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FOURTH ORDER DIFFERENCE EQUATIONS: OSCILLATION AND NONOSCILLATION

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1. Introduction. This paper studies various properties of solutions of the difference equation

$$\Delta^2(P_n\Delta^2 U_n) - Q_{n+1}\Delta^2 U_{n+1} - R_{n+2}U_{n+2} = 0, \tag{e}$$

where P_n, Q_n and R_n define real sequences satisfying $P_n > 0$, $Q_n \geq 0$, $R_n > 0$ for each $n \geq 1$, and Δ denotes the difference operator of the finite calculus, $\Delta U_n = U_{n+1} - U_n$.

By a *solution* of (e) is meant a sequence $U = \{U_n\}$ satisfying (e) for all positive integers n. The *graph* of a sequence U is the polygonal path joining the points (n, U_n), $n \geq 1$. Clearly, the graph of a sequence is a continuous function defined by $U^*(t)$, satisfying $U^*(t) = U_t$ for all positive integers t. The zeros of $U^*(t)$ are called *nodes*. A nontrivial solution U of (e) is *oscillatory* if it has arbitrarily large nodes; otherwise, it is *nonoscillatory*. Solution will mean nontrivial solution. The variables n, m, N, M, i, j, k represent positive integers. The variables n, m, N, M when subscripted represent positive integers. Furthermore, when referring to *intervals* such as $[a, b]$ or $[a, \infty)$, a, b are positive integers and $[a, b] = \{a, a+1, \dots, b\}$ while $[a, \infty) = \{a, a+1, a+2, \dots\}$. This is the same notation used by Hartman [**3**].

Remark. In the literature the second order equation

$$\Delta(P_n\Delta U_n) - R_{n+1}U_{n+1} = 0 \tag{1}$$

is commonly referred to as a self-adjoint equation. See, for example, [**5,9**]. If one actually derives the adjoint of (1), according to Fort's book [**2**], one obtains

$$\Delta(P_{n+1}\Delta V_n) - R_{n+2}V_{n+1} = 0. \tag{2}$$

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Reducing n, on the coefficients of (2), we find the reduced adjoint of (1)

$$\Delta(P_n \Delta V_n) - R_{n+1} V_{n+1} = 0.$$

Following this same convention, the reduced adjoint of (e) is the equation

$$(\mathrm{e}^*) \qquad \Delta^2(P_n \Delta^2 V_n - Q_n V_{n+1}) - R_{n+2} V_{n+2} = 0.$$

Thus, (e) is self-adjoint if and only if Q is constant.

Very few studies have appeared in the literature on fourth order difference equations. The special case of (e) where $P_n \equiv 1$ and $Q_n \equiv 0$ has recently been studied in detail by Smith and Taylor [**8**], and by Hooker and Patula [**4**]. Other works on fourth order difference equations include those by Cheng [**1**] and by Smith and Taylor [**7**].

Following [**4**], we say that a solution U of (e) has a *generalized zero at* m if one of the following holds:

(i) $U_m = 0$, $m \geq 1$;

(ii) $U_{m-1} U_m < 0$, $m > 1$;

(iii) There exists an integer k, $1 < k \leq m$, such that

$$(-1)^k U_{m-k} U_m > 0 \quad \text{and} \quad U_{m-1} = U_{m-2} = \cdots = U_{m-k+1} = 0.$$

A generalized zero of a solution of (e) is of order 0, 1, $k > 1$ according to whether (i), (ii) or (iii), respectively, holds. A nontrivial solution U of (e) satisfying $U_{m-1} = U_m = 0$, or having a generalized zero of order 2 at m, has a *double zero* at m. Similarly, a solution U satisfying $U_{m-2} = U_{m-1} = U_m = 0$, or having a generalized zero of order 3 at m, has a *triple zero* at m. If i and j are such that $i + j = 4$ and no solution of (e) has an (i,j)-distribution of generalized zeros, (e) is (i,j)-*disconjugate*, meaning no nontrivial solution of (e) has a pair of generalized zeros of multiplicities i and j, respectively.

2. Disconjugacy properties. In this section we discuss the disconjugacy properties of (e) and examine the distribution of generalized zeros of various solutions, specifically those having a generalized zero of order greater than one. The asymptotic behavior of these solutions is also studied.

Our first result shows the effect of initial conditions on the asymptotic behavior of certain solutions of (e).

