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EXTENSIONS AND INTERNAL STRUCTURE

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ABSTRACT. This paper introduces a kind of grid which helps to organize the information about T_0-extensions of topological spaces. The kinds of extensions a space can support are closely controlled by its internal structure. For example, a T_2-space cannot have a *compact* T_2-extension unless it is also completely regular. But every extension induces a trace system of filters on the original space; moreover, these filter systems can be arranged into proximity classes. Thus, the proximities form a kind of x-axis, and each proximity class is a kind of y-axis.

Using this grid, we can "plot" the Stone-Čech, the Wallman, and the one-point compactification. They all turn out to have the same "height." In addition, we can identify new classes of compactifications. For example, each proximity class has a largest filter system, which consists of all the open filters which are in some sense compatible with the proximity. By taking only the filters which are minimal in some sense related to the proximity, we obtain a compactification which is highly separated. If the proximity is dense and separated, then this compactification is the unique T_2-compactification induced by the proximity. Of course, in such a case the original space must be T_2 and completely regular. This is just a sample of the kinds of results obtained. It is hoped that this idea of a grid will continue to shed light on the ways a space can be extended.

1. Filter systems and extensions. Let X be a T_0-space. An extension (e, Y) induces a system of open filters on X via the pullbacks of the neighborhood filters under e. This system is known as the *trace system* of (e, Y). It turns out that each system of open filters on X which includes all the neighborhood filters is the trace system for some T_0-extension. This leads to the following.

Definition 1.1. A *filter system* on a T_0-topological space X is a family of open filters which includes all the neighborhood filters.

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Remark 1.2. Filter systems are closely related to the nearness structures of Herrlich [**6**] and to the merotopic structures of Katětov [**8**]. In fact, a filter system can be thought of as the *dual* of a nearness structure on X. The open hulls of filters with duals in a given nearness constitute a filter system; conversely, the duals of the filters of a given filter system generate a nearness. (For a detailed discussion of the duality between grills and filters, see Thron [**13**].) This correspondence between filter systems and nearnesses is not in general $1-1$, since more than one filter system can generate the same nearness.

Similarly, there is a correspondence between filter systems and merotopic structures. Given a filter system Θ, we can obtain a merotopic structure by taking all collections $\mathcal{A}$ for which $[\mathcal{A}]$ contains some filter in Θ. By $[\mathcal{A}]$ is meant the set of all supersets of members of $\mathcal{A}$. Conversely, a merotopic structure Γ on X gives rise to a filter system if we take the open hulls of the filters in Γ.

Finally, every filter system is the trace system of an extension of X. The construction of one such extension follows.

Construction 1.3. *Let Θ be a filter system on a T_0-space X. Let Y denote the set of filters in Θ. The map $e : X \to Y$ is defined by*

$$e(x) = \mathbf{N}_x, \qquad \textit{the neighborhood filter at } x.$$

For $A \subseteq X$, we define $\hat{A} = \{\mathcal{F} \in \Theta : A \in \mathcal{F}\}$. $\mathcal{T}^\wedge$ is generated by $\{\hat{G}$: G is an open subset of $X\}$. Finally, $\kappa_\Theta = (e, (Y, \mathcal{T}^\wedge))$.

This construction yields a T_0-extension κ_Θ with trace system Θ. Moreover, $\mathcal{T}^\wedge$ is the smallest topology on Θ with this property. In Banaschewski and Maranda [**2**] and again in Banaschewski [**1**] there is a concise but thorough development of the properties of κ_Θ, called by them the *strict* extension with trace system Θ. Basing his work on that of Kowalsky [**9**], Thron gives a clear and detailed account in [**12**, Chapter 17]. Thron refers to this extension as the *principal* extension belonging to Θ. Bowing to long-established usage, we will refer to this extension as the *strict* extension determined by Θ.

Theorem 1.4. *If Θ is a filter system on a T_0-space X, then κ_Θ is a T_0-extension of X with trace system Θ.*

Proof. See Thron [**12**, Theorem 17.4 and Remark 17.4]. □

This indicates that T_0-extensions of X can be studied using the internal structure of X; namely, its filter systems. We will now consider some properties of the filter system which guarantee desirable properties of the corresponding extension.

Theorem 1.5. *Let Θ be a filter system on a T_0-space X.*

(1) *κ_Θ is T_1 if and only if each filter in Θ is minimal in Θ.*

(2) *κ_Θ is T_2 if and only if distinct members of Θ are disjoint*; *i.e., they have no proper upper bound.*

Proof. For any filter $\mathcal{F}$ in Θ, let $\mathbf{N}_\mathcal{F}$ denote the set of $\mathcal{T}^\wedge$-neighborhoods of $\mathcal{F}$. Note that for $\mathcal{F}$ and $\mathcal{G}$ in Θ we have that $\mathcal{F} \preceq \mathcal{G}$ if and only if $\mathbf{N}_\mathcal{F} \preceq \mathbf{N}_\mathcal{G}$.

