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FUNCTIONS WITH GIVEN MOMENTS AND WEIGHT FUNCTIONS FOR ORTHOGONAL POLYNOMIALS

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ABSTRACT. We give a constructive technique to find smooth functions with given moments. We apply this technique to find weight functions for classical polynomials. In particular, we give some weight functions for Bessel polynomials.

Introduction. Given any sequence of complex numbers $(a_n)_n$, called moments, satisfying

$$\Delta_n = \det\,[a_{i+j}]_{i,j=0}^n \neq 0,$$

there correspond the set of Chebychev polynomials defined by $p_0(x) = 1$ and

$$p_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} a_0 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{2n-1} \\ 1 & \cdots & x^n \end{vmatrix}, \qquad n \geq 1.$$

They are mutually orthogonal with respect to any linear functional w defined on polynomials generating the moments, that is,

$$\langle w, x^n \rangle = a_n, \qquad n \geq 0.$$

Such a functional is called a weight for the polynomials $(p_n(x))_n$.

A known result by Boas and Polya [**2, 17**] guarantees that for every sequence $(a_n)_n$ there is a function of bounded variation such that

$$\int_0^\infty t^n\, df(t) = a_n, \qquad n \geq 0. \tag{0.1}$$

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More recently, [**4**], the author proved that it is possible to find a very regular function f (f belonging to the Schwartz space S and vanishing for $t < 0$) such that (0.1) holds.

That is, every set of Chebychev polynomials has a weight function. But how are these weight functions found?

Recently there have been many attempts to interpret the formal weight w in some generalized sense. R.D. Morton and A.M. Krall [**16**] have introduced the formal functional Taylor series (δ series in short) of w:

$$w(x) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{n!} \delta^{(n)}.$$

Assuming suitable restrictions on the growth of a_n, these series have been used to obtain distributional weight for Chebychev polynomials. S.S. Kim and K.H. Kwon [**13**] have interpreted the δ series in the hyperfunctional sense.

However, an old open problem remained unsolved: to find a weight function for the Bessel polynomials.

This problem appeared the first time in the paper by H.L. Krall and O. Frink [**14**]. More recently, the same problem can be found in [**15, 16**] and elsewhere. The problem is to find a function f which satisfies:

$$\int_{\mathbf{R}} t^n f(t)\, dt = \frac{(-2)^{n+1}}{(n+1)!} \qquad \text{for all } n \geq 0.$$

It has been shown [**16**], that the δ series

$$w(x) = -\sum_{n=0}^{\infty} \frac{2^{n+1}}{n!(n+1)!} \delta^{(n)} \tag{0.2}$$

serves as a distributional weight for the Bessel polynomials. But a function f which generates the Bessel moments was unknown.

In this paper we give a constructive technique to find a function with given moments. Moreover, we will construct these functions in the Schwartz spaces S or S^+ (S^+ is the Schwartz space on the interval $(0, +\infty)$).

In Section 1, we give the technique using a certain class of continuous linear maps from the space $S(I)$ ($S(I)$ is the Schwartz space on the interval I) into itself; for example, the Fourier transform and the Hankel transform belong to this class. Any operator in this class would satisfy that its transpose operator transform the functions t^n into the derivatives of the Dirac's delta (up to known constants). Hence, the problem of finding a function with given moments is equivalent to the well-known Borel Theorem of finding a function whose derivatives at the origin are given.

In Section 2, we use the above technique to find weight functions for the classical orthogonal polynomials. Thus, we find the following weight functions for the Bessel polynomials:

$$\begin{aligned} f_1(t) = \frac{-1}{\pi c}\int_{-\infty}^{+\infty} &((2ix)^{-1/2} J_1(\sqrt{8ix})\cdot \\ &\cdot\int_{-1}^{x}(\chi_{[-1,0]}(u)-\chi_{(0,1]}(u))e^{-1/u^2-1/(1-u^2)}\,du)e^{-ixt}\,dx \end{aligned}$$

$$\begin{aligned} f_2(t) = -\frac{1}{2c}\int_0^{\infty} &\Bigg(\Bigg(\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}(n!)^2(n+1)!}\Bigg)\cdot \\ &\cdot\int_{-1}^{x}(\chi_{[-1,0]}(u)-\chi_{(0,1]}(u))e^{-1/u^2-1/(1-u^2)}\,du\Bigg)J_0(\sqrt{xt})\,dx \end{aligned}$$

($c=\int_{-1}^{0}(\chi_{[-1,0]}(u)-\chi_{(0,1]}(u))e^{-1/u^2-1/(1-u^2)}\,du$), which belong to the space S and S^+, respectively, and satisfy

$$\int_{\mathbf{R}} t^n f_1(t)\,dt = \frac{(-2)^{n+1}}{(n+1)!}, \qquad \int_0^{\infty} t^n f_2(t)\,dt = \frac{(-2)^{n+1}}{(n+1)!} \qquad n\geq 0.$$

Finally, in Section 3 we prove that the δ series (0.2) is not a tempered distribution with support contained in $[0,+\infty)$. This is a partial negative answer to the question in [**15**], whether there exists a function of bounded variation f such that $w=df(x)/dx$ (also see [**16**, p. 624]).