Theorem 2.1. *If* U *is a nontrivial solution of (e) satisfying*

$$U_m \geq 0, \qquad \Delta U_m \geq 0, \qquad P_m \Delta^2 U_m \geq 0, \qquad \Delta(P_m \Delta^2 U_m) \geq 0,$$

for some choice of $m \geq 1$*, then*

$$U_n > 0, \qquad \Delta U_n > 0, \qquad P_n \Delta^2 U_n > 0, \qquad \Delta(P_n \Delta^2 U_n) > 0,$$

for each $n \geq m+3$*.*

Proof. We assume without loss of generality that $\Delta(P_m \Delta^2 U_m) > 0$. Then the following inequalities hold:

$$\begin{aligned} U_{m+1} &= U_m + \Delta U_m \geq 0, \\ \Delta U_{m+1} &= \Delta U_m + \Delta^2 U_m \geq 0, \\ P_{m+1} \Delta^2 U_{m+1} &= P_{m+1}(\Delta^2 U_m + \Delta^3 U_m) \\ &= P_m \Delta^2 U_m + \Delta(P_m \Delta^2 U_m) \\ &\geq \Delta(P_m \Delta^2 U_m) > 0. \end{aligned}$$

From (e),

$$\begin{aligned} \Delta(P_{m+1}\Delta^2 U_{m+1}) = \Delta(P_m \Delta^2 U_m) + (&Q_{m+1}\Delta^2 U_{m+1} \\ &+ R_{m+2}(U_{m+1} + \Delta U_{m+1})) > 0. \end{aligned}$$

Similarly,

$$\begin{gathered} U_{m+2} \geq 0, \qquad \Delta U_{m+2} > 0, \qquad P_{m+2}\Delta^2 U_{m+2} > 0, \\ \Delta(P_{m+2}\Delta^2 U_{m+2}) > 0, \end{gathered}$$

and

$$\begin{gathered} U_{m+3} > 0, \qquad \Delta U_{m+3} > 0, \qquad P_{m+3}\Delta^2 U_{m+3} > 0, \\ \Delta(P_{m+3}\Delta^2 U_{m+3}) > 0. \end{gathered}$$

A simple induction argument completes the proof. □

Lemma 2.2. *If U is a nontrivial solution of* (e) *satisfying*

$$U_{m+1} \geq 0, \qquad \Delta U_m \leq 0, \qquad P_m \Delta^2 U_m \geq 0, \qquad \Delta(P_m \Delta^2 U_m) \leq 0$$

for some choice of $m > 1$, then

$$U_n > 0, \qquad \Delta U_n < 0, \qquad P_n \Delta^2 U_n > 0, \qquad \Delta(P_n \Delta^2 U_n) < 0$$

for each $n = 1, 2, \dots, m-1$.

Proof. From (e), we find

$$\Delta^2(P_{m-1}\Delta^2 U_{m-1}) = Q_m \Delta^2 U_m + R_{m+1} U_{m+1} \geq 0.$$

Thus,

$$\Delta(P_m \Delta^2 U_m) - \Delta(P_{m-1}\Delta^2 U_{m-1}) \geq 0,$$

and hence

$$\Delta(P_{m-1}\Delta^2 U_{m-1}) \leq \Delta(P_m \Delta^2 U_m) < 0.$$

It follows

$$P_m \Delta^2 U_m - P_{m-1}\Delta^2 U_{m-1} < 0.$$

So we see $P_{m-1}\Delta^2 U_{m-1} > 0$. From this it follows that $\Delta U_m - \Delta U_{m-1} > 0$ in which case $\Delta U_{m-1} < 0$. Finally, $\Delta U_m = U_{m+1} - U_m \leq 0$ implies $U_m \geq 0$. Repeating this process for each $1 \leq n < m-1$ proves the lemma. □

The previous result allows us to examine solutions with multiple zeros.

Theorem 2.3. *If U is a solution of* (e) *with a triple zero at $m+2$, then none of the terms*

$$U_n,\ \Delta U_n,\ P_n \Delta^2 U_n,\ \Delta(P_n \Delta^2 U_n),$$

changes sign on the interval $[1, m-2]$. Furthermore,

$$\operatorname{sgn} U_n = \operatorname{sgn} P_n \Delta^2 U_n \neq \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta(P_n \Delta^2 U_n)$$

for each $n = 1, 2, \dots, m-2$ *and*

$$\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} P_n \Delta^2 U_n = \operatorname{sgn} \Delta(P_n \Delta^2 U_n)$$

for each $n \geq m+3$.

