mathcal{G}$ contains a limit point g_{E_0}. Note that $A(E_0)$ is the arc in the element E_0 of $\mathcal{E}'$ associated with the limit point g_{E_0} of $\mathcal{G}$ (see Figure 15).

By (14.3) and (14.4), $f(q_k(E_0)) \in (q_k(E_0), q_{k+1}(E_0))$ for each odd integer k, $1 \leq k \leq 11$.

Let $\{Q_j : 1 \leq j \leq 12\}$ be a collection of disjoint disks in $\mathbf{R}^2$ such that

(14.5) $$q_j(E_0) \in \operatorname{Int} Q_j \qquad \text{for each integer } j,\ 1 \leq j \leq 12,$$

(14.6) $$Q_k \subset J \text{ and } Q_{k+1} \subset G_n \qquad \text{for each odd integer } k,\ 1 \leq k \leq 11,$$

and

(14.7) $$(Q_{k-1} \cup Q_k \cup [q_{k-1}(E_0), q_k(E_0)]) \cap f(L \cap Q_k) = \phi \quad \text{for each odd integer } k,\ 3 \leq k \leq 11.$$

For each integer j, $1 \leq j \leq 12$, let R_j be a circular disk in $\operatorname{Int} Q_j$ centered on $q_j(E_0)$ such that

(14.8) the $q_j(E_0)$-component Λ_j of $Q_j \cap A(E_0)$ contains $R_j \cap A(E_0)$.

Let S be the rectangular region $\{(x, y) \in \mathbf{R}^2 : 1 \leq x \leq 12 \text{ and } -1 \leq y \leq 1\}$.

For each integer j, $1 \leq j \leq 12$, let T_j be the vertical interval $\{(x, y) \in \mathbf{R}^2 : j = x \text{ and } -1 \leq y \leq 1\}$.

For each integer j, $1 \leq j \leq 11$, let U_j be the rectangular region $\{(x, y) \in \mathbf{R}^2 : j \leq x \leq j + 1 \text{ and } -1 \leq y \leq 1\}$.

By (14.5), (14.6), and (14.7), there exists a homeomorphism h of $\mathbf{R}^2$ onto $\mathbf{R}^2$ that sends $\{(x, y) \in \mathbf{R}^2 : 1 \leq x \leq 12 \text{ and } y = 0\}$ onto $A(E_0)$ such that

(14.9) $$h((j, 0)) = q_j(E_0) \text{ and } h(T_j) \subset R_j \text{ for each integer } j,\ 1 \leq j \leq 12,$$

(14.10) $$h(S) \cap R_1 = h(U_1) \cap R_1 \text{ and } h(S) \cap R_{12} = h(U_{11}) \cap R_{12}$$

(recall (14.8)), and

(14.11) $$(Q_k \cup Q_{k+1} \cup h(U_k)) \cap f(L \cap Q_{k+1}) = \phi \quad \text{for each even integer } k,\ 2 \leq k \leq 10$$

(recall (14.7)).

Let η be a positive number less than $\rho(A(E_0), \mathbf{R}^2\backslash(R_1 \cup R_{12} \cup h(S)))$, the radius of R_1, the radius of R_{12}, and the minimum of $\{\rho(R_j, A(E_0)\backslash\Lambda_j) : j = 1, 2, \ldots, 12\}$ (recall (14.8)).

Let g_{E_1} and g_{E_2} be distinct elements of $\mathcal{G}\backslash\{g_{E_0}\}$ such that $\mu(g_{E_0}, \{g_{E_1}, g_{E_2}\}) < \eta$.

For $i = 0, 1$, and 2, define $g_i = g_{E_i} g_{E_0}^{-1}$.

Note that, for $i = 0, 1$, and 2,

$$g_i(A(E_0)) \subset \operatorname{Int}(R_1 \cup R_{12} \cup h(S)), \tag{14.12}$$

$$g_i(q_j(E_0)) \in R_j \qquad \text{for } j = 1 \text{ and } 12, \tag{14.13}$$

$$\begin{gathered}\text{the } g_i(q_j(E_0))\text{-component of } Q_i \cap g_i(A(E_0)) \text{ contains} \\ R_j \cap g_i(A(E_0)) \text{ for } j = 1, 2, \ldots, 12,\end{gathered} \tag{14.14}$$

and

$$\begin{gathered}\text{for all integers } j \text{ and } k,\ 1 \leq j < k \leq 12, \text{ every point of } R_j \cap g_i(A(E_0)) \\ \text{precedes every point of } R_k \cap g_i(A(E_0)) \text{ with respect} \\ \text{to the order on } g_i(A(E_0)).\end{gathered} \tag{14.15}$$

For $i = 0, 1$, and 2, let $\sigma_{i,1}$ be the last point of $g_i(A(E_0))$ that belongs to $\operatorname{Bd} R_1$, and let $\sigma_{i,12}$ be the first point of $g_i(A(E_0))$ that belongs to $\operatorname{Bd} R_{12}$ (recall (14.13)).

By (14.12),

$$(\sigma_{i,1}, \sigma_{i,12}) \subset \operatorname{Int} h(S)\backslash(R_1 \cup R_{12}) \qquad \text{for } i = 0, 1, \text{ and } 2. \tag{14.16}$$

Let V_1 and V_2 be the two arcs in $h(\operatorname{Bd} S)$ that are irreducible between R_1 and R_{12} (recall (14.9) and (14.10)).