1. Functions with given moments. First we recall the definition of the Schwartz spaces $S(I)$,

$$S(I)=\{f\in\mathcal{C}^{\infty}(I) : ||f||_{I,k,n}=\sup_{t\in I}|t^k f^{(n)}(t)|<\infty \text{ for all } k,n\in\mathbf{N}\}.$$

Here, $I = \mathbf{R}$ or $I = (0, +\infty)$. We put $S = S(\mathbf{R})$ and $S^+ = S((0, +\infty))$. It is easy to prove that a function f belongs to S^+ if and only if f is the restriction to $(0, +\infty)$ of a function which belongs to the space S.

We will endow these spaces with the Frechet topology generated by the semi-norms $(||f||_{I,k,n})_{k,n\geq 0}$. Their dual spaces are the space of tempered distribution S' and the space of tempered distributions with support contained in $[0, +\infty)$ $(S^+)'$, respectively.

Now, we give a construction of a function in the space $S(I)$ with given moments:

Let $(f_n^I)_n$ be a sequence in $S(I)$ such that

$$\int_I t^k f_n^I(t)\, dt = \delta_{k,n}, \qquad k, n \geq 0 \tag{1.1}$$

where $\delta_{k,n}$ is the Kronecker delta.

Given a sequence of complex numbers $(a_n)_n$, the function

$$f(t) = \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n^{n+1}} f_n^I\left(\frac{t}{\lambda_n}\right) \tag{1.2}$$

formally satisfies

$$\int_I t^k f(t)\, dt = a_k, \qquad k \geq 0 \tag{1.3}$$

for all sequences $(\lambda_n)_n$ of positive numbers.

The formula (1.2) was suggested to the author by Jaak Peetre, and it has been used in [**7**] to extend some results on a Stieltjes moment problem with complex exponents for Banach space valued functions.

So, first we will give a sequence $(f_n^I)_n$ satisfying (1.1).

We denote by T_I any continuous linear map from $S(I)$ into itself which satisfies the following conditions:

Condition C.1. For all $h \in S(I)$ and $\lambda > 0$,

$$T_I(h(t))\left(\frac{x}{\lambda}\right) = T_I(\lambda h(\lambda t))(x).$$

Condition C.2. Its transpose mapping T_I^t defined by

$$\begin{gathered} T_I^t : S'(I) \to S'(I) \\ \langle T_I^t(u), \phi \rangle = \langle u, T_I(\phi) \rangle \end{gathered}$$

satisfies that there exists a complex sequence $(d_n^I)_n$ such that $T_I^t(t^n) = d_n^I \delta^{(n)}$ and $d_n^I \neq 0$ for all $n \geq 0$.

In order to construct the sequence $(f_n^I)_n$, we consider a function g which satisfies

$$\begin{gathered} g \in \mathcal{C}^\infty(\mathbf{R}), \quad \operatorname{supp}(g) \text{ is a compact set}, \quad 0 \in \operatorname{int}(\operatorname{supp} g), \\ g(0) = 1 \quad \text{and} \quad g^{(n)}(0) = 0 \qquad \text{for } n \geq 1. \end{gathered} \tag{1.4}$$

Remark 1.1. A function satisfying (1.4) is the following one:

$$g(x) = \frac{1}{c} \int_{-1}^{x} (\chi_{[-1,0]}(t) - \chi_{(0,1]}(t)) e^{-1/t^2 - 1/(1-t^2)} \, dt$$

where $c = \int_{-1}^{0} e^{-1/t^2 - 1/(1-t^2)} \, dt$.

We define the functions f_n^I by $f_n^I(t) = C_n^I T_I(t^n g(t))$ where the constants C_n^I are defined by

$$C_n^I = \frac{(-1)^n}{n! d_n^I} \tag{1.5}$$

and g is a function satisfying (1.4).