Proof. Without loss of generality, suppose U is a solution of (e) satisfying

$$U_{m-1} < 0, \qquad U_m = 0, \qquad U_{m+1} = 0, \qquad U_{m+2} \geq 0$$

for some choice of $m \geq 1$. Clearly,

$$U_m = 0, \qquad \Delta U_{m-1} > 0, \qquad P_{m-1} \Delta^2 U_{m-1} = P_{m-1} U_{m-1} < 0.$$

Moreover,

$$\Delta(P_{m-1} \Delta^2 U_{m-1}) = \Delta(P_{m-1} U_{m-1}) = -P_{m-1} U_{m-1} > 0.$$

Hence, by Lemma 2.2,

$$\operatorname{sgn} U_n = \operatorname{sgn} P_n \Delta^2 U_n \neq \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta(P_n \Delta^2 U_n)$$

for each $n = 1, 2, \dots, m-2$. We also note that

$$\begin{aligned} U_m = 0, \qquad \Delta U_m = 0, \qquad P_m \Delta^2 U_m &= P_m(U_{m+2} - 2U_{m+1} + U_m) \\ &= P_m U_{m+2} \geq 0. \end{aligned}$$

Because of Theorem 2.1, to complete our proof it will be enough to show $\Delta(P_m \Delta^2 U_m) > 0$. However, this is clear. From (e), we find

$$\begin{aligned} \Delta(P_m \Delta^2 U_m) &= \Delta(P_{m-1} \Delta^2 U_{m-1}) + Q_m \Delta^2 U_m + R_{m+1} U_{m+1} \\ &= \Delta(P_{m-1} \Delta^2 U_{m-1}) + Q_m \Delta^2 U_m + R_{m+1}(U_m + \Delta U_m) \\ &= \Delta(P_{m-1} \Delta^2 U_{m-1}) + Q_m \Delta^2 U_m \\ &= \Delta(P_{m-1} U_{m-1}) + Q_m \Delta^2 U_m \\ &= -P_{m-1} U_{m-1} + Q_m \Delta^2 U_m > 0. \qquad \square \end{aligned}$$

Corollary 2.4. *Equation* (e) *is* $(1,3)$*-disconjugate and* $(3,1)$*-disconjugate.*

Theorem 2.1 implies equation (e) always has unbounded nonoscillatory solutions; however, Lemma 2.2 can be used to construct a bounded nonoscillatory solution for (e).

A proof of the following result may be obtained using standard iterative techniques. See, for example, [**4**].

Theorem 2.5. *There exists a solution* W *of* (e) *such that*

$$\operatorname{sgn} W_n = \operatorname{sgn} P_n\Delta^2 W_n \neq \operatorname{sgn} \Delta W_n = \operatorname{sgn} \Delta(P_n\Delta^2 W_n)$$

for all n*, and*

$$\lim_{n\to\infty} \Delta W_n = \lim_{n\to\infty} P_n\Delta^2 W_n = \lim_{n\to\infty} \Delta(P_n\Delta^2 W_n) = 0.$$

Whether or not the solution W is essentially unique remains an open question.

Lemma 2.6. *Suppose* U *is a solution of* (e) *satisfying*

$$\begin{gathered} U_m \geq 0, \qquad \Delta U_m \geq 0, \\ P_{m-1}\Delta^2 U_{m-1} \geq 0, \qquad \Delta(P_{m-2}\Delta^2 U_{m-2}) > 0, \end{gathered}$$

for some integer $m > 2$*. Then,*

$$\begin{gathered} U_n > 0, \qquad \Delta U_n > 0, \\ P_{n-1}\Delta^2 U_{n-1} > 0, \qquad \Delta(P_{n-2}\Delta^2 U_{n-2}) > 0 \end{gathered}$$

for each $n \geq m+2$*. Furthermore,*

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \Delta U_n = \lim_{n\to\infty} P_n\Delta^2 U_n = \infty.$$

Proof. From (e), we find

$$\begin{aligned} P_m U_{m+2} = \Delta(P_{m-2}\Delta^2 U_{m-2}) &+ P_{m-1}\Delta^2 U_{m-1} \\ &+ Q_{m-1}\Delta^2 U_{m-1} + P_m\Delta U_m + R_m U_m + P_m U_{m+1}. \end{aligned}$$

Thus,

$$P_m \Delta U_{m+1} = \Delta(P_{m-2}\Delta^2 U_{m-2}) + P_{m-1}\Delta^2 U_{m-1} \\ + Q_{m-1}\Delta^2 U_{m-1} + P_m \Delta U_m + R_m U_m$$

and

$$P_m \Delta^2 U_m = \Delta(P_{m-2}\Delta^2 U_{m-2}) + P_{m-1}\Delta^2 U_{m-1} \\ + Q_{m-1}\Delta^2 U_{m-1} + R_m U_m.$$