For $i = 1$ and 2, let c_i be the end point of V_i in R_1, and let d_i be the end point of V_i in R_{12}.

By (14.9) and (14.10), V_1 and V_2 each contain an end point of $h(T_5)$. By (14.16), $(\sigma_{i,1}, \sigma_{i,12}) \cap h(T_5) \neq \phi$ for $i = 0, 1$, and 2 [**37**, Theorem 28, p. 156]. Hence, there exist arc segments Φ and Ψ in $\operatorname{Bd} R_1 \cup \operatorname{Bd} R_{12}$ such that $\{\sigma_{i,1} : i = 0, 1, 2\} \subset \Phi$, $\{\sigma_{i,12} : i = 0, 1, 2\} \subset \Psi$, and the end points of Φ and Ψ are c_1, c_2 and d_1, d_2, respectively [**37**, Theorem 28, p. 156].

Assume without loss of generality that $\sigma_{0,1}$ separates $\sigma_{1,1}$ from $\sigma_{2,1}$ in Φ. Then $\sigma_{0,12}$ separates $\sigma_{1,12}$ from $\sigma_{2,12}$ in Ψ [**37**, Theorem 28, p. 156].

For $j = 1$ and 12, let Σ_j be the arc segment in $\Phi \cup \Psi$ with end points $\sigma_{1,j}$ and $\sigma_{2,j}$.

For $j = 2, 3, \dots, 11$, let Σ_j be an arc segment in $h(T_j)$ whose closure is irreducible between $[\sigma_{1,1}, \sigma_{1,12}]$ and $[\sigma_{2,1}, \sigma_{2,12}]$. Note that $[\sigma_{0,1}, \sigma_{0,12}]$ intersects each Σ_j [**37**, Theorem 28, p. 156].

For $i = 1, 2$ and $j = 2, 3, \dots, 11$, let $\sigma_{i,j}$ be the end point of Σ_j that belongs to $[\sigma_{i,1}, \sigma_{i,12}]$.

For $i = 1, 2$ and $j = 1, 2, \dots, 11$, by (14.9) and (14.15), $\sigma_{i,j}$ precedes $\sigma_{i,j+1}$ with respect to the order on $g_i(A(E_0))$.

For $j = 1, 2, \dots, 6$, let $B_j = \Sigma_{2j-1}$.

For $i = 1, 2$ and $j = 1, 2, \dots, 6$, let $a_{i,j} = \sigma_{i,2j-1}$.

Let x denote the point $q_1(E_0)$.

For $j = 1, 2, 3$, and 4,

(14.17)
every arc in $\mathbf{P}_x$ that is ordered from B_{j+1} to B_j intersects B_{j+2}.

To see this, assume there is an arc H in $\mathbf{P}_x \backslash B_{j+2}$ that is ordered from B_{j+1} to B_j. The simple closed curve $\Sigma_{2j} \cup B_{j+2} \cup [\sigma_{1,2j}, a_{1,j+2}] \cup [\sigma_{2,2j}, a_{2,j+2}]$ separates B_j from B_{j+1} in $\mathbf{R}^2$. Since $\mathbf{P}_x$ misses $[a_{1,1}, a_{1,6}] \cup [a_{2,1}, a_{2,6}]$, the arc H intersects Σ_{2j}. Let v be the first point of H that belongs to Σ_{2j}. Let u be the last point of H that precedes v and belongs to B_{j+1}.

Let Δ be the disk in $\mathbf{R}^2$ bounded by $\Sigma_{2j} \cup B_{j+1} \cup [\sigma_{1,2j}, a_{1,j+1}] \cup [\sigma_{2,2j}, a_{2,j+1}]$. By (14.14) and (14.15), $\Sigma_{2j} \cup B_{j+1} \cup [\sigma_{1,2j}, a_{1,j+1}] \cup [\sigma_{2,2j}, a_{2,j+1}]$ is in $Q_{2j} \cup Q_{2j+1} \cup h(U_{2j})$. Hence, $Q_{2j} \cup Q_{2j+1} \cup h(U_{2j})$ contains Δ. The arc $[u, v]$ misses B_{j+2}. Thus, $[u, v] \subset \Delta$. Since $u \in L \cap B_{j+1} \subset J$ and $v \in L \cap \Sigma_{2j} \subset G_n$, it follows from (14.3) and

(14.4) that $f(u) \in [u, v]$, and this contradicts (14.11). Hence, (14.17) is true.

Furthermore,

(14.18) every arc in $\mathbf{P}_x$ that is ordered from B_4 to B_3 intersects B_1.

To see this, let H be an arc in $\mathbf{P}_x$ that is ordered from B_4 to B_3. Let v be the first point of H that belongs to B_3. Let u be the last point of H that precedes v and belongs to B_4. By (14.17), $[u,v] \cap B_5 \neq \phi$. Note that $[u,v] \cap B_4 = \{u\}$ and $[u,v] \cap ([a_{1,1}, a_{1,4}] \cup [a_{2,1}, a_{2,4}]) = \phi$. Since $B_1 \cup B_4 \cup [a_{1,1}, a_{1,4}] \cup [a_{2,1}, a_{2,4}]$ separates B_3 from B_5 in $\mathbf{R}^2$, it follows that $[u,v] \cap B_1 \neq \phi$. Hence, (14.18) is true.