Since $t^n g(t) \in S(I)$, it follows that $f_n^I \in S(I)$, and we get:

$$\begin{aligned} \int_I t^k f_n^I(t) \, dt &= \langle t^k, f_n^I(t) \rangle \\ &= \langle t^k, C_n^I T_I(t^n g(t)) \rangle \\ &= \langle T_I^t(t^k), C_n^I t^n g(t) \rangle \\ &= \langle d_n^I \delta^{(k)}, C_n^I t^n g(t) \rangle = \delta_{k,n}. \end{aligned}$$

We simplify the expression for the function f. From Condition C.1, formally the function (1.2), whose moments are $(a_n)_n$ can be written as

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n^{n+1}} f_n^I\left(\frac{t}{\lambda_n}\right) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n^{n+1}} C_n^I T_I(\lambda_n^{n+1} x^n g(\lambda_n x))(t) \\ &= T_I\left(\sum_{n=0}^{\infty} a_n C_n^I x^n g(\lambda_n x)\right)(t). \end{aligned}$$

Now we prove that for a suitable choice of the sequence $(\lambda_n)_n$, the function f is in $S(I)$ and (1.3) holds.

Indeed, taking $\lambda_n^I = n + \sum_{m=0}^{n} |a_m C_m^I m!|$, we get (see [**3**, p. 50]) that the function

$$h(x) = \sum_{n=0}^{\infty} a_n C_n^I x^n g(\lambda_n^I x) \tag{1.6}$$

is a $\mathcal{C}^\infty$ function with compact support and which satisfies $h^{(n)}(0) = a_n C_n^I n!$. So $h \in S(I)$ and, hence, $f(t) = T_I(h)(t) \in S(I)$. Moreover, we get

$$\begin{aligned} \int_I t^k f(t)\, dt &= \langle t^k, f(t) \rangle \\ &= \langle t^k, T_I(h) \rangle = \langle T_I(t^k), h(x) \rangle \\ &= \langle d_k^I \delta^{(k)}, h(x) \rangle = a_k \qquad k \geq 0 \end{aligned} \tag{1.7}$$

Remark 1.2. Notice that the keys in the above construction are the formula (1.2) and Conditions C.1, C.2. From Conditions C.1 and C.2, it follows that if a function f is regular enough, its moments are the derivatives of the function $T_I f$ at the origin (modified by some known constants). So, to give a function with given moments is equivalent to giving a function whose derivatives at the origin are given, that is the well-known Borel Theorem. From here, the formula (1.2) which gives a function with given moments is equivalent to the formula given by Garding ([**3**, p. 50]) to prove the Borel Theorem.

Now, we give some operators T_I which satisfy Conditions C.1, C.2.

If $I = \mathbf{R}$, we consider the Fourier transform defined by

$$T_{\mathbf{R}} : S \to S$$
$$(T_{\mathbf{R}} f)(x) = (\mathcal{F} f)(x) = \int_{\mathbf{R}} f(t) e^{-ixt}\, dt.$$

If $I = (0, +\infty)$, we consider the Hankel transform defined by

$$T_{(0,+\infty)} : S^+ \to S^+$$
$$(T_{(0,+\infty)} f)(x) = (\mathcal{H}_0 f)(x) = \frac{1}{2} \int_0^\infty f(t) J_0(\sqrt{xt})\, dt$$

(J_0 is the Bessel function of first kind). In both cases T_I is an isomorphism from $S(I)$ onto itself (In the case $I = \mathbf{R}$, this result is well known. For the case $I = (0, +\infty)$, see [**5**]). Its transpose T_I^t is also an isomorphism from $S'(I)$ onto itself, which satisfies the following formulas:

If $I = \mathbf{R}$,

$$(T_{\mathbf{R}}^t)^2(u) = 2\pi \check{u} \qquad \text{for all } u \in S' \tag{1.8}$$

($\check{u}$ is the distribution defined by $\langle \check{u}, \phi \rangle = \langle u, \phi(-x) \rangle$)

$$T_{\mathbf{R}}^t(\delta^{(n)}) = i^n x^n \tag{1.9}$$

$$T_{\mathbf{R}}^t(t^n) = 2\pi i^n \delta^{(n)} \tag{1.10}$$

and if $I = (0, +\infty)$ (see [**6**, Lemma 3])

$$(T_{(0,+\infty)}^t)^2 = \mathrm{Id}\,. \tag{1.11}$$

(Here, Id denotes the identity map)

$$T_{(0,+\infty)}^t(\delta^{(n)}) = \frac{1}{n!} \left(\frac{1}{2}\right)^{2n+1} x^n. \tag{1.12}$$