So we find

$$\Delta(P_{m-1}\Delta^2 U_{m-1}) = \Delta(P_{m-2}\Delta^2 U_{m-2}) + Q_{m-1}\Delta^2 U_{m-1} + R_m U_m.$$

Hence,

$$U_m + \Delta U_m = U_{m+1} \geq 0, \qquad \Delta U_{m+1} > 0, \\ P_m \Delta^2 U_m > 0, \qquad \Delta(P_{m-1}\Delta^2 U_{m-1}) > 0.$$

Similarly,

$$U_{m+2} > 0, \qquad \Delta U_{m+2} > 0, \\ P_{m+1}\Delta^2 U_{m+1} > 0, \qquad \Delta(P_m \Delta^2 U_m) > 0.$$

An induction argument will complete the proof. □

Theorem 2.7. *If U is a solution of* (e) *with a double zero at $m+2$, where $m \geq 2$, then none of the terms*

$$U_n,\ \Delta U_n,\ P_n \Delta^2 U_n,\ \Delta(P_n \Delta^2 U_n)$$

changes sign on the interval $[1, m-1]$ and

$$\operatorname{sgn} U_n = \operatorname{sgn} P_n \Delta^2 U_n \neq \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta(P_n \Delta^2 U_n),$$

or none of the terms

$$U_n,\ \Delta U_n,\ P_{n-1}\Delta^2 U_{n-1},\ \Delta(P_{n-2}\Delta^2 U_{n-2})$$

changes sign on the interval $[m+2, \infty)$ and

$$\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} P_{n-1}\Delta^2 U_{n-1} = \operatorname{sgn} \Delta(P_{n-2}\Delta^2 U_{n-2})$$

for each $n \geq m+4$.

Proof. Without loss of generality, suppose U is a solution of (e) satisfying

$$U_m > 0, \qquad U_{m+1} = 0, \qquad U_{m+2} \geq 0,$$

for some integer $m \geq 2$. Clearly, $U_{m+1} = 0$, $\Delta U_m < 0$, $P_m \Delta^2 U_m = P_m(U_{m+2} + U_m) > 0$. If $\Delta(P_m \Delta^2 U_m) < 0$, Lemma 2.2 applies and the first alternative of the theorem holds. If $\Delta(P_m \Delta^2 U_m) \geq 0$, then $P_{m+1}\Delta^2 U_{m+1} \geq P_m \Delta^2 U_m > 0$. Moreover, $U_{m+1} + \Delta U_{m+1} = U_{m+2} \geq 0$ and $\Delta U_{m+2} = \Delta U_{m+1} + \Delta^2 U_{m+1} > 0$. By Lemma 2.6, the second alternative of the theorem holds for each $n \geq m+4$. This completes the proof of the theorem. □

Corollary 2.8. *Equation* (e) *is* $(1,2,1)$*-disconjugate.*

3. Properties of nonoscillatory solutions.

Lemma 3.1. *If U is a nonoscillatory solution of* (e)*, then U_n, ΔU_n, $\Delta^2 U_n$ do not change sign for all n sufficiently large.*

Proof. Suppose U is a nonoscillatory solution of (e). Under the transformation $T_n = P_n \Delta^2 U_n$, equation (e) becomes

$$\Delta^2 T_n - \frac{Q_{n+1}}{P_{n+1}} T_{n+1} = G_n, \tag{3}$$

where $G_n = R_{n+2} U_{n+2}$. Since U is a nonoscillatory solution of (e), G_n is eventually of one sign. By [**6**], equation (3) is nonoscillatory, thus $T_n = P_n \Delta^2 U_n$ is eventually sign definite. This implies ΔU_n is eventually of one sign, and the product $U_n \Delta U_n \Delta^2 U_n$ does not change sign for all n sufficiently large. □

The following result specifies all possible behaviors of nonoscillatory solutions of (e). The proof is straightforward and will be omitted.

Theorem 3.2. *If U is a nonoscillatory solution of* (e)*, then one of the following is true*:

(i) $\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta^2 U_n = \operatorname{sgn} \Delta(P_n \Delta^2 U_n)$ *for all n sufficiently large.*

(ii) $\operatorname{sgn} U_n = \operatorname{sgn} \Delta^2 U_n \neq \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta(P_n \Delta^2 U_n)$ *for each* $n \geq 1$.