To establish (2), note that if $\mathcal{F} \in \Theta$ and $\mathcal{U}$ is an ultrafilter, then $\mathcal{F} \preceq \mathcal{U}$ if and only if $e(\mathcal{U}) \to \mathcal{F}$. Recall that a space is T_2 if and only if each ultrafilter has a unique limit. Thus κ_Θ is T_2 if and only if filters in Θ are disjoint. □

Since we intend to treat compactifications in this paper, we would like to obtain a condition on Θ which guarantees that κ_Θ is compact. Intuitively, a filter system determines a compact extension if it is closed in some sense; that is, if it "includes" all "nearby" filters.

Definition 1.6. A filter $\mathcal{F}$ is *close* to a filter system Θ if and only if every finite family of open subsets of x which meets every filter in Θ also meets $\mathcal{F}$. Let $[\Theta]$ denote the set of all filters which contain a member of Θ. We say that Θ is *contigual* if and only if $[\Theta]$ includes every filter which is close to Θ.

This definition of contigual was designed to be the dual of the corresponding term in the context of nearness spaces. (See Remark 1.2.) And, in fact, it is easy to check that a filter system is contigual if and only if its dual nearness is contigual.

The definition of *close* given here has an interesting relationship to the definition of a *micromeric* filter given in Bentley and Herrlich [**3**, page 145]. To see the relationship, we need the concept of the *contigual hull* of a nearness, developed in Reed [**10**, Proposition 1.22]. Every nearness has a kind of contigual closure, which is the smallest contigual nearness which contains the given one. It turns out that a filter which is *close* to a filter system Θ has its dual in the *contigual hull* of the dual nearness, whereas a *micromeric* filter has its dual in the dual nearness itself. If the filter system Θ is contigual, then a filter is close to Θ if and only if it is micromeric with respect to Θ.

Theorem 1.7 *Let Θ be a filter system on X. Then Θ is contigual if and only if κ_Θ is compact.*

Proof. ($\Rightarrow$). Suppose Θ is contigual. Let $\mathcal{U}$ be an ultrafilter on Y_Θ. We wish to show that $\mathcal{U}$ converges. Set

$$\mathcal{F} = \{A : \hat{A} \in \mathcal{U}\}.$$

It is straightforward to check that $\mathcal{F}$ is a filter on X. Notice that the operation $A \mapsto \hat{A}$ is preserved under intersections and containment.

(1) The filter $\mathcal{F}$ is close to Θ. Let $\mathcal{A}$ be a finite family of open subsets of X. Suppose $\mathcal{A} \cap \mathcal{F} = \varnothing$. We wish to find a filter $\mathcal{G}$ in Θ such that $\mathcal{A} \cap \mathcal{G} = \varnothing$.

For $A \in \mathcal{A}$, we have $A \notin \mathcal{F}$ and so $\hat{A} \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, $Y \backslash \hat{A} \in \mathcal{U}$. But $\mathcal{A}$ is finite, so $\cap\{Y\backslash\hat{A} : A \in \mathcal{A}\} \neq \varnothing$. Let $\mathcal{G}$ be a member. Then $\mathcal{A} \cap \mathcal{G} = \varnothing$.

(2) Since Θ is contigual, we may choose $\mathcal{F}^*$ in Θ such that $\mathcal{F} \succeq \mathcal{F}^*$. We claim that $\mathcal{U} \to \mathcal{F}^*$. Clearly, if G is an open set in $\mathcal{F}^*$, then $G \in \mathcal{F}$ and so, by definition of $\mathcal{F}$, we have that $\hat{G} \in \mathcal{U}$. Clearly then $\mathcal{U} \to \mathcal{F}^*$ as desired.

($\Leftarrow$). Now suppose that κ_Θ is compact. Let $\mathcal{F}$ be a filter close to Θ. We need to show that $\mathcal{F}$ contains a filter in Θ. Let

$$\mathcal{S} = \{Y\backslash\hat{G} : G \text{ is open and } G \notin \mathcal{F}\}.$$

(1) The family $\mathcal{S}$ has the finite intersection property. Let $\mathcal{A}$ be a finite family of open sets with $\mathcal{A} \cap \mathcal{F} = \varnothing$. Since $\mathcal{F}$ is close to Θ, there

must be some $\mathcal{G}$ in Θ such that $\mathcal{A} \cap \mathcal{G} = \varnothing$. Clearly, then for $G \in \mathcal{A}$ we have $G \notin \mathcal{G}$, and so $\mathcal{G} \notin \hat{G}$. Thus we have $\mathcal{G} \in \cap\{Y \backslash \hat{G} : G \in \mathcal{A}\}$.