A change of variables gives Condition C.1 for the Fourier and Hankel transforms. In these cases, the constant C_n^I defined in (1.5) are $C_n^I = i^n/(2\pi n!)$ if $I = \mathbf{R}$ and

$$C_n^I = \frac{(-1)^n}{2^{2n+1}(n!)^2} \quad \text{if } I = (0, +\infty).$$

Remark 1.3. When $I=(0,+\infty)$, we can extend the above technique as follows: Let $\alpha>-1$, and let $T^{\alpha}_{(0,+\infty)}$ be the operator defined by

$$T^{\alpha}_{(0,+\infty)}: S^+ \to S^+$$
$$T^{\alpha}_{(0,+\infty)}(f)(x)=\frac{1}{2}\int_0^{\infty} f(t)(tx)^{-\alpha/2}t^{\alpha}J_{\alpha}(\sqrt{xt})\,dt$$

(J_α is the Bessel function of first kind) where it is an isomorphism from S^+ onto itself and satisfies $(T^{\alpha}_{(0,+\infty)})^2=\mathrm{Id}$ (see [**5**]). Its transpose $(T^{\alpha}_{(0,+\infty)})^t$ is also an isomorphism from $(S^+)'$ onto itself. The following formula can be found in [**6**, Lemma 3]:

$$(T^{\alpha}_{(0,+\infty)})^t(t^{n+\alpha})=\Gamma(\alpha+n+1)2^{\alpha+2n+1}\delta^{(n)}.$$

So, taking

$$C_n^{I,\alpha}=\frac{(-1)^n}{2^{\alpha+2n+1}\Gamma(\alpha+n+1)n!} \quad \text{and} \quad \lambda_n^{\alpha}=n+\sum_{m=0}^{n}|a_nC_n^{I,\alpha}|,$$

we get that the function

$$f(t)=T^{\alpha}_{(0,+\infty)}\Bigg(\sum_{n=0}^{\infty}a_nC_n^{I,\alpha}x^ng(\lambda_n^{\alpha}x)\Bigg)(t)$$

satisfies that $\int_0^{\infty} t^nt^{\alpha}f(t)\,dt=a_n$ for $n\geq 0$.

In this way, the following theorem has been proved:

Theorem 1. *Let $(a_n)_n$ be a sequence of complex numbers and $\alpha>-1$. Then the functions*:

$$f_1(t)=\int_{-\infty}^{+\infty}\Bigg(\sum_{n=0}^{\infty}a_nC_n^{\mathbf{R}}x^ng(\lambda_n^{\mathbf{R}}x)\Bigg)e^{-ixt}\,dx \tag{1.13}$$

(1.14)
$$f_2(t)=\frac{1}{2}\int_0^{+\infty}\Bigg(\sum_{n=0}^{\infty}a_nC_n^{(0,+\infty),\alpha}x^ng(\lambda_n^{\alpha}x)\Bigg)(tx)^{-\alpha/2}x^{\alpha}J_{\alpha}(\sqrt{xt})\,dx$$

satisfy

$$\int_{-\infty}^{+\infty} t^n f_1(t)\,dt = a_n, \qquad \int_0^{+\infty} t^n t^\alpha f_2(t)\,dt = a_n \qquad n \geq 0,$$

where

$$C_n^{\mathbf{R}} = \frac{i^n}{2\pi n!}, \qquad C_n^{(0,+\infty),\alpha} = \frac{(-1)^n}{2^{\alpha+2n+1}\Gamma(\alpha+n+1)n!},$$

$\lambda_n^I = n + \sum_{m=0}^n |a_m C_m^I m!|$, *and* g *is a function satisfying* (1.4).

2. Weight functions for orthogonal polynomials. The second part of this paper is devoted to showing some applications of Theorem 1. We study some cases in which the expressions (1.13) and (1.14) can be simplified. They are used to compute weight functions for classical polynomials.

Case 2.I. *We assume that the function* $\phi(x) = \sum_{n=0}^\infty a_n C_n^I x^n$ *belongs to the space* $S(I)$.

In this case, we can put $g = 1$ in the formulas (1.13) and (1.14). Indeed, since the function $\phi \in S(I)$, if we change the function h (see (1.6)) to the function ϕ, (1.7) remains valid.

Example 2.I.1. *Hermite polynomials.* The moments for the Hermite polynomials are $a_{2n} = \sqrt{\pi}(2n)!/4^n n!$, $a_{2n+1} = 0$ for $n = 0, 1, 2, \dots$.