(iii) $\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta^2 U_n \neq \operatorname{sgn} \Delta(P_n \Delta^2 U_n)$ *for all* n *sufficiently large.*

(iv) $\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n \neq \operatorname{sgn} \Delta^2 U_n$ *for all* n *sufficiently large.*

Remark. An eventually positive solution of (e) satisfying the relations (i) of Theorem 2.3 we call *strongly increasing*. An eventually positive solution of (e) satisfying the relations (ii) of Theorem 2.3 is termed *strongly decreasing*. Note that in (iv) of Theorem 3.2 no information is known about the behavior of $\Delta(P_n \Delta^2 U_n)$ since $\Delta^2(P_n \Delta^2 U_n)$ may not be of one sign.

In the next result, W represents the solution of (e) whose existence is assured by Theorem 2.5.

Theorem 3.3. *Suppose* U *is an oscillatory solution of* (e) *such that* $U_m \neq 0$ *for some positive integer* m*. Let* t *be a real number such that* $W_m - tU_m = 0$*. Then* $X_n = W_n - tU_n$ *defines an oscillatory solution of* (e) *and* $X_m = 0$*.*

Proof. Suppose not, then X_n satisfies one of the conditions of Theorem 3.2. Clearly, X_n is not strongly decreasing, since it has a zero. From the remaining possibilities, X_n must be increasing. But, examining X_n along the nodes of U, we see that X_n is decreasing. This contradiction completes the proof of the theorem. □

Theorem 3.4. *If* (e) *has an oscillatory solution and* U *is a nonoscillatory solution of* (e) *which has a generalized zero of order zero in common with some oscillatory solution of* (e)*, then there exists an integer* N *such that*

$$\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta^2 U_n = \operatorname{sgn} \Delta(P_n \Delta^2 U_n)$$

for all $n \geq N$*.*

Proof. Let Z be an oscillatory solution of (e) and suppose that $U_m = Z_m = 0$, where U is a nonoscillatory solution of (e). Since U is a nonoscillatory solution of (e), there exists an integer $M_0 \geq m$ such that U_n, ΔU_n, $\Delta^2 U_n$, $\Delta(P_n \Delta^2 U_n)$ are sign definite for each $n \geq M_0$. Let

$$X_n = U_n Z_{n-1} - Z_n U_{n-1},$$

and suppose $n_1 < n_2$ are consecutive generalized zeros of Z which are greater than M_0. Without loss of generality, suppose $Z_{n_1 - 1} > 0$, $Z_{n_1} < 0$, then

$$X_{n_1} = U_{n_1} Z_{n_1 - 1} - Z_{n_1} U_{n_1 - 1} > 0$$

and

$$X_{n_2} = U_{n_2} Z_{n_2 - 1} - Z_{n_2} U_{n_2 - 1} \leq 0.$$

Define $Y_n = U_s Z_n - Z_s U_n$, where s is the first integer between n_1 and $n_2 - 1$ such that $X_s > 0$ and $X_{s+1} \leq 0$. Note that $Y_{s-1} = X_s > 0$, $Y_s = 0$, $Y_{s+1} = -X_{s+1} \geq 0$. Clearly,

$$Y_s = 0, \qquad \Delta Y_s \geq 0, \qquad \Delta^2 Y_{s-1} > 0.$$

Moreover, $\Delta(P_{s-2} \Delta^2 Y_{s-2}) \geq 0$, for if $\Delta(P_{s-2} \Delta^2 Y_{s-2}) < 0$ we would have the following lineup

$$Y_{s-1} > 0, \qquad \Delta Y_{s-2} < 0, \qquad \Delta^2 Y_{s-2} > 0, \qquad \Delta(P_{s-2} \Delta^2 Y_{s-2}) < 0.$$

By Lemma 2.2, Y would be of constant sign for all n such that $1 \leq n < s - 3$, contradicting the fact that $Y_m = 0$. From Lemma 2.6, Y_n, ΔY_n $P_n \Delta^2 Y_n$ all tend to ∞ as n tends to ∞. To complete the proof, it is enough to show $\operatorname{sgn} P_n \Delta^2 U_n = \operatorname{sgn} \Delta(P_n \Delta^2 U_n)$ for all $n \geq M_0$. However, this is clear, for $\operatorname{sgn} P_n \Delta^2 U_n \neq \operatorname{sgn} \Delta(P_n \Delta^2 U_n)$ implies $P_n \Delta^2 U_n$ is bounded for all n sufficiently large. This is seen to be impossible by examining the graph of $P_n \Delta^2 Y_n$ along an infinite sequence of nodes of $P_n \Delta^2 Z_n$. □

Corollary 3.5. *If* (e) *is oscillatory, a nonoscillatory solution of* (e) *satisfies either* (i), (ii), *or* (iii) *of Theorem* 3.2.