(2) Since κ_Θ is compact, we may conclude that $\cap \mathcal{S}$ is nonempty. Let $\mathcal{F}^*$ be a member. We claim that $\mathcal{F} \succeq \mathcal{F}^*$. Let G be open and suppose that $G \notin \mathcal{F}$. Then $Y \backslash \hat{G} \in \mathcal{S}$, and so $\mathcal{F}^* \in Y \backslash \hat{G}$. Thus $G \notin \mathcal{F}^*$. This establishes that $\mathcal{F} \succeq \mathcal{F}^*$. □

2. Proximity classes of filter systems. We have just seen that T_0-extensions of X can be analyzed by using filter systems on X. Each filter system induces a kind of proximity on the subsets of X. Using this proximity relation, we can group the filter systems into proximity classes, and thus obtain a grid for the filter systems. The x-axis of the grid consists of the proximity relations, and each proximity class of filter systems is a kind of y-axis.

Each proximity class has a largest filter system Θ_L, which is trivially contigual. A more separated system, Θ_G, can be obtained by using only filters which are minimal in some sense related to the proximity class. In case the proximity is an EF-proximity, Θ_G turns out to be the trace system for the T_2-compactification associated with the proximity. Finally, a system Θ_W can be obtained by using members of Θ_G which are contained in ultra closed filters. This system gives the y-coordinate for the Wallman, the Stone-Čech, and the Alexandroff compactifications. In what follows, let Θ be a filter system on a T_0-space X.

Definition 2.1. Let Φ denote the family of filters $\mathcal{F}$ on X such that $\mathcal{F} \to x$ whenever $\mathcal{F} \preceq \dot{x}$, the ultrafilter generated by $\{x\}$. Recall that the *dual* of a filter consists of all the sets whose complements are not in the filter. For A and B subsets of X, we define

$$A \, \pi_\Theta \, B \quad \text{if and only if} \quad A \cap \operatorname{cl} B \neq \varnothing, \quad \text{or}$$

$$\text{there exists an} \quad \mathcal{F} \in \Theta \cap \Phi \quad \text{with} \quad A, B \in d\mathcal{F}.$$

The relation π_Θ is not quite a proximity, since it may not be symmetric. It satisfies a weakened Lodato condition, which will be stated below.

Proposition 2.2. *Let A, B, and C be subsets of X, and let $x \in X$. Let π denote π_Θ. Then the following conditions hold.*

(P1) $\varnothing \not\pi A \not\pi \varnothing$.

(P2) $A \,\pi\, B \subseteq C \Rightarrow A \,\pi\, C$.

(P3) $C \supseteq B \,\pi\, A \Rightarrow C \,\pi\, A$.

(P4) $A \,\pi\, (B \cup C) \Rightarrow A \,\pi\, B$ or $A \,\pi\, C$.

(P5) $(B \cup C) \,\pi\, A \Rightarrow B \,\pi\, A$ or $C \,\pi\, A$.

(P6) $A \cap B \neq \varnothing \Rightarrow A \,\pi\, B$.

(P7) $A \,\pi\, \mathrm{cl}\, B \Rightarrow A \,\pi\, B$.

(P8) $x \in \mathrm{cl}\, A$ if and only if $\{x\} \pi\, A$.

Proof. The first five properties follow easily from the properties of ultra filters and grills. Recall that the dual of a filter is a grill, and that a grill is a union of ultra filters. Note also that if $\mathcal{U}$ is an ultrafilter with $A \cup B \in \mathcal{U}$, then either $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

Clearly, from the definition of π we have that if $A \cap \mathrm{cl}\, B \neq \varnothing$ then $A \,\pi\, B$. This establishes (P6) and parts of (P7) and (P8). Now suppose that $A \,\pi\, \mathrm{cl}\, B$ and $A \cap \mathrm{cl}\, B = \varnothing$. Let $\mathcal{F}$ be a filter in $\Phi \cap \Theta$ such that A and $\mathrm{cl}\, B$ are in $d\mathcal{F}$. Since $\mathcal{F}$ is open, B is also in $d\mathcal{F}$, and so $A \,\pi\, B$.

Finally, suppose $\{x\}\pi A$. We wish to show that $x \in \mathrm{cl}\, A$. Suppose that $\mathcal{F} \in \Phi \cap \Theta$ and $\{x\}, A \in d\mathcal{F}$. Since $\{x\} \in d\mathcal{F}$ we have $\mathcal{F} \preceq \dot{x}$. But $\mathcal{F} \in \Phi$ and so $\mathcal{F} \to x$. But then since $A \in d\mathcal{F}$, we have $X \backslash A \notin \mathbf{N}_x$. From this it follows that $x \in \mathrm{cl}\, A$. □

Definition 2.3. A relation π on the subsets of a T_0-space X will be called a *quasi-Lodato proximity* on X if and only if it satisfies the conditions of Proposition 2.2.