Moreover,

(14.19) every arc in $\mathbf{P}_x$ that is ordered from B_1 to B_4 intersects B_2.

To see this, let H be an arc in $\mathbf{P}_x$ that is ordered from B_1 to B_4. Let v be the first point of H that belongs to B_4. Let u be the last point of H that precedes v and belongs to B_1. By (14.17), with $j = 4$, the arc $[u,v]$ misses B_5. Since $[u,v] \cap ([a_{1,2}, a_{1,5}] \cup [a_{2,2}, a_{2,5}]) = \phi$ and $B_2 \cup B_5 \cup [a_{1,2}, a_{1,5}] \cup [a_{2,2}, a_{2,5}]$ separates B_1 from B_4 in $\mathbf{R}^2$, it follows that $B_2 \cap [u,v] \neq \phi$. Hence, (14.19) is true.

Let z be the first point of $A(E_0)$ that belongs to B_4. Let y be the last point of $A(E_0)$ that precedes z and belongs to B_1. Since $\mathbf{L}_x = L$ and $A(E_0) \cap B_2 \neq \phi$, it follows from (14.17), (14.18), and (14.19) that there exist a point u of $B_1 \cap \mathbf{P}_z$ and a point w of $B_4 \cap \mathbf{P}_u$ such that (y,u) misses B_1 and intersects B_2 and B_3, and (u,w) misses B_4 and intersects B_2 and B_3 (see Figure 2 in Section 4).

Let C_1 be an arc segment in B_1 such that $\operatorname{Cl} C_1$ is irreducible between y and $[u,w]$.

Let v be the end point of C_1 opposite y. Let Ω be the complementary domain of $C_1 \cup [y,v]$ that contains w.

As in Section 4, for $i = 1$ and 2, let $A_i = [a_{i,1}, a_{i,4}]$. By (4.4), $C_1 \cup [y,v]$ separates A_1 from A_2 in $\mathbf{R}^2$. By (4.7), $\mathbf{P}_w \subset \Omega$. Since $A_1 \cup A_2 \subset \mathbf{L}_x$, this contradicts the fact that $\mathbf{L}_x = \mathbf{L}_w$. Hence, f has a fixed point. $\square$

15. Applications to arc-component-preserving maps.

FIGURE 16.

Theorem 15. *Suppose M is a plane continuum that does not contain a simple closed curve. Then every arc-component-preserving map of M has a fixed point* [**19,** Q. 4.24, **29**, Theorem 12.11, p. 149].

Proof. Let $\mathcal{D}$ in Theorem 14.1 be the collection of arc components of M. □

Corollary 15.2. *If M is a uniquely arcwise connected plane continuum, then M has the fixed-point property* [**17**].

Proof. Let $\mathcal{D} = \{M\}$. □

These results do not extend from the plane to all 2-manifolds. Figure 16 is Young's uniquely arcwise connected continuum [**45**] without the fixed-point property embedded in a torus.

16. The Poincare-Bendixson theorem. Let ψ be a continuous flow on the plane $\mathbf{R}^2$. As in Section 2, for each real number t, let

$\psi_t(p) = \psi(t,p)$.

Suppose M is an invariant continuum under ψ.

Note that

(16.1) M contains a closed orbit or an equilibrium point of ψ.

To see this, let $\mathcal{D}$ be the collection of orbits of ψ in M. Assume M does not contain a closed orbit. It follows from (2.2) that no element of $\mathcal{D}$ contains a simple closed curve. Thus, each element of $\mathcal{D}$ is uniquely arcwise connected. By Theorem 14.1, the restriction of each ψ_t to M has a fixed point. For each positive integer n, let A_n be the nonempty closed subset of M consisting of the fixed points of $\psi_{1/2^n}$. By (2.2), $A_1, A_2, \ldots$ is a decreasing sequence. Let F be the nonempty set $\cap\{A_n : n = 1, 2, \ldots\}$. Every point of F is fixed under ψ_t for all t of the form $1/2^n$ with $n \geq 1$. By (2.2), every point of F is fixed under ψ_t for all dyadic rationals $t = m/2^n$. Since the dyadic rationals are dense in $\mathbf{R}$, each point of F is fixed under ψ_t for all t. Hence, (16.1) is true. $\square$

We are now ready to prove the Poincare-Bendixson theorem:

Theorem 16.2. *Every nonempty compact limit set of a planar continuous flow* ψ *that does not contain an equilibrium point is a closed orbit* [**24**, p. 248].

Proof. Assume the limit set $L_\omega(p)$ of ψ is a nonempty compact set that does not contain an equilibrium point.

Let $\mathbf{P}_p$ denote the set $\{\psi_\alpha(p) : 0 \leq \alpha < \infty\}$. For each point q of $\mathbf{P}_p$, let $\mathbf{P}_q$ denote $\{\psi_\beta(p) : \alpha \leq \beta < \infty \text{ and } \psi_\alpha(p) = q\}$.