In the following result, W represents the solution of (e) whose existence is guaranteed by Theorem 2.5.

Theorem 3.6. *The following two statements are equivalent.*

(a) *Equation* (e) *has an oscillatory solution.*

(b) *If* Y_n *is a nonoscillatory solution of* (e), *then* Y_n *satisfies either* (i) *or* (ii) *of Theorem* 3.2.

Proof. Assume (a) holds, and let U be a nonoscillatory solution of (e). Then there exists a number M_0 such that U_n, ΔU_n, $\Delta^2 U_n$, $\Delta(P_n\Delta^2 U_n)$ are of constant sign for all $n \geq M_0$.

Note. If $\operatorname{sgn} U_n = \operatorname{sgn}\Delta^2 U_n \neq \operatorname{sgn}\Delta U_n = \operatorname{sgn}\Delta(P_n\Delta^2 U_n)$ for all $n \geq M_0$, then $\operatorname{sgn} U_n = \operatorname{sgn}\Delta^2 U_n \neq \operatorname{sgn}\Delta U_n = \operatorname{sgn}\Delta(P_n\Delta^2 U_n)$ for each $n \geq 1$. Let Z be an oscillatory solution of (e) and $m > M_0$ such that $Z_m = 0$. Let K_1 be a constant with

$$U_m + K_1 W_m = 0.$$

Let $Y_n = U_n + K_1 W_n$ and, without loss of generality, suppose $U_n > 0$, for each $n \geq M_0$. If U_n does not satisfy condition (b), then either (iii) or (iv) of Theorem 3.2 is satisfied and in either case Y is nonoscillatory. Since $Y_m = Z_m = 0$, by Theorem 3.4, there exists an integer M_1 such that $\operatorname{sgn} Y_n = \operatorname{sgn}\Delta Y_n = \operatorname{sgn}\Delta^2 Y_n = \operatorname{sgn}\Delta(P_n\Delta^2 Y_n)$, on $[M_1,\infty)$. Indeed, it follows that $\lim_{n\to\infty}|P_n\Delta^2 Y_n| = \infty$ since $\operatorname{sgn}\Delta(P_n\Delta^2 Y_n) = \operatorname{sgn}\Delta^2(P_n\Delta^2 Y_n)$ for $n > M_1$. But this is a contradiction since both $P_n\Delta^2 U_n$ and $P_n\Delta^2 W_n$ are bounded for $n \geq M_0$. This completes the first part of the theorem.

Suppose now condition (b) holds. Let $\{Z_n^0\}, \{Z_n^1\}, \{Z_n^2\}, \{Z_n^3\}$ denote solutions of (e) defined by

$$\Delta^j Z_0^i = \delta_{ij}, \Delta(P_0\Delta^2 Z_0^i) = \delta_{i3}$$

for $i = 0, 1, 2, 3$, $j = 0, 1, 2$, where δ_{ij} is the Kronecker delta. For each natural number m, let b_{0m}, b_{3m}, c_{2m}, and c_{3m} be numbers satisfying

$$\begin{aligned}(b_{0m})^2 + (b_{3m})^2 = (c_{2m})^2 + (c_{3m})^2 = 1,\\ b_{0m}Z_m^0 + b_{3m}Z_m^3 = 0,\\ c_{2m}Z_m^2 + c_{3m}Z_m^3 = 0.\end{aligned}$$

Define U_n^m and V_n^m by

$$\begin{aligned}U_n^m &= b_{0m}Z_n^0 + b_{3m}Z_n^3,\\ V_n^m &= c_{2m}Z_n^2 + c_{3m}Z_n^3.\end{aligned}$$

By the compactness of the unit ball in R^2, there exists a sequence $\{m_k\}$ of natural numbers, and numbers b_0, b_3, c_2, c_3, such that the sequences $\{b_{0m_k}\}$, $\{b_{3m_k}\}$, $\{c_{2m_k}\}$, and $\{c_{3m_k}\}$ converge to b_0, b_3, c_2, c_3, respectively, where