Example 2.4. Not every quasi-Lodato proximity on X supports a filter system. For example, let X be the reals with the usual topology. Define $A \,\pi\, B$ if and only if $A \cap \mathrm{cl}\, B \neq \varnothing$. It is easy to check that π is a quasi-Lodato proximity on X. We claim that π is not obtained from any filter system. Let Θ be a filter system on X. Choose A and B so that $A \cap \mathrm{cl}\, B = \varnothing$ and $\mathrm{cl}\, A \cap \mathrm{cl}\, B \neq \varnothing$. Note that $A \not\pi B$. We claim that $A \,\pi_\Theta\, B$. Choose $x \in \mathrm{cl}\, A \cap \mathrm{cl}\, B$. Then A and B are in the dual of $\mathbf{N}_x$. Note that since X is a T_1-space, we have $\mathbf{N}_x \in \Phi$. Thus $A \,\pi_\Theta\, B$.

Definition 2.5. Let π be a quasi-Lodato proximity on X. We say π is *admissible* if and only if there is a filter system Θ such that $\pi = \pi_\Theta$. The *proximity class* of π consists of all filter systems Θ such that $\pi = \pi_\Theta$. In case X is T_1, then every neighborhood filter is in $\Theta \cap \Phi$. This forces π_Θ to be symmetric. Thus an admissible proximity on a T_1-space is a generalized proximity in the sense of Bentley and Herrlich [**3**].

Next we will construct several filter systems which are good candidates for a given proximity class. Under the proper conditions, all these systems will in fact turn out to be members of the class.

Definition 2.6. A filter $\mathcal{F}$ is a π-*filter* if and only if for A and B in $d\mathcal{F}$ we have $A \,\pi\, B$. We remark that a π-filter is simply the dual of a π-clan in the sense of Gagrat and Thron [**5**]. Moreover, minimal π-filters are the duals of maximal π-clans.

Construction 2.7. *Let π be a quasi-Lodato proximity on X. Let* $\mathbf{N}$ *denote the set of all neighborhood filters. We define*

$$\Theta_L(\pi) = \{\mathcal{F} : \mathcal{F} \textit{ is open and } \mathcal{F} \textit{ is a } \pi\textit{-filter or there exists an } x \textit{ with } \mathcal{F} \preceq \mathbf{N}_x\}.$$

$$\Theta_G(\pi) = \{\mathcal{F} : \mathcal{F} \textit{ is a minimal open } \pi\textit{-filter}\} \cup \mathbf{N}.$$

$$\Theta_W(\pi) = \{\mathcal{F} : \mathcal{F} \in \Theta_G(\pi) \textit{ and } \exists \mathcal{U} \textit{ ultraclosed with } \mathcal{F} \preceq \mathcal{U}\} \cup \mathbf{N}.$$

$$\Theta_S(\pi)\{\mathcal{U}^i \cap \mathcal{V}^i : \mathcal{U} \textit{ and } \mathcal{V} \textit{ are ultrafilters and } \mathcal{U}^i \cap \mathcal{V}^i \textit{ is a } \pi\textit{-filter}\} \cup \mathbf{N}.$$

Let π_L, π_G, π_W, and π_S denote the respective proximities. We note that minimal open π-filters are the same as minimal π-filters. This follows essentially from (P7) and (P8). Moreover, the dual of Θ_G is the nearness ν_G generated by all the π-clans. It is the trace nearness of the extension of Gagrat and Naimpally [**4**, Theorem 4.4].

Lemma 2.8. *Let π be a quasi-Lodato proximity on X, and let Θ be any filter system on X.*

(1) *Every neighborhood filter which is a π-filter is a* minimal π-filter.

(2) *Every open π-filter is in Φ.*

(3) *If $\pi_\Theta \subseteq \pi$, then every filter in $\Theta \cap \Phi$ is a π-filter.*

(4) *If π is admissible and $\mathbf{N}_x \in \Phi$, then $\mathbf{N}_x$ is a π-filter.*

(5) *Every open π-filter contains a minimal open π-filter.*

Proof. (1) Suppose $\mathbf{N}_x$ is a π-filter, and there is a π-filter $\mathcal{F}$ such that $\mathcal{F} \preceq \mathbf{N}_x$. We will show that $\mathcal{F} \to x$, so that in fact $\mathcal{F} = \mathbf{N}_x$. Let G be an open set with $x \in G$. Then $\{x\} \not\pi (X \backslash G)$. Now, $X\backslash\{x\} \notin \mathcal{F}$, since $\mathcal{F} \preceq \mathbf{N}_x$. Hence $\{x\} \in d\mathcal{F}$. Since $\mathcal{F}$ is a π-filter, $X\backslash G \notin d\mathcal{F}$, and so $G \in \mathcal{F}$. Thus we have $\mathcal{F} \to x$ as desired.