We assume that $\mathbf{P}_p$ is a ray; for otherwise, $\mathbf{P}_p$ contains a simple closed curve J, and by (2.2), $J = L_\omega(p)$.

By (2.2) and the continuity of ψ, the limit set $L_\omega(p)$ is an invariant continuum under ψ.

By (16.1),

(16.3) $L_\omega(p)$ contains a closed orbit J_1.

FIGURE 17.

To complete this proof, we will show that

$$J_1 = L_\omega(p). \tag{16.4}$$

Suppose $J_1 \cap \mathbf{P}_p \neq \phi$. Then, since J_1 is an orbit, $\mathbf{P}_p \subset J_1$. Thus, $L_\omega(p) \subset J_1$, and (16.4) follows from (16.3).

Therefore, we assume

$$J_1 \cap \mathbf{P}_p = \phi. \tag{16.5}$$

Let $a_{1,0}$ be a point of J_1. Let τ be one-sixth the period of J_1.

For $j = 1, 2, 3, 4$, and 5, let $a_{1,j}$ denote the point $\psi_{j\tau}(a_{1,0})$ on J_1 (see Figure 17).

For each point q of $\mathbf{R}^2$ and each positive number α, let $\mathbf{A}(q, \alpha)$ denote the arc $\{\psi_t(q) : 0 \leq t \leq \alpha\}$.

By the continuity of ψ, there exist seven disks $Y_0, Y_1, \dots, Y_6$ in $\mathbf{R}^2$ such that

$$a_{1,j} \in \operatorname{Int} Y_j \qquad \text{for } j = 0, 1, 2, 3, 4, \text{ and } 5, \tag{16.6}$$

$$Y_0 \subset Y_6, \tag{16.7}$$

$$Y_j \cap J_1 \text{ is an arc for } j = 1, 2, 3, 4, 5, \text{ and } 6, \tag{16.8}$$

$$\text{the elements of } \{Y_j : 1 \leq j \leq 6\} \text{ are pairwise disjoint,} \tag{16.9}$$

$$\psi_\tau(Y_j) \subset Y_{j+1} \qquad \text{for } j = 0, 1, 2, 3, 4, \text{ and } 5, \tag{16.10}$$

$$\mathbf{A}(q,\tau) \text{ misses } \cup\{Y_j : 2 \leq j \leq 5\} \text{ for each point } q \text{ of } Y_0, \tag{16.11}$$

and

$$\begin{gathered}\text{for } j = 1, 2, 3, 4, \text{ and } 5, \text{ if } q \text{ is a point of } Y_j, \text{ then } \mathbf{A}(q,\tau)\\ \text{misses } \cup\{Y_k : 1 \leq k \leq 6 \text{ and } j \neq k \neq j+1\}.\end{gathered} \tag{16.12}$$

Let Z be an open set in $\mathbf{R}^2$ that contains J_1. To establish (16.4), it suffices to prove the existence of a point w of $\mathbf{P}_p$ such that $\mathbf{P}_w \subset Z$.

Using polar coordinates, we define Q to be the open annulus $\{(r,\theta) \in \mathbf{R}^2 : 1 < r < 3\}$.

For $j = 1, 2$, and 3, let R_j be the circle $\{(r,\theta) \in \mathbf{R}^2 : r = j\}$.

For $j = 0, 1, 2, 3, 4$, and 5, let S_j be the interval $\{(r,\theta) \in \mathbf{R}^2 : 1 \leq r \leq 3 \text{ and } \theta = j\pi/3\}$.

Let h be a homeomorphism of $\mathbf{R}^2$ onto $\mathbf{R}^2$ such that $h(Q) \subset Z$; $h(R_2) = J_1$; and $h((2, j\pi/3)) = a_{1,j}$ and $h(S_j) \subset Y_j$ for $j = 0, 1, 2, 3, 4$, and 5.

By (16.3) and (16.5), $\mathbf{P}_p$ intersects exactly one component T of $h(Q)\backslash J_1$. Note that T is an open annulus whose boundary contains $h(R_1)$ or $h(R_3)$. Assume without loss of generality that $h(R_1) \subset \operatorname{Bd} T$.

For $j = 0, 1, 2, 3, 4$, and 5, let U_j denote $\{(r,\theta) \in \mathbf{R}^2 : 1 < r < 3$ and $(j-1)\pi/3 < \theta < (j+1)\pi/3\}$.

Since $a_{1,0} \in L_\omega(p)$ and ψ is continuous, there is a point q of $\mathbf{P}_p$ in $h(U_0)$ close to $a_{1,0}$ such that

$$\mathbf{A}(q, 6\tau) \subset T, \tag{16.13}$$

$$\psi_{6\tau}(q) \in h(U_0), \tag{16.14}$$

and

$$\text{for } j = 1, 2, 3, 4, \text{ and } 5, \text{ the arc } \mathbf{A}(q, 6\tau) \text{ intersects } h(S_j) \text{ and no arc in } \mathbf{A}(q, 6\tau) \text{ is ordered from } Y_{j+1} \text{ to } Y_j. \tag{16.15}$$

By (16.13), (16.15), and [**37**, Theorem 28, p. 156], there is a subarc of $\mathbf{A}(q, 6\tau)$ in the disk $h(\mathrm{Cl}\, U_2)$ that separates $a_{1,2}$ from $h((1, 2\pi/3))$ in $h(\mathrm{Cl}\, U_2)$. Hence, by (16.14), (16.15), and [**37**, Theorem 32, p. 181], there is a simple closed curve J_2 in $h(U_0) \cup \mathbf{A}(q, 6\tau)$ that separates J_1 from $h(R_1)$ in $\mathbf{R}^2$.

