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FINITE GROUPS ACTING ON BORDERED SURFACES AND THE REAL GENUS OF A GROUP

COY L. MAY

1. Introduction. Group actions on compact bordered surfaces have recently received considerable attention. The surface has often carried a dianalytic structure and been considered a Klein surface or, equivalently, a real algebraic curve with real points. The new monograph [**4**] is an excellent general reference for the work on bordered Klein surfaces.

Many results have concerned the size and structure of the automorphism group of a bordered surface of algebraic genus greater than one. Let X be a compact bordered Klein surface of algebraic genus $g \geq 2$. Then the automorphism group G of X is finite, and the order of G is at most $12(g-1)$ [**16**]. If the order of G is the largest possible, then G is called an *M^*-group*. M^*-groups have received special attention. There is certainly a wealth of these groups and, in fact, many important finite groups are M^*-groups.

The general upper bound for the size of the automorphism group can be improved, of course, in special cases. For example, if the group G is nilpotent, then its order is at most $8(g-1)$ [**19**]. Cyclic groups acting on bordered surfaces were studied in [**18, 6, 4**]. Abelian groups were considered briefly in [**4**].

Other research has concentrated on group actions on bordered surfaces of a fixed algebraic or topological genus [**5, 3, 2**]. In some of the older research [**12, 13**], only orientable surfaces and orientation-preserving automorphisms were considered.

Many of these results, however, hold for arbitrary topological group actions on bordered surfaces, not just actions by dianalytic groups. Perhaps the language of Klein surfaces has obscured the generality of

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these results. In fact, Tucker's similar remarks about the symmetric genus and Riemann surfaces [**24**] provided some motivation for our work here.

In connection with group actions on bordered surfaces, then, there is a natural parameter associated with each finite group. A finite group G can be represented as a group of automorphisms of a compact bordered Klein surface, that is, G acts on a bordered surface. We define the *real genus* $\rho(G)$ to be the minimum algebraic genus of any bordered surface on which G acts. There are, of course, other genus parameters for the group G. Two of the most important are the graph-theoretical genus [**25**] and the symmetric genus [**24**]. We use the adjective *real* for our parameter because of the important correspondence between compact Klein surfaces and real algebraic curves; the bordered surfaces correspond to curves with real points.

We initiate our study of the real genus of a group by applying the work done on automorphism groups of compact Klein surfaces. This yields upper and lower bounds for the real genus of a group in terms of its order. One consequence is that for each $\rho \geq 2$ there are only a finite number of groups of real genus ρ. Another consequence is a formula for the real genus of an M^*-group. This is used to give genus formulas for three important families of groups, the large symmetric and alternating groups and almost all the projective special linear groups $PSL(2, q)$.

We also establish some relationships between the real genus and other genus parameters and classify the groups with real genus $\rho \leq 3$. There are infinitely many groups of real genus zero and also infinitely many of real genus one. However, there are no groups with $\rho = 2$ and exactly two with $\rho = 3$. In addition, we find genus formulas for three other families of groups, the elementary abelian 2-groups and 3-groups and the dicyclic groups. Finally, we determine $\rho(G)$ for each group G with order less than 16.

Some of these results are pretty well known in some form or another. But this is not surprising, considering the recent activity in the field along with its older roots. We have attempted to present the material in a new and interesting context, and certainly some of the results are new. We close with some natural open problems about the real genus parameter.

2. Preliminaries. We shall assume that all surfaces are compact. If X is a bordered surface, then X is characterized topologically by orientability, the number k of components of the boundary ∂X and the topological genus p.

The bordered surface X can carry a dianalytic structure [**1**, p. 46] and be considered a Klein surface or a nonsingular algebraic curve over $\mathbf{R}$. Thus X has an algebraic genus g, which is given by the following important relation:

$$g = \begin{cases} 2p + k - 1 & \text{if } X \text{ is orientable} \\ p + k - 1 & \text{if } X \text{ is nonorientable.} \end{cases}$$

The algebraic genus appears naturally in bounds for the order of an automorphism group of a Klein surface [**16, 18, 19, 6, 4**], and, in fact, the integer g has often been referred to simply as *the genus* of the surface.

Non-Euclidean crystallographic (NEC) groups have been quite useful in investigating group actions on bordered Klein surfaces. Let $\mathcal{L}$ denote the group of automorphisms of the open upper half-plane U, and let $\mathcal{L}^+$ denote the subgroup of index 2 consisting of the orientation-preserving automorphisms. An NEC group is a discrete subgroup Γ of $\mathcal{L}$ (with the quotient space U/Γ compact). If $\Gamma \subset \mathcal{L}^+$, then Γ is called a *Fuchsian* group. Otherwise, Γ is called a *proper NEC group*; in this case Γ has a canonical Fuchsian subgroup $\Gamma^+ = \Gamma \cap \mathcal{L}^+$ of index 2.

Associated with the NEC group Γ is its *signature*, which has the form

$$(2.1) \qquad (p; \pm; [\lambda_1, \dots, \lambda_r]; \{(\nu_{11}, \dots, \nu_{1s_1}), \dots, (\nu_{k1}, \dots, \nu_{ks_k})\}).$$

The quotient space $X = U/\Gamma$ is a surface with topological genus p and k holes. The surface is orientable if the plus sign is used and nonorientable if the minus sign is used. The integers $\lambda_1, \dots, \lambda_r$, called the *ordinary periods*, are the ramification indices of the natural quotient mapping from U to X in fibers above interior points of X. The integers $\nu_{i1}, \dots, \nu_{is_i}$, called the *link periods*, are the ramification indices in fibers above points on the ith boundary component of X. Associated with the signature (2.1) is a presentation for the NEC group Γ, although the form of the presentation depends on whether the plus or minus sign is present. For these presentations and more information about signatures, see [**14, 22, 4**].