(2) Now let $\mathcal{G}$ be any open π-filter. We wish to show that $\mathcal{G} \in \Phi$. Suppose $\mathcal{G} \preceq \dot{x}$ for some x. Then since $\mathcal{G}$ is open, we have $\mathcal{G} \preceq \mathbf{N}_x$. Thus, $\mathbf{N}_x$ contains a π-filter, and so it must already itself be a π-filter. But we have just seen that every neighborhood π-filter is a *minimal* π-filter. Thus, $\mathcal{G} = \mathbf{N}_x$. This establishes that $\mathcal{G} \in \Phi$, as desired.

(3) Now suppose that $\pi_\Theta \subseteq \pi$, and let $\mathcal{F}$ be any filter in $\Theta \cap \Phi$. Clearly then $\mathcal{F}$ is a π_Θ-filter, and hence a π-filter.

(4) If π is admissible and $\mathbf{N}_x \in \Phi$, then we can apply the preceding result to conclude that $\mathbf{N}_x$ is a π-filter. Note that $\mathbf{N}_x$ is in every filter system in the proximity class of π.

(5) Let $\mathcal{G}$ be an open π-filter. Set

$$\mathcal{A} = \{\mathcal{H} : \mathcal{H} \text{ is an open } \pi\text{-filter and } \mathcal{H} \preceq \mathcal{G}\}.$$

Observe that the intersection of any chain of open π-filters is itself an open π-filter. Thus by Zorn's lemma we may conclude that $\mathcal{A}$ has a minimal element $\mathcal{M}$. It is easy to see that $\mathcal{M}$ must be a minimal open π-filter. □

Theorem 2.9. *Let π be an admissible proximity on X. Then Θ_S, Θ_G, and Θ_L are in the proximity class of π. Moreover, $\pi_W \subseteq \pi$.*

Proof. (1) Let $\Theta_1 \in \{\Theta_L, \Theta_G, \Theta_W, \Theta_S\}$, and let π_1 be the proximity class of Θ_1. We claim that $\pi_1 \subseteq \pi$. Suppose $A\, \pi_1\, B$. If $A \cap \operatorname{cl} B \neq \varnothing$, then since π is a quasi-Lodato proximity we have $A\, \pi\, B$. Now suppose that there exists an $\mathcal{F} \in \Theta_1 \cap \Phi$ with $A, B \in d\mathcal{F}$. We claim that $\mathcal{F}$ is a π-filter, so that $A \,\pi\, B$. The only difficulty occurs when $\mathcal{F} \in \mathbf{N}$. Suppose that $\mathcal{F} \preceq \mathbf{N}_x$ for some x. Since $\mathcal{F} \in \Phi$ we have $\mathcal{F} = \mathbf{N}_x$. But since π is admissible we can apply the fourth part of Lemma 2.8 to conclude that $\mathbf{N}_x$ is a π-filter, and hence $A \,\pi\, B$.

(2) We will show that $\pi \subseteq \pi_G \cap \pi_S$. Let $A \,\pi\, B$. Let Θ be any filter system in the proximity class of π. Then $A \,\pi_\Theta\, B$. If $A \cap \operatorname{cl} B \neq \varnothing$, then clearly $A \,\pi_G\, B$, and $A \,\pi_S\, B$.

Suppose now that A and B are in $d\mathcal{F}$ for some $\mathcal{F} \in \Theta \cap \Phi$. Then, by the third part of Lemma 2.8, we have that $\mathcal{F}$ is a π-filter. Again by Lemma 2.8, $\mathcal{F}$ must contain a minimal open π-filter $\mathcal{G}$. Then A and B are in $d\mathcal{G}$ and $\mathcal{G} \in \Theta_G$. By Lemma 2.8, $\mathcal{G}$ is also in Φ, since $\mathcal{G}$ is an open π-filter. Hence $A \,\pi_G\, B$.

To see that $A \,\pi_S\, B$, note that there exist ultra filters $\mathcal{U}$ and $\mathcal{V}$ such that $A \in \mathcal{U}$, $B \in \mathcal{V}$, and $\mathcal{F} \preceq \mathcal{U} \cap \mathcal{V}$. This follows from the fact that A and B are in $d\mathcal{F}$. Let $\mathcal{G} = \mathcal{U}^i \cap \mathcal{V}^i$. Note that $\mathcal{F} \preceq G$, so that $\mathcal{G}$ is a π-filter. By Lemma 2.8, $\mathcal{G} \in \Phi$. Note also that A and B are in $d\mathcal{G}$ by its construction. Since $\mathcal{G} \in \Theta_S \cap \Phi$, then we have $A \,\pi_S\, B$.