Let V be the open annulus cobounded by J_1 and J_2.

For $j = 1, 2, 3, 4,$ and 5, let B_j be the arc segment in $h(S_j) \cap V$ that has $a_{1,j}$ and a point $a_{2,j}$ of J_2 as end points. Note that $\{a_{2,1}, a_{2,5}\} \subset \mathbf{P}_q$.

Let x be a point of $\mathbf{P}_{a_{2,5}}$ in Y_0 close enough to $a_{1,0}$ so that

$$\psi_{6\tau}(x) \in Y_0 \tag{16.16}$$

and

$$\mathbf{A}(x, 12\tau) \subset V. \tag{16.17}$$

For $j = 1, 2,$ and 3,

$$\text{every arc in } \mathbf{P}_x \text{ that is ordered from } B_{j+1} \text{ to } B_j \text{ intersects } B_{j+2}. \tag{16.18}$$

To see this, assume there is an arc H in $\mathbf{P}_x \backslash B_{j+2}$ that is ordered from B_{j+1} to B_j. Let z be the first point of H in B_j. Let y be the last point of H in B_{j+1} that precedes z. Let Γ be the arc in $J_1 \backslash \{a_{1,0}\}$ from $a_{1,j}$ to $a_{1,j+2}$. Let Λ be the subarc of Γ with end points $a_{1,j}$ and $a_{1,j+1}$. Since $\mathbf{P}_q$ contains $[a_{2,j}, a_{2,j+2}]$ and does not contain a simple closed curve, it follows from (16.5) that H misses $\Gamma \cup B_{j+2} \cup [a_{2,j}, a_{2,j+2}]$. Hence, the disk bounded by the simple closed curve $\Lambda \cup B_j \cup B_{j+1} \cup [a_{2,j}, a_{2,j+1}]$ contains $[y, z]$.

Since $z \in B_j \subset Y_j$, it follows from (16.12) that $\mathbf{A}(y,\tau) \subset [y,z]$. By (16.10), $\mathbf{A}(y,\tau) \cap Y_{j+2} \neq \phi$. By (16.6), (16.8), and (16.9), $\Lambda \cap Y_{j+2} = \phi$. Since $\Lambda \cup B_j \cup B_{j+1} \cup [a_{2,j}, a_{2,j+1}]$ separates (y,z) from B_{j+2}, it follows from (16.9) that $Y_{j+2} \cap [a_{2,j}, a_{2,j+1}] \neq \phi$. Hence, there is an arc in $[a_{2,j}, a_{2,j+1}]$ that is ordered from Y_{j+2} to Y_{j+1}, and this contradicts (16.15). Therefore, (16.18) holds.

Furthermore,

(16.19) every arc in $\mathbf{P}_x$ that is ordered from B_4 to B_3 intersects B_1.

To see this, let H be an arc in $\mathbf{P}_x$ that is ordered from B_4 to B_3. Let z be the first point of H that belongs to B_3. Let y be the last point of H that precedes z and belongs to B_4. By (16.18), $[y,z] \cap B_5 \neq \phi$. Let Γ be the arc in $J_1 \backslash \{a_{1,0}\}$ from $a_{1,1}$ to $a_{1,4}$. Note that $[y,z] \cap B_4 = \{y\}$ and $[y,z] \cap (\Gamma \cup [a_{2,1}, a_{2,4}]) = \phi$. Since $\Gamma \cup B_1 \cup B_4 \cup [a_{2,1}, a_{2,4}]$ separates B_3 from B_5 in $\mathbf{R}^2$, it follows that $[y,z] \cap B_1 \neq \phi$. Hence, (16.19) is true.

Moreover,

(16.20) every arc in $\mathbf{P}_x$ that is ordered from B_1 to B_4 intersects B_2.

To see this, assume there is an arc $[y,u]$ in $\mathbf{P}_x \backslash B_2$ that is ordered from B_1 to B_4. Let Γ be the arc in $J_1 \backslash \{a_{1,0}\}$ from $a_{1,2}$ to $a_{1,5}$. The simple closed curve $\Gamma \cup B_2 \cup B_5 \cup [a_{2,2}, a_{2,5}]$ separates B_1 from B_4 in $\mathbf{R}^2$. Since $\mathbf{P}_q$ contains $[a_{2,2}, a_{2,5}]$ and does not contain a simple closed curve, $\mathbf{P}_x \cap [a_{2,2}, a_{2,5}] = \phi$. By (16.5), $\mathbf{P}_x \cap \Gamma = \phi$. Hence, $[y,u] \cap B_5 \neq \phi$.