Taking $I = \mathbf{R}$, we obtain that the function

$$\phi(x) = \frac{1}{2\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n (2n)!}{4^n (2n)! n!} x^n = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$$

belongs to S, and so a weight function for the Hermite polynomials is

$$f(t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2/4} e^{-ixt}\,dx = e^{-t^2}.$$

Example 2.I.2. *Laguerre and generalized Laguerre polynomials.* The moments for the generalized Laguerre polynomials are $a_n =$

$\Gamma(n+\alpha+1)/\Gamma(\alpha+1)$ for $n \geq 0$ and $\alpha > -1$. In 2.III.2, we shall give weight functions also when $\alpha < -1$, $\alpha \neq -1, -2, \dots$).

Taking $I = (0, +\infty)$, we obtain that the function

$$\phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\alpha+1)}{2^{\alpha+2n+1}\Gamma(n+\alpha+1)\Gamma(\alpha+1)n!} x^n = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} e^{-x/4}$$

belongs to S^+, and so a weight function for the generalized Laguerre polynomials is (see [**11**, p. 185 (30)])

$$\begin{aligned} t^\alpha f(t) &= t^\alpha \frac{1}{2^{\alpha+2}\Gamma(\alpha+1)} \int_0^{+\infty} e^{-x/4} (tx)^{-\alpha/2} x^\alpha J_\alpha(\sqrt{xt})\, dx \\ &= \frac{t^\alpha}{\Gamma(\alpha+1)} e^{-t}. \end{aligned}$$

Case 2.II. *We assume that there exists a distribution u with compact support such that $T_I^t u = \sum_{n=0}^{\infty} a_n C_n^I x^n$.*

Here, when $I = (0, +\infty)$, we assume $\alpha = 0$.

In this case, we can obtain a distribution with compact support v such that $\langle v, t^n \rangle = a_n$.

Indeed, taking $v = T_I^t(\sum_{n=0}^{\infty} a_n C_n^I x^n)$, from (1.8) and (1.11), it follows that $v = 2\pi \check{u}$ if $I = \mathbf{R}$ and $v = u$ if $I = (0, +\infty)$. From the following Lemma, (1.9) and (1.12), it follows that $\langle v, t^n \rangle = a_n$.

Lemma. *Let w_1, w_2 be two distributions with compact support. Then $\langle w_1, T_I^t w_2 \rangle = \langle w_2 T_I^t w_1 \rangle$.*

Proof. Let u be a distribution with compact support K, φ a $\mathcal{C}^\infty$ function with compact support such that $\varphi(x) = 1$ if $x \in K$ and F and entire function. Since the series which appears in the right side of the following formula $F(x)\varphi(x) = \sum_{n=0}^{\infty} F^{(n)}(0) x^n \varphi(x)/n!$ converges uniformly on compact sets of $\mathbf{R}$, we get

$$\langle u, F(x) \rangle = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \langle u, x^n \varphi(x) \rangle = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \langle u, x^n \rangle.$$

It is not hard to prove that $T_I^t u(x) = \langle u(t), e^{-ixt}\rangle$ if $I = \mathbf{R}$ and $T_I^t u(x) = \langle u(t),\, J_0(\sqrt{xt})/2\rangle$ if $I = (0, +\infty)$.

From these previous results, it follows that $T_I^t w_i$ is an entire function and $(T_I^t w_i)^{(n)}(0) = \langle w_i, \alpha_n^I t^n\rangle$ for $i = 1, 2$, where $\alpha_n^I = (-i)^n$ if $I = \mathbf{R}$ and $\alpha_n^I = (-1)^n/(2^{2n+1} n!)$ if $I = (0, +\infty)$. The same argument gives

$$\langle w_1, T_I^t w_2\rangle = \sum_{n=0}^{\infty} \frac{\alpha_n^I}{n!}\langle w_2, x^n\rangle\langle w_1, x^n\rangle$$

and

$$\langle w_2, T_I^t w_1\rangle = \sum_{n=0}^{\infty} \frac{\alpha_n^I}{n!}\langle w_2, x^n\rangle\langle w_1, x^n\rangle$$

finishing the proof of the Lemma. □

Example 2.II.1. *Jacobi polynomials.* The moments for the Jacobi polynomials are

$$a_n = \sum_{m=0}^{n} \binom{n}{m} (-1)^m 2^m \frac{\Gamma(b+m)\Gamma(a+b)}{\Gamma(b)\Gamma(a+b+m)} \qquad \text{for } n \geq 0 \text{ and } a, b > 0.$$