(3) We claim now that $\pi_S = \pi_G = \pi_L = \pi$. From (1) and (2) we have that $\pi_S = \pi_G = \pi$. Note that $\pi_G \subseteq \pi_L \subseteq \pi$, and so $\pi_L = \pi$ as well. □

The system $\Theta_L(\pi)$ is trivially contigual, since $\{X\}$ is a member. In the corresponding extension, all filters converge to the same point; namely, $\{X\}$. They may have other points of convergence as well. In order to reduce the number of duplicates and obtain as much separation of points as possible in the extension, we will restrict the size of Θ.

Definition 2.10. A filter system Θ on X is *pruned* if and only if the only filters in Θ which are not in Φ are the neighborhood filters. A quasi-proximity π on X is *contigual* if and only if it has a pruned contigual filter system in its proximity class.

Example 2.11. Not every admissible proximity is contigual. Let Z be the integers, with the right-hand ray topology. Set

$$G_m = \{n \in Z : n \geq m\}.$$

The open sets of Z consist of the G_m's, together with Z and $\varnothing$. Let π be the proximity defined by $A\,\pi\,B$ if and only if $A \cap \operatorname{cl} B \neq \varnothing$. It is easy to check that π is a quasi-Lodato proximity on Z. We maintain that *every* filter system is in the proximity class of π.

Notice that for $x > y$ we have $\mathbf{N}_y \preceq \dot{x}$. Hence no neighborhood filter can be in Φ. Notice that each set in an open filter $\mathcal{F}$ must contain a set of the form G_m, for some m. From this it follows easily that if $A, B \in d\mathcal{F}$ then $A \cap \operatorname{cl} B \neq \varnothing$. Hence for *any* filter system Θ we have $\pi_\Theta = \pi$. Thus π is admissible. We claim that π is not contigual.

We will show that the filter $\{Z\}$ is close to $\mathbf{N}$, and hence close to any filter system on Z. Suppose that $\mathcal{A}$ is a finite family of open sets and $\mathcal{A} \cap \{Z\} = \varnothing$. Then each nonempty set in $\mathcal{A}$ must be of the form G_m, for some m. Since $\mathcal{A}$ is finite there is an integer b such that $b < m$ for G_m in $\mathcal{A}$. Clearly $\mathbf{N}_b \cap \mathcal{A}$ is empty.

Now $\{Z\}$ cannot contain any member of any pruned filter system on Z. For clearly $\{Z\} \notin \Phi$, and $\{Z\}$ does not converge. From this it follows that there are no pruned contigual filter systems at all on Z, and so π cannot be contigual.

Theorem 2.12. *Let π be an admissible proximity on X. Then π is contigual if and only if $\Theta_G(\pi)$ is contigual.*

Proof. Let Θ_G denote $\Theta_G(\pi)$.

($\Leftarrow$). Suppose Θ_G is contigual. Using Theorem 2.9, we see that Θ_G is a contigual filter system in the proximity class of π. To see that Θ_G is pruned, let $\mathcal{F}$ be in Θ_G and suppose that $\mathcal{F} \notin \mathbf{N}$. Then clearly $\mathcal{F}$ is an open π-filter. By Lemma 2.8 then $\mathcal{F} \in \Phi$.

($\Rightarrow$). Suppose π is contigual. Let Θ be a pruned contigual filter system such that $\pi_\Theta = \pi$. We need to show that Θ_G is contigual. Suppose $\mathcal{F}$ is close to Θ_G and $\mathcal{F}$ does not converge. We wish to show that $\mathcal{F}^i$ is a π-filter. Then, using Lemma 2.8, we may conclude that $\mathcal{F}$ contains a member of Θ_G.

Let A and B be members of $d\mathcal{F}^i$. We need to show that $A\,\pi\,B$. Note that $\operatorname{cl} A$ and $\operatorname{cl} B$ are both in $d\mathcal{F}$. Now set

$$\mathcal{C} = \{G : G \text{ is open and } G \notin \mathcal{F}\}.$$

Since $\mathcal{F}$ does not converge, $\mathcal{C}$ is an open cover of X. Let C^c denote the set of complements of the sets in $\mathcal{C}$.

Case I. Suppose that $\mathcal{C}^c \cup \{\mathrm{cl}\, A, \mathrm{cl}\, B\}$ does not have the finite intersection property. Then there is a finite subset $\mathcal{D}$ of $\mathcal{C}$ such that $\mathcal{D}$ covers $\mathrm{cl}\, A \cap \mathrm{cl}\, B$. Set

$$\mathcal{A} = \mathcal{D} \cup \{X \backslash \mathrm{cl}\, A, X \backslash \mathrm{cl}\, B\}.$$

Then $\mathcal{A} \cap \mathcal{F} = \varnothing$. Since $\mathcal{F}$ is close to Θ_G we may choose $\mathcal{H} \in \Theta_G$ such that $\mathcal{A} \cap \mathcal{H} = \varnothing$. We claim that $\mathcal{H}$ is a π-filter, for which $A, B \in d\mathcal{H}$.