$$(b_0)^2 + (b_3)^2 = (c_2)^2 + (c_3)^2 = 1.$$

Let S_n and T_n be solutions of (e) given by

$$S_n = b_0 Z_n^0 + b_3 Z_n^3,$$
$$T_n = c_2 Z_n^2 + c_3 Z_n^3.$$

We wish to show that S_n and T_n are oscillatory. Suppose S_n is not oscillatory, and assume without loss of generality that $S_n > 0$ for all $n \geq N$, for some N. Then S_n satisfies either (i) or (ii) of Theorem 3.2. But, $\Delta S_1 = 0$; therefore, S_n does not satisfy (ii) of Theorem 3.2. Thus, there exists N_0 such that $S_n > 0$, $\Delta S_n > 0$, $\Delta^2 S_n > 0$, $\Delta(P_n \Delta^2 S_n) > 0$, for all $n \geq N_0$. Suppose $M > N_0$, where M is an integer. Since $\{U_M^{m_k}\}$, $\{\Delta U_M^{m_k}\}$, $\{\Delta^2 U_M^{m_k}\}$ and $\{\Delta(P_M^{m_k} \Delta^2 U_M^{m_k})\}$ converge to S_M, ΔS_M, $\Delta^2 S_M$, and $\Delta(P_M \Delta^2 S_M)$, respectively, there exists a natural number N_1 such that $U_M^{m_k} > 0$, $\Delta U_M^{m_k} > 0$, $\Delta^2 U_M^{m_k} > 0$, $\Delta(P_M^{m_k} \Delta^2 U_M^{m_k}) > 0$, for all $m_k > N_1$. Hence, by Theorem 2.1, $U_n^{m_k} > 0$, $\Delta U_n^{m_k} > 0$, $\Delta^2 U_n^{m_k} > 0$, $\Delta(P_n^{m_k} \Delta^2 U_n^{m_k}) > 0$, for all $n > M$, $m_k > N_1$. But this is a contradiction since $U_{m_k}^{m_k} = 0$, for all m_k. Therefore, S_n is oscillatory. Similarly, T_n is oscillatory. This completes the proof of the theorem. □

4. Oscillation criteria. In addition to the standing conditions on P_n, Q_n and R_n, we assume for the remainder of this paper that $\sum^\infty 1/P_n = \infty$.

Theorem 4.1. *If Q_n is bounded and $\sum^\infty R_n = \infty$, then* (e) *is oscillatory.*

Proof. It is sufficient to establish that a nonoscillatory solution of (e) satisfies either (i) or (ii) of Theorem 3.2. Let Y_n be a nonoscillatory solution of (e) and assume $Y_n > 0$ for all n sufficiently large. Let n_0 be such that $Y_n \Delta Y_n \Delta^2 Y_n$ has no nodes on $[n_0, \infty)$. We show first that

$\Delta^2 Y_n > 0$ on $[n_0, \infty)$. Suppose it could happen that $\Delta^2 Y_n < 0$ on $[n_0, \infty)$. Then we would have $\Delta Y_n > 0$ on $[n_2, \infty)$. Let α be an upper bound of Q_n. Then summing from n_0 to n, we obtain

$$\sum_{n_0}^{n} Q_{m+1}\Delta^2 Y_{m+1} \geq \alpha \sum_{n_0}^{n} \Delta^2 Y_{m+1} = \alpha[\Delta Y_{n+2} - \Delta Y_{n_0}].$$

Since $\lim_{n\to\infty} \Delta Y_n$ exists, we conclude that $\sum^{\infty} Q_{m+1}\Delta^2 Y_{m+1}$ is finite. Summing (e) from n_0 to n and using the fact that Y_n is increasing on $[n_0, \infty)$, we have

$$\begin{aligned}\Delta(P_{n+1}\Delta^2 Y_{n+1}) - \Delta(P_{n_0}\Delta^2 Y_{n_0})& \\ &= \sum_{m=n_0}^{n} Q_{m+1}\Delta^2 Y_{m+1} + \sum_{m=n_0}^{n} R_{m+2}Y_{m+2} \\ &\geq \sum_{m=n_0}^{n} Q_{m+1}\Delta^2 Y_{m+1} + Y_{n_0+2}\sum_{m=n_0}^{n} R_{m+2}.\end{aligned}$$

Since $\sum^{\infty} R_{m+2} = \infty$, it follows that $\Delta(P_n\Delta^2 Y_n) \to \infty$ as $n \to \infty$. But this implies $P_n\Delta^2 Y_n \to \infty$, contradicting our assumption that $\Delta^2 Y_n < 0$. Thus, $\Delta^2 Y_n > 0$ on $[n_0, \infty)$. Since $Y_n > 0$ and $\Delta^2 Y_n > 0$, it follows from (e) that $\Delta^2(P_n\Delta^2 Y_n) > 0$. Thus, $\Delta(P_n\Delta^2 Y_n)$ is eventually of one sign. If $\Delta(P_n\Delta^2 Y_n) > 0$ on $[n_1, \infty)$, $n_1 \geq n_0$, then $P_n\Delta^2 Y_n \to \infty$ as $n \to \infty$ and, in particular, $P_n\Delta^2 Y_n \geq 1$ on $[n_2, \infty)$, $n_2 \geq n_1$. Therefore,

$$\Delta Y_{n+1} - \Delta Y_{n_2} = \sum_{m=n_2}^{n} \Delta^2 Y_m \geq \sum_{m=n_2}^{n} 1/P_m.$$