In 2.III.2 we shall give weight functions also when $b \neq 0$, $b < 0$ and $a < 0$, $a \neq 0, -1, -2, \dots$). Taking $I = \mathbf{R}$, we obtain the function

$$\phi(x) = \frac{1}{2\pi} e^{-ix}\, {}_1F_1(b, a+b; 2ix)$$

(${}_1F_1(-,-;-)$ is the hypergeometric function). So, we obtain the following distribution with compact support

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)2^{a+b-1}}(1-x)^{b-1}(1+x)^{a-1}\chi_{[-1,1]}.$$

Remark . Notice that when the function $\phi(x) = \sum_{n=0}^{\infty} a_n C_n^I x^n$ can be written as the sum of two functions $\sum_{n=0}^{\infty} a_n C_n^I x^n = \phi_1 + \phi_2$, such that $\phi_1 \in S(I)$ and there exists a distribution with compact support such that $T_I u = \phi_2$, we can put $g = 1$ in the formulas (1.13) and (1.14) to obtain the weight $T_I\phi_I + T_I^t\phi_2$ which is a sum of a function in $S(I)$ and a distribution with compact support.

Example 2.II.2. *Laguerre type polynomials.* The moments for the Laguerre type polynomials are $a_0 = (R+1)/R$, $a_n = \Gamma(n+\alpha+1)/\Gamma(\alpha+1)$ for $n \geq 1$ and $\alpha > -1$. Taking $I = (0,+\infty)$, we obtain the function $\phi(x) = 1/R + e^{-x/4}/2$. So, we can take $\phi_2 = 1/R$ and $\phi_1 = e^{-x/4}/2$ to obtain the weight $\delta/R + e^{-x}$.

Case 2.III. We assume that

$$\limsup_n \sqrt[n]{|a_n C_n^I|} = R < +\infty. \tag{2.1}$$

Here, in the formulae (1.13) and (1.14), we can put $\lambda_n = 1$, choosing the function g (which appears in those formulae) such that $\operatorname{supp} g \subset (-1/R, 1/R)$. This choice guarantees that the function $h(x) = \sum_{n=0}^{\infty} a_n C_n^I x^n g(x)$ belongs to the space $S(I)$ and so (1.7) holds.

Example 2.III.1. *Bessel polynomials.* The moments for the Bessel polynomials are $a_n = (-b)^{n+1}\Gamma(a)/\Gamma(a+n)$ for $n \geq 0$ and $b \neq 0$, $a \neq 0, -1, -2, \dots$. Since

$$\limsup_n \sqrt[n]{\left|\frac{(-b)^{n+1}\Gamma(a)}{\Gamma(a+n)} C_n^I\right|} = 0,$$

taking $I = \mathbf{R}$, we obtain that a weight function for the Bessel polynomials is

$$f(t) = \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi}\sum_{n=0}^{\infty} \frac{(-b)^{n+1} i^n \Gamma(a)}{\Gamma(a+n)n!} x^n\right) g(x) e^{-ixt}\, dx$$

where g is a function satisfying (1.4). Now, from the Taylor expansion for the Bessel functions and taking the function of Remark 1.1, we obtain that the above function can be written as follows

$$f(t) = \frac{-b\Gamma(a)}{2\pi c} \int_{-\infty}^{+\infty} \Bigg((bix)^{(1-a)/2} J_{a-1}(\sqrt{4bix}) \cdot \int_{-1}^{x} (\chi_{[-1,0](u)} - \chi_{(0,1]}(u)) e^{-1/u^2 - 1/(1-u^2)}\, du\Bigg) e^{-ixt}\, dx$$

Also, taking $I = (0, +\infty)$, we obtain that the function

$$f(t) = \frac{1}{2c} \int_0^\infty \left(\left(\sum_{n=0}^\infty \frac{(-b)^{n+1}\Gamma(a)}{\Gamma(a+n)} \frac{(-1)^n}{2^{2n+1} n! n!} x^n \right) \right.$$
$$\left. \cdot \int_{-1}^x (\chi_{[-1,0](u)} - \chi_{(0,1]}(u)) e^{-1/u^2 - 1/(1-u^2)}\, du \right) J_0(\sqrt{xt})\, dx$$

is a weight function for Bessel polynomials.