Let z be the last point of $[y,u]$ in B_5. Since $u \in B_4 \subset Y_4$, it follows from (16.12) that $\mathbf{A}(z,\tau) \subset [z,u]$. By (16.10), $\mathbf{A}(z,\tau) \cap Y_6 \neq \phi$. By (16.6)–(16.9), $\Gamma \cap Y_6 = \phi$. Since $\Gamma \cup B_2 \cup B_5 \cup [a_{2,2}, a_{2,5}]$ separates (z,u) from B_0, it follows from (16.9) that $Y_6 \cap [a_{2,2}, a_{2,5}] \neq \phi$. Hence, there is an arc in $[a_{2,2}, a_{2,5}]$ that is ordered from Y_6 to Y_5, and this contradicts (16.15). Therefore, (16.20) holds.

As in Section 4, for $i = 1$ and 2, let A_i be the arc in J_i that goes from $a_{i,1}$ to $a_{i,4}$ and contains $a_{i,2}$.

By (16.10), (16.12), (16.16) and (16.17), there exist points y and u of $B_1 \cap \mathbf{A}(x, 8\tau)$ such that (y,u) misses B_1 and intersects B_2 and B_3 (see Figure 2 in Section 4). Furthermore, there exists a point w of

$B_4 \cap \mathbf{A}(u, 4\tau)$ such that $(u, w) \cap B_4 = \phi$. By (16.18) and (16.20), (u, w) intersects B_2 and B_3.

Let C_1 be an arc segment in B_1 such that $\mathrm{Cl}\, C_1$ is irreducible between y and $[u, w]$. Let v be the end point of C_1 opposite y. By (4.4), the simple closed curve $C_1 \cup [y, v]$ separates A_1 from A_2 in the annulus $\mathrm{Cl}\, V$. Hence, J_1 and $C_1 \cup [y, v]$ cobound an open annulus W in V.

Let Ω be the complementary domain of $C_1 \cup [y, v]$ in $\mathbf{R}^2$ that contains w. By (4.7), $\mathbf{P}_w \subset \Omega$. Thus, $L_\omega(p) \subset \mathrm{Cl}\, \Omega$. Since $J_1 \subset L_\omega(p)$ and $J_1 \cap (C_1 \cup [y, v]) = \phi$, it follows that $J_1 \subset \Omega$. Note that $v \in \mathrm{Bd}\, \Omega$ and $(v, w) \subset \Omega \backslash J_1$. Therefore, $w \in W$. Since $\mathbf{P}_w \cap (J_1 \cup C_1 \cup [y, v]) = \phi$, it follows that $\mathbf{P}_w \subset W$. Since W is in the arbitrarily chosen open set Z, (16.4) is true. This completes the proof of Theorem 16.2. □

Appendix

A summary of the proof of our principal theorem. Let M be a plane continuum and $\mathcal{D}$ be a decomposition of M whose elements are uniquely arcwise connected. Suppose f is a map of M that preserves the elements of $\mathcal{D}$.

To prove that f has a fixed point, assume the contrary. For each point x of M, there exists a Borsuk ray $\mathbf{P}_x$ in M that has a limit $\mathbf{L}_x$ with the properties described in Section 10. The Borsuk ray $\mathbf{P}_x$ is the path of the dog in Bing's dog-chases-rabbit arguments [**10**, p. 123].

It is convenient to assume that M is embedded in a 2-sphere $\mathbf{S}^2$. In Section 11, we use Zorn's lemma to define a subcontinuum L of M such that for each point x of L either $\mathbf{P}_x \not\subset L$ or $\mathbf{L}_x = L$.

We define $\mathcal{E}$ to be the collection of elements of $\mathcal{D}$ that are contained in L. The proof breaks down into two cases.

Case 1. Suppose $\mathcal{E}$ is countable. By definition, L is either the intersection of a nested sequence $\mathbf{L}_{x_1}, \mathbf{L}_{x_2}, \ldots$ of limits of Borsuk rays (Case 12.2a) or the limit of one Borsuk ray (Case 12.2b).

In our proof, a complete argument is given that rules out Case 12.2a. We summarize this argument in the next eight paragraphs. This argument can easily be modified to eliminate Case 12.2b.

If $L = \cap\{\mathbf{L}_{x_n} : n = 1, 2, \ldots\}$, we define $\mathbf{Q}_{x_1}$ to be $\cup\{\mathbf{P}_{x_n} : n =$

$1, 2, \dots\}$ with the linear order induced by the order on each of the $\mathbf{P}_{x_n}$'s. For each positive integer n, we have $\mathrm{Cl}\,\mathbf{P}_{x_n} \supset \mathbf{L}_{x_n} \supset \mathrm{Cl}\,\mathbf{P}_{x_{n+1}}$.

According to Lemmas 6.1 and 6.47, there exist a component Δ of $\mathbf{S}^2 \backslash L$ and a point y of $\mathbf{P}_{x_1}$ such that $y \in \Delta$ and $\mathbf{P}_y \subset \mathrm{Cl}\,\Delta$. The set $\mathbf{P}_y$ consists of y and all points of $\mathbf{P}_{x_1}$ that follow y. Similarly, $\mathbf{Q}_y$ is defined to be y and all points of $\mathbf{Q}_{x_1}$ that follow y. Note that $\mathbf{Q}_y \subset \mathrm{Cl}\,\Delta$.