First, we will establish that $\mathcal{H} \notin \mathbf{N}$. Choose $x \in X$. Clearly if $x \notin \mathrm{cl}\, A \cap \mathrm{cl}\, B$, then $X \backslash \mathrm{cl}\, A$ or $X \backslash \mathrm{cl}\, B$ is a neighborhood of x which is not in $\mathcal{H}$. And if $x \in \mathrm{cl}\, A \cap \mathrm{cl}\, B$, then x is in some member U of $\mathcal{D}$. Then U is a neighborhood of x not in $\mathcal{H}$. Thus, $\mathcal{H} \notin \mathbf{N}$, and so $\mathcal{H}$ must be a π-filter. Since $\mathcal{H}$ is open, with $\mathrm{cl}\, A$, $\mathrm{cl}\, B$ in $d\mathcal{H}$, we have $A, B \in d\mathcal{H}$, and so $A\, \pi\, B$, as desired.

Case II. Suppose that $\mathcal{C}^c \cup \{\mathrm{cl}\, A, \mathrm{cl}\, B\}$ has the finite intersection property. Let $\mathcal{U}$ be an ultrafilter containing this family of sets. We claim that $\mathcal{U}$ is close to $\mathbf{N}$.

Suppose $\mathcal{A}$ is a finite family of open sets, and $\mathcal{A} \cap \mathcal{U} = \varnothing$. Then since $\mathcal{U}$ is an ultrafilter, $\mathcal{A}^c \subseteq \mathcal{U}$, and so $\cap \mathcal{A}^c \neq \varnothing$. Let x be a member. Then $\mathbf{N}_x$ is a neighborhood filter which misses $\mathcal{A}$. This establishes that $\mathcal{U}$ is close to $\mathbf{N}$. Clearly then $\mathcal{U}$ is close to Θ, since $\mathbf{N} \subseteq \Theta$. But Θ is contigual, so $\mathcal{U}$ must contain a filter $\mathcal{H}$ in Θ. We claim that $\mathcal{H}$ is a π-filter and A and B are in $d\mathcal{H}$.

Note first that $\mathcal{U}$ is nonconvergent, since $\mathcal{F}$ does not converge. Thus $\mathcal{H}$ is also nonconvergent. Now recall that $\mathcal{H}$ is a member of a pruned filter system. Thus, $\mathcal{H}$ must be a member of Φ. Then $\mathcal{H} \in \Theta \cap \Phi$ and so by Lemma 2.8, $\mathcal{H}$ is a π-filter.

Finally, recall that $\mathrm{cl}\, A$ and $\mathrm{cl}\, B$ are in $\mathcal{U}$. Thus $\mathrm{cl}\, A$ and $\mathrm{cl}\, B$ are in $d\mathcal{H}$. Since $\mathcal{H}$ is open, we have $A, B \in d\mathcal{H}$. Thus again $A\, \pi\, B$. This establishes $\mathcal{F}^i$ as a π-filter and completes the proof that Θ_G is contigual. $\square$

Now let us consider the Wallman systems $\Theta_W(\pi)$. These turn out to be the trace systems for some very familiar compactifications. However, to guarantee that Θ_W is in the proximity class of π we need a restriction on π.

Definition 2.13. A *Wallman* filter is a filter which is contained in some ultraclosed filter. The *Wallman system* of π is defined to be $\Theta_W(\pi)$. Note that the Wallman system of π consists of the minimal Wallman π-filters along with the neighborhood filters. A proximity π is *covered* if and only if whenever $A\,\pi\,B$ and $A \cap \operatorname{cl} B = \varnothing$ then A and B are in the dual of some open π-filter $\mathcal{F}$ such that $\mathcal{F}$ is either a Wallman or a neighborhood filter.

Theorem 2.14. *An admissible proximity π includes its Wallman system if and only if π is covered.*

Proof. Let Θ_W denote $\Theta_W(\pi)$.