Example 2.III.2. Here we can give weight functions for generalized Laguerre polynomials when $\alpha < -1$ and $\alpha \neq -1, -2, \dots$. Indeed, the moments satisfy (2.1). So, if we take the least positive integer k_α such that $-1 < \alpha + k_\alpha$, we get the following weight function

$$f(t) = \frac{1}{2} \int_0^\infty \left(\sum_{n=0}^\infty \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1) w^{\alpha+k_\alpha+2n+1} \Gamma(n+\alpha+k_\alpha+1) n!} x^n \right) g(x)$$
$$\cdot (tx)^{-(\alpha+k_\alpha)/2} x^{\alpha+k_\alpha} J_{\alpha+k_\alpha}(\sqrt{xt})\, dx,$$

that is,

$$f(t) = \frac{1}{2^{\alpha+k_\alpha+2}\Gamma(\alpha+k_\alpha+1)c} \int_0^\infty {}_1F_1(\alpha+1, \alpha+k_\alpha+1; -x/4)$$
$$\cdot \left(\int_{-1}^x (\chi_{[-1,0](u)} - \chi_{(0,1]}(u)) e^{-1/u^2 - 1/(1-u^2)}\, du \right)$$
$$\cdot (tx)^{-(\alpha+k_\alpha)/2} x^{\alpha+k_\alpha} J_{\alpha+k_\alpha}(\sqrt{xt})\, dx.$$

Finally, for the Jacobi polynomials, when $b \neq 0$ and $a \neq 0, -1, -2, \dots,$ we obtain the following weight function

$$f(t) = \frac{1}{2\pi c} \int_{-\infty}^{+\infty} {}_1F_1(b, a+b; 2ix)$$
$$\cdot \left(\int_{-1}^x (\chi_{[-1,0](u)} - \chi_{(0,1]}(u)) e^{-1/u^2 - 1/(1-u^2)}\, du \right) e^{-ix(1+t)}\, dx.$$

APPENDIX

Here we prove that the δ series

$$w = -\sum_{n=0}^{\infty} \frac{2^{n+1}}{n!(n+1)!}\delta^{(n)} \tag{3.1}$$

is not a tempered distribution with support contained in $[0, +\infty)$ (that is, $w \notin (S^+)'$). More exactly, there exists a dense subspace A of S^+ (so $S^+)' \subset A'$), such that the δ series w belongs to A' (that is, the series which defines w converges in the weak topology of A') and $w \neq u$ in A' for all $u \in (S^+)'$.

To prove this, we consider the Gelfand Shilov space S_1^{+0} defined by (see [**8**]):

$$S_1^{+0} = \{f \in \mathcal{C}^\infty((0,+\infty)) : \exists\, C, A, B > 0 \text{ such that } \sup_{t>0} |t^k f^{(p)}(t)| \le CA^k B^p k^k \quad \forall\, k, p \ge 0\}.$$

The space S_1^{+0} can be regarded as the union with respect to $A, B > 0$ of the Banach spaces

$$S_{1,A}^{+0,B} = \{f \in \mathcal{C}^\infty((0,+\infty)) : \exists\, C > 0 \text{ such that } \sup_{t>0} |t^k f^{(p)}(t)| \le CA^k B^p k^k \quad \forall\, k, p \ge 0\}$$

with the norm $||f||_{A,B} = \inf_{k,p\ge 0} \sup_{t>0} |t^k f^{(p)}(t)|/(A^k B^p K^k)$.

The space S_1^{+0} is a dense subspace of the Schwartz space S^+, and the inclusion is a continuous map, so we get $(S^+)' \subset (S_1^{+0})'$.

Now the δ series w defined by (3.1) belongs to $(S_1^{+0})'$, and the series which defines w converges in the weak topology of $(S_1^{+0})'$. Indeed, it is sufficient to prove that for all $f \in S_1^{+0}$ the series $-\sum_{n=0}^{\infty}(-1)^n 2^{n+1} f^{(n)}(0)/n!(n+1)!$ converges. But this follows since $|f^{(n)}(0)| \le CA^n$ for $n \ge 0$ and certain $A > 0$ for all $f \in S_1^{+0}$ (see [**8**, Proposition 4.11]).

We consider the Laguerre orthogonal system in $L^2((0,+\infty))$ defined by:

$$\phi_n(x) = L_n(x)e^{-x/2} \tag{3.2}$$

where $(L_n(x))_n$ are the Laguerre polynomials.

Since $\phi_n \in S_1^{+0}$, the Laguerre-Fourier coefficients for an element $u \in (S_1^{+0})'$ can be defined by the formula $\langle u, \phi_n \rangle$.

The following characterization of these Laguerre-Fourier coefficients can be found in [**8**, Corollary 3.1]:

Theorem A. *If* $u \in (S_1^{+0})'$, *then* $\limsup_n \sqrt[n]{|\langle u, \phi_n \rangle|} \leq 1$.