Borsuk's proof [**13**] that every hereditarily unicoherent arcwise connected continuum has the fixed-point property is based on a related lemma. The limit of each ray in a hereditarily unicoherent arcwise connected continuum is a point. Hence, for such a continuum $\mathrm{Cl}\,\Delta$ is $\mathbf{S}^2$.

To establish the existence of Δ, we first show that $\mathbf{P}_{x_1}$ cannot enter a complementary domain Ω of L, leave $\mathrm{Cl}\,\Omega$, and then return to Ω. Thus, if Δ does not exist, $\mathbf{P}_{x_1}$ runs through infinitely many complementary domains of L without returning to any one. Since $L \subset \mathbf{L}_{x_1}$, it follows that $\mathbf{P}_{x_1}$ must keep doubling back. Hence, L contains an indecomposable continuum and this contradicts Lemma 5.1.

There exists an arc segment Σ in Δ contained in an element of $\mathcal{D}$ that has one endpoint in L. The nonseparating plane continuum $\mathbf{S}^2 \backslash \Delta$ is the intersection of a nested sequence of polygonal disks. Two of the polygonal disks are drawn in Figure 10 in Section 7. In this figure, $\mathbf{T}_x(L) = \mathbf{S}^2 \backslash \Delta$.

Let R be the annular region that is the complement of $\mathbf{S}^2 \backslash \Delta$ relative to a polygonal disk. The region R is divided into sections by a collection of disjoint half-open arcs that run from L to the boundary of the polygonal disk. This collection of sections is called a frame. One of the dividing half-open arcs is in $\mathrm{Cl}\,\Sigma$. According to Lemma 7.7, we can assume that the intersection of L and the closure of a section is always a proper subcontinuum of L.

At most one ray in $\mathbf{Q}_y$ intersects Σ. Therefore, instead of circling around $\mathbf{S}^2 \backslash \Delta$, the rays of $\mathbf{Q}_y$ must run back and forth through the consecutive sections of a frame as they get closer and closer to L. Property 12.9 involves a sequence of frames on $\mathbf{S}^2 \backslash \Delta$ with this condition. In Bing's terminology, this condition asserts that the dog and the rabbit are repeatedly in one of the sections of the frame at the same time.

Consequently, there exists a proper subcontinuum of $\mathbf{S}^2\backslash\Delta$ that has Property 12.9.

Using the Brouwer reduction theorem, we define a compact connected set X in $\mathbf{S}^2\backslash\Delta$ that is irreducible with respect to Property 12.9. Either $f(\mathrm{Bd}\,X) \subset \mathrm{Bd}\,X$ or $\mathrm{Bd}\,X \subset f(\mathrm{Bd}\,X)$. Since f is fixed-point free, X is not a point. Therefore, X is a nonseparating plane continuum and the above argument for $\mathbf{S}^2\backslash\Delta$ can be modified to show that X has a proper subcontinuum with Property 12.9. This contradiction of the irreducibility of X rules out Case 1.

Case 2. Suppose $\mathcal{E}$ is uncountable. We define disjoint open subsets J and G_n of $\mathbf{S}^2$ that intersect L and an uncountable subcollection $\mathcal{E}'$ of $\mathcal{E}$ with the following property. For each point x of $J \cap (\cup\mathcal{E}')$, the arc $[x, f(x)]$ is in $\mathbf{P}_x$ and misses G_n.

For each point x of $L \cap (\cup\mathcal{E}')$, we have that $\mathbf{L}_x = L$. Hence, each element E of $\mathcal{E}'$ contains an arc $A(E)$ that starts in J, ends in G_n, and runs back and forth five times between J and G_n. The order of each $A(E)$ agrees with the order of a Borsuk ray. Since there are uncountably many of these disjoint arcs, we can find three that run back and forth in the parallel manner pictured in Figure 15 in Section 14. Two of these arcs are the sides of a zig-zagging strip S that contains the third arc $A(E_0)$.

Let x be the first point of $A(E_0)$. Since S is very narrow and the rabbit cannot get too far ahead of the dog in S, each time $\mathbf{P}_x$ passes through S, it enters through the disk Q_1 at one end of S and leaves through the disk Q_{12} at the other end of S (see Figure 15). Thus, $A(E_0)$ is not in $\mathbf{L}_x$, and this contradicts the fact that $\mathbf{L}_x = L$. Hence, Case 2 is impossible. Therefore, f has a fixed point.

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REFERENCES

1. J.M. Aarts and M. Martens, *Flows on one-dimensional spaces*, Fund. Math. **131** (1988), 53–67.

2. J.M. Aarts and L. Oversteegen, *Flowbox manifolds*, Trans. Amer. Math. Soc. **327** (1991), 449–463.

3. V.N. Akis, *Fixed point theorems and almost continuity*, Fund. Math. **121** (1984), 133–142.

4. W.L. Ayres, *Some generalizations of the Scherer fixed-point theorem*, Fund. Math. **16** (1930), 332–336.

5. H. Bell, *On fixed point properties of plane continua*, Trans. Amer. Math. Soc. **128** (1967), 539–548.

6. ———, *A fixed point theorem for plane homeomorphisms*, Fund. Math. **100** (1978), 119–128. See also, Bull. Amer. Math. Soc. **82** (1976), 778–780.