($\Leftarrow$). Suppose π is covered. Recall that, by Theorem 2.9, we have $\pi_W \subseteq \pi$. Now suppose that $A\,\pi\,B$. We need to show that $A\,\pi_W\,B$. If we have $A \cap \operatorname{cl} B \neq \varnothing$, then clearly $A\,\pi_W\,B$. Now suppose that $A \cap \operatorname{cl} B = \varnothing$. Then since π is covered, there is an open π-filter $\mathcal{F}$ such that $A, B \in d\mathcal{F}$, and $\mathcal{F}$ is either a Wallman or a neighborhood filter. By Lemma 2.8, we have that $\mathcal{F} \in \Phi$. If $\mathcal{F}$ is a neighborhood filter, then clearly $\mathcal{F} \in \Theta_W \cap \Phi$ and so $A\,\pi_W\,B$. If $\mathcal{F}$ is a Wallman π-filter, then, by Lemma 2.8, $\mathcal{F}$ must contain a minimal π-filter $\mathcal{G}$ which is open. Clearly, then, $\mathcal{G}$ is in Θ_W. Since $\mathcal{G}$ is an open π-filter, we have $\mathcal{G} \in \Phi$, by Lemma 2.8. Thus $\mathcal{G} \in \Theta_W \cap \Phi$. Clearly, then, G is a π_W-filter, and hence $A\,\pi_W\,B$.

($\Rightarrow$). Conversely, suppose $\pi_W = \pi$. Let $A\,\pi\,B$, and suppose that $A \cap \operatorname{cl} B = \varnothing$. Since $A\,\pi_W\,B$ there is a filter $\mathcal{F}$ in $\Theta_W \cap \Phi$ such that A and B are in $d\mathcal{F}$. Clearly, then, by Lemma 2.8, $\mathcal{F}$ is an open π-filter. But by the definition of Θ_W, $\mathcal{F}$ is either a Wallman or a neighborhood filter. Hence π is covered. $\square$

In the last part of this section we will show that the Wallman, the Stone-Čech and the Alexandroff compactifications all have trace systems of the form $\Theta_W(\pi)$ for a suitably chosen Lodato proximity π. From now on, we will assume that X is a T_1-space. This last part relies heavily on the notation and results of Reed [**10**] and [**11**].

Theorem 2.15. *Let $d\Theta$ denote the set of duals of filters in a T_1-filter system Θ on X. Then*

(1) *$d\Theta$ generates a nearness ν on X.*

(2) *The clusters of ν are the members of $d\Theta$.*

(3) *κ_ν is equivalent to κ_Θ.*

Proof. Note that for $x \in X$ we have $\dot{x} \preceq d\mathbf{N}_x$. From this it is easy to check that $d\Theta$ generates a nearness which is compatible with the topology on X. Since Θ is T_1, it follows easily from Theorem 1.5 that every member of $d\Theta$ is a cluster of ν. Conversely, since ν is grill-generated, then every cluster must be in $d\Theta$. It is straightforward to check that the map $\mathcal{F} \mapsto d\mathcal{F}$ defines an equivalence between the two extensions. □

Notation 2.16. Let π_W denote the "Wallman" proximity on X:

$$A \, \pi_W \, B \quad \text{if and only if} \quad \operatorname{cl} A \cap \operatorname{cl} B \neq \varnothing.$$

Let π_A denote the "Alexandroff" proximity:

$A \, \pi_A \, B$ if and only if $\operatorname{cl} A \cap \operatorname{cl} B \neq \varnothing$ or $\operatorname{cl} A$ and $\operatorname{cl} B$ are noncompact.

Theorem 2.17. *Let π be a Lodato proximity on X.*

(1) *If $\pi = \pi_W$, then $\Theta_W(\pi)$ is the trace system for the Wallman compactification of X.*

(2) *If $\pi = \pi_A$, then $\Theta_W(\pi)$ is the trace system of the Alexandroff compactification of X.*

(3) *If π is an Efremovich proximity, then $\Theta_W(\pi)$ is the trace system for the T_2-compactification associated with π.*

Proof. In Reed [**11**] the concept of a Wallman nearness was introduced. Given a Lodato proximity, π, the nearness $\nu_W(\pi)$ is defined to be the nearness generated by all the Wallman π-clans; i.e., those π-clans which contain ultraclosed filters. It is easy to check that the clusters of $\nu_W(\pi)$ are simply the duals of the filters in $\Theta_W(\pi)$. Thus the corresponding extensions are equivalent, by the preceding theorem.

In Reed [**11**, Theorem 2.18] it was shown that the extension obtained from $\nu_W(\pi_W)$ is the usual Wallman compactification of X; hence, its trace system must be $\Theta_W(\pi_W)$. Similarly, in the same theorem it was shown that the extension obtained from $\nu_W(\pi_A)$ is the Alexandroff compactification of X; its trace system is therefore $\Theta_W(\pi_A)$. Finally, if π is an EF-proximity, then the extension obtained from $\nu_W(\pi)$ is a T_2-compactification; its trace system is $\Theta_W(\pi)$. □

This theorem locates the Wallman, the Alexandroff, and the T_2-compactifications at the same "height" in the proximity class. We have now seen that if the proximity class is sufficiently nice it contains a largest member Θ_L, a "highly separated" member Θ_G, a "Wallman" member Θ_W, and a very small member Θ_S. Hopefully, this provides a useful way to organize information about extensions.

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