And the following characterization on the Laguerre-Fourier coefficients for tempered distribution with support contained in $[0, +\infty)$ can be found in [**9**, Theorem 3.9]:

Theorem B. *If* $u \in (S^+)'$, *then there exist two constants* C, $k > 0$ *such that* $|\langle u, \phi_n \rangle| \leq C(n+1)^k$.

From Theorem B, it follows that if there exists a tempered distribution u with support contained in $[0, +\infty)$, such that $\langle u, f \rangle = \langle w, f \rangle$ for all $f \in S_1^{+0}$, then

$$|\langle w, \phi_n \rangle| \leq C(n+1)^k \tag{3.3}$$

for certain constants C, $k > 0$.

Now, by computing the sequence $(\langle w, \phi_n \rangle)_n$, we prove that (3.3) does not hold, and so the δ series w is not a tempered distribution with support contained in $[0, +\infty)$.

The following formula which gives the Laguerre-Fourier coefficients of $\delta^{(k)}$ can be found in [**9**, Ex. 3.1]:

$$\langle \delta^{(k)}, \phi_n \rangle = \sum_{m=0}^{k} \left(\frac{1}{2}\right)^{k-m} \binom{k}{m} \binom{n}{m}.$$

From this formula, it follows that:

$$
\begin{aligned}
\langle w, \phi_n \rangle &= -\sum_{k=0}^{\infty} \frac{2^{k+1}}{k!(k+1)!} \langle \delta^{(k)}, \phi_n(x) \rangle \\
&= -\sum_{k=0}^{\infty} \frac{2^{k+1}}{(k)!(k+1)!} \sum_{m=0}^{k} \left(\frac{1}{2}\right)^{k-m} \binom{k}{m} \binom{n}{m} \\
&= -\sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \sum_{m=0}^{k} \frac{2^{m+1} k! n!}{m!(k-m)!m!(n-m)!} \\
&= -n! \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \sum_{m=0}^{k} \frac{2^{m+1}}{m!(k-m)!m!(n-m)!}
\end{aligned}
$$

So, for all $k \geq 0$ and $0 \leq m \leq k$, we get

$$
|\langle w, \phi_n \rangle| \geq \frac{n! 2^{m+1}}{(k+1)!(k-m)!(n-m)!m!m!}.
$$

If we take $m = k = \log n$, and by using the Stirling formula, we obtain the following estimate

$$
\begin{aligned}
&\frac{n! 2^{\log n+1}}{(\log n+1)!(n-\log n)!(\log n)!(\log n)!} \\
&\qquad \sim \frac{\alpha_1 n^{\alpha_2} n^n e^{-n}}{(n-\log n)^{n-\log n} e^{-n} (\log n)^{3 \log n}} \\
&\qquad \geq \alpha_1 n^{\alpha_2} \frac{n^n}{(\log n)^{3 \log n} n^n n^{-\log n}} \\
&\qquad = \alpha_1 n^{\alpha_2} \frac{n^{\log n}}{(\log n)^{3 \log n}}.
\end{aligned} \tag{3.4}
$$

where α_1 and α_2 are constants which do not depend on n.

From (3.4), we deduce that it is not possible that the Laguerre-Fourier coefficients of w satisfy (3.3).

Using a similar technique, the author has tried (without success) to prove that the δ series w is not a tempered distribution. In this case, we have changed the space S_1^{+0} to the Gelfand Shilov space $S_{1/2}^{1/2}$ (see [**12**]), and the Laguerre orthonormal system to the Hermite orthonormal system. Similar results to Theorems A and B can be found in [**1, 10, 18**], respectively. However, the expression for the Hermite Fourier coefficients of w is more involved than the expression for the Laguerre Fourier coefficients. For example, the even Hermite Fourier coefficients of w are:

$$\langle w, \phi_{2n}\rangle = \frac{(-1)^{n+1}}{\sqrt[4]{\pi}2^n}((2n)!)^{1/2}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{k+1}}{(2k+1)!} \cdot \sum_{m=0}^{\min(k,n)}\frac{2^{3m}}{(n-m)!(k-m)!(2m)!}.$$

The factor $(-1)^k$, which appears in the series on the left side in the above equality, makes it difficult to prove the result with this approach.

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Noted added November 1992. After this papr was written in June 1990 containing the first weight functions for the Bessel polynomials, K.H. Kwon, S.S. Kim and S.S. Han have found (January 1991) other weight functions for these polynomials (see their paper *Orthogonalizing weights of Tchebychev sets of polynomials*, Bull. London Math. Soc. **24** (1992), 361–367).

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