7. D.P. Bellamy, *Composants of Hausdorff indecomposable continua; a mapping approach*, Pacific J. Math. **47** (1973), 303–309.

8. ———, *A tree-like continuum without the fixed point property*, Houston J. Math. **6** (1979), 1–13.

9. R.H. Bing, *A simple closed curve is the only homogeneous bounded plane continuum that contains an arc*, Canad. J. Math. **12** (1960), 209–230.

10. ———, *The elusive fixed point property*, Amer. Math. Monthly **76** (1969), 119–132.

11. ———, *Commentary on problem* 107, *The Scottish book*, Birkhauser, Boston, 1981, 190–192.

12. K. Borsuk, *Einige Satze uber stetige Streckenbilder*, Fund. Math. **18** (1932), 198–213.

13. ———, *A theorem on fixed points*, Bull. Acad. Polon. Sci. **2** (1954), 17–20.

14. C.L. Hagopian, *A fixed point theorem for plane continua*, Bull. Amer. Math. Soc. **77** (1971), 351–354. Addendum, Bull. Amer. Math. Soc. **78** (1972), 627–628.

15. ———, *Another fixed point theorem for plane continua*, Proc. Amer. Math. Soc. **31** (1972), 627–628. Also see *Mapping theorems for plane continua*, Topology Proc. **3** (1978), 117–122.

16. ———, *λ connectivity and mappings onto a chainable indecomposable continuum*, Proc. Amer. Math. Soc. **45** (1974), 132–136.

17. ———, *Uniquely arcwise connected plane continua have the fixed-point property*, Trans. Amer. Math. Soc. **248** (1979), 85–104.

18. ———, *The fixed-point property for deformations of uniquely arcwise connected continua*, Topology Appl. **24** (1986), 207–212.

19. ———, *Fixed points of arc-component-preserving maps*, Trans. Amer. Math. Soc. **306** (1988), 411–420.

20. ———, *Fixed-points of tree-like continua*, Contemp. Math. **72** (1988), 131–137.

21. O.H. Hamilton, *Fixed points under transformations of continua which are not connected im kleinen*, Trans. Amer. Math. Soc. **44** (1938), 18–24.

22. ———, *A fixed point theorem for pseudo-arcs and certain other metric continua*, Proc. Amer. Math. Soc. **2** (1951), 173–174.

23. M.W. Hirsch and C.C. Pugh, *Cohomology of chain recurrent sets*, Ergodic Theory Dynamical Systems **8** (1988), 73–80.

24. M.W. Hirsch and S. Smale, *Differential equations, dynamical systems, and linear algebra*, Academic Press, New York, 1974.

25. W. Holsztynski, *Fixed points of arcwise connected spaces*, Fund. Math. **64** (1969), 289–312.

26. S. Iliadis, *Positions of continua on the plane and fixed points*, Vestnik Mosc. Univ., Ser. Math. Meh. **4** (1970), 66–70.

27. F.B. Jones, *Concerning non-aposyndetic continua*, Amer. J. Math. **70** (1948), 403–413.

28. J.L. Kelley, *General topology*, Van Nostrand, Princeton, New Jersey, 1955.

29. V. Klee and S. Wagon, *New and old unsolved problems in plane geometry and number theory*, Dolciani Math. Expositions, No. 11, Math. Assoc. Amer., 1991.

30. J. Krasinkiewicz, *Concerning the boundaries of plane continua and the fixed point property*, Bull. Acad. Polon. Sci. **21** (1973), 427–431.

31. ———, *Concerning the accessibility of composants of indecomposable plane continua*, Bull. Acad. Polon. Sci. **21** (1973), 621–628.

32. K. Kuratowski, *Topology*, Vol. II, Academic Press, New York, 1968.

33. I.W. Lewis, *Continuum theory problems*, Topology Proc. **8** (1983), 361–394.

34. P. Minc, *A fixed point theorem for weakly chainable plane continua*, Trans. Amer. Math. Soc. **317** (1990), 303–312.

35. L. Mohler and L.G. Oversteegen, *Open and monotone fixed point free maps on uniquely arcwise connected continua*, Proc. Amer. Math. Soc. **95** (1986), 476–482.

36. R.L. Moore, *Concerning triodic continua in the plane*, Fund. Math. **13** (1929), 261–263.

37. ———, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 13, Amer. Math. Soc., Providence, RI, 1962.

38. L.G. Oversteegen and J.T. Rogers, Jr., *An inverse limit description of an atriodic tree-like continuum and an induced map without a fixed point*, Houston J. Math. **6** (1980), 549–564.

39. ———, *Fixed-point-free maps on tree-like continua*, Topology Appl. **13** (1982), 85–95.

40. H.E. Schlais, *Non-aposyndesis and non-hereditary decomposability*, Pacific J. Math. **63** (1968), 257–278.

41. K. Sieklucki, *On a class of plane acyclic continua with the fixed point property*, Fund. Math. **63** (1968), 257–278.

42. J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.

43. H. Whitney, *Regular families of curves*, Ann. Math. **34** (1933), 244–270.

44. G.T. Whyburn, *Analytic topology*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, RI, 1963.

45. G.S. Young, *Fixed-point theorems for arcwise connected continua*, Proc. Amer. Math. Soc. **11** (1960), 880–884.

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