Letting n tend to ∞ we conclude $\lim_{n\to\infty} \Delta Y_n = \lim_{n\to\infty} Y_n = \infty$, implying that Y_n is strongly increasing. If $\Delta(P_n\Delta^2 Y_n) < 0$ on $[n_1, \infty)$, $n_1 \geq n_0$, then we claim that $\Delta Y_n < 0$ on $[n_0, \infty)$. For if $\Delta Y_n > 0$ on $[n_0, \infty)$, then Y is increasing on $[n_0, \infty)$ and from (e)

$$\Delta^2(P_n\Delta^2 Y_n) = Q_{n+1}\Delta^2 Y_{n+1} + R_{n+2}Y_{n+2} \geq Y_{n_0+2}R_{n+2}.$$

Summing this inequality, we obtain

$$\Delta(P_n\Delta^2 Y_n) - \Delta(P_{n_0}\Delta^2 Y_{n_0}) \geq Y_{n_0+2}\sum_{m=n_0}^{n-1} R_{m+2},$$

so that $\Delta(P_n\Delta^2 Y_n) \to \infty$ as $n \to \infty$, which contradicts $\Delta(P_n\Delta^2 Y_n) < 0$ on $[n_1, \infty)$. Thus, $\Delta(P_n\Delta^2 Y_n) < 0$ on $[n_1, \infty)$ implies that the inequalities $Y_n > 0$, $\Delta Y_n < 0$, $P_n\Delta^2 Y_n > 0$, $\Delta(P_n\Delta^2 Y_n) < 0$ hold on $[n_1, \infty)$. From Lemma 2.2, these same inequalities hold on $[0, n_1)$ and (ii) of Theorem 3.2 holds.

Theorem 4.2. *If* $R_{n+2} - 2Q_{n+1} > 0$ *and either*

$$\liminf_{n\to\infty} Q_n > 0$$

or

$$\liminf_{n\to\infty}(R_{n+2} - 2Q_{n+1}) > 0,$$

then (e) *is oscillatory.*

Proof. Let Y be a nonoscillatory solution of (e) and assume $Y_n > 0$ for all n sufficiently large. Note that (e) can be written as

$$\Delta^2(P_n\Delta^2 Y_n) = Q_{n+1}Y_{n+3} + (R_{n+2} - 2Q_{n+1})Y_{n+2} + Q_{n+1}Y_{n+1}.$$

Clearly, from our hypothesis, it follows for sufficiently large n, the product $Y_n\Delta Y_n\Delta^2 Y_n\Delta(P_n\Delta^2 Y_n)$ has no nodes. Since $\Delta^2(P_n\Delta^2 Y_n) > 0$, it is clear that $\Delta(P_n\Delta^2 Y_n)$ is increasing and is eventually of one sign. Suppose $\Delta(P_n\Delta^2 Y_n) < 0$ for large n, then $P_n\Delta^2 Y_n > 0$ for large n, for otherwise we arrive at a contradiction. Similarly, $\Delta Y_n < 0$ for large n and therefore Y_n is strongly decreasing. If $\Delta(P_n\Delta^2 Y_n) > 0$ for large n, then it follows easily that Y_n is strongly increasing using the methods of the previous theorem. □

In conclusion, we present some examples.

Example 1. The equation

$$\Delta^4 Y_n - (n+1)\Delta^2 Y_{n+1} - (20 + 4n)Y_{n+2} = 0$$

satisfies the conditions of Theorem 4.2 where $P_n = 1$, $Q_n = n$ and $R_n = 12 + 4n$. In fact, $Y_n = (-1)^n$ defines an oscillatory solution of this equation.

Example 2. The general solution of the equation

$$\Delta^4 Y_n - (1/4)\Delta^2 Y_{n+1} - (1/8)Y_{n+2} = 0$$

is defined by

$$Y_n = C_1 2^n + C_2(1/2)^n + C_2 \sin n\theta + C_4 \cos n\theta$$

where $\tan\theta = (15)^{1/2}/7$. Clearly, $Y_n = 2^n$ and $Y_n = (1/2)^n$ define strongly increasing and strongly decreasing solutions, respectively, of this difference equation. Note that the conditions of Theorem 4.2 are not satisfied; however, the conditions of Theorem 4.1 are satisfied.

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