Let Γ be an NEC group with signature (2.1) and assume $k \geq 1$ so that the quotient space U/Γ is a bordered surface. The non-Euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature [**22**, p. 235]:

$$\mu(\Gamma)/2\pi = \gamma - 1 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\left(1 - \frac{1}{n_{ij}}\right), \tag{2.2}$$

where γ is the algebraic genus of the quotient space U/Γ. If Λ is a subgroup of finite index in Γ, then

$$[\Gamma : \Lambda] = \mu(\Lambda)/\mu(\Gamma). \tag{2.3}$$

An NEC group K is called a *surface group* if the quotient map from U to U/K is unramified. Fuchsian surface groups contain no elements of finite order. If the quotient space U/K has a nonempty boundary, then K is called a *bordered surface group*. Bordered surface groups contain reflections but no other elements of finite order.

Let X be a bordered Klein surface of algebraic genus $g \geq 2$, and let G be a group of dianalytic automorphisms of X. Then X can be represented as U/K where K is a bordered surface group with $\mu(K) = 2\pi(g-1)$. Further, there are an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$ [**17**]. The group $G \cong \Gamma/K$, so that from (2.3) we obtain

$$g = 1 + o(G) \cdot \mu(\Gamma)/2\pi. \tag{2.4}$$

Minimizing g is therefore equivalent to minimizing $\mu(\Gamma)$. Among the NEC groups Γ for which G is a quotient of Γ by a bordered surface group, then, we want to identify one for which $\mu(\Gamma)$ is as small as possible.

Let G be a finitely presented group. If the generating set has the minimum size, then this number of generators is called the *rank* of G. If G' is a quotient group of G, then clearly $\text{rank}\,(G) \geq \text{rank}\,(G')$. In particular, suppose the group H acts on a bordered surface, and we represent H as a quotient of the NEC group Γ. Then $\text{rank}\,(\Gamma)$ gives an upper bound for $\text{rank}\,(H)$.

The canonical presentation for an NEC group almost always involves redundant generators, however. Let Γ be an NEC group with signature (2.1) and associated canonical presentation. Suppose $k \geq 1$ and exactly l of the k period cycles are empty. Regardless of whether the plus or minus sign is present, the number of generators in the presentation is

$$N = r + S + k + \gamma + 1,$$

where $S = s_1 + \cdots + s_k$ and γ is the algebraic genus of the quotient space U/Γ. Of these, $Q = 1 + (k - l)$ are clearly redundant. Thus, Γ has a simplified presentation with $N - Q$ generators, and $\operatorname{rank}(\Gamma) \leq N - Q$, that is,

$$\operatorname{rank}(\Gamma) \leq \gamma + r + S + l. \tag{2.5}$$

Of the elements in the simplified presentation, clearly at most $\gamma + r$ can have order larger than two.

3. The real genus. We begin by showing that each finite group acts on a bordered surface. This result is certainly not surprising and should be considered something of a folk theorem. The corresponding result about Riemann surfaces is in [**9**, p. 572].

Theorem 1. *Let G be a finite group. Then there is a bordered Klein surface X such that G is a group of automorphisms of X.*

Proof. If the group G is cyclic or dihedral, then certainly G acts on the disc (with its unique dianalytic structure). Suppose G has generators $z_1, \ldots, z_r$ where $r \geq 2$ and $o(z_i) = m_i$. If $r = 2$, then further assume that m_1 or m_2 is at least 3 (if $m_1 = m_2 = 2$, then G is a dihedral group). Then let Γ be the NEC group with signature $(0; +; [m_1, \ldots, m_r]; \{()\})$. The group Γ is generated by $x_1, \ldots, x_r$, e and the reflection c with defining relations

$$(x_i)^{m_i} = c^2 = ece^{-1}c = x_1 x_2 \cdots x_r e = 1.$$

Define a homomorphism $\phi : \Gamma \to G$ by $\phi(x_i) = z_i$, $\phi(e) = (z_1 z_2 \cdots z_r)^{-1}$, $\phi(c) = 1$. Let $K = \operatorname{kernel} \phi$. Then $c \in K$ and K contains reflections but no other elements of finite order. Thus, K is a bordered surface group, and $G \cong \Gamma/K$ is a group of automorphisms of the bordered Klein surface U/K. $\square$

Thus each finite group G acts on a bordered surface (as a group of homeomorphisms), and it is natural to seek the surface of smallest genus. We define the *real genus* $\rho(G)$ to be the minimum algebraic genus of any bordered surface on which G acts. The real genus, in one guise or another, has been studied by several mathematicians working on Klein surfaces (for example, see [**4**, Section 4]). However, they have usually added the restriction that the surface have algebraic genus at least two. We feel it is more natural to allow actions on surfaces of genus zero and one. This agrees with the conventions for the graph-theoretic genus [**25**] and the symmetric genus [**24**].

If H is a subgroup of G, then obviously $\rho(H) \leq \rho(G)$. Also, suppose Q is a quotient group of G and X is a bordered surface of genus $\rho = \rho(G)$ on which G acts. Then Q acts on a quotient surface Y of X, and the genus of Y is at most ρ, just from the Riemann-Hurwitz formula. Hence, $\rho(Q) \leq \rho(G)$.

A nice consequence of our proof of Theorem 1 is an upper bound for $\rho(G)$. The similar upper bound for the graph-theoretical genus of a group is in [**25**, p. 93].

Corollary. *Let G be a finite group with generators $z_1, \dots, z_r$ and $o(z_i) = m_i$. Then*

$$\rho(G) \leq 1 + o(G)\left[r - 1 - \sum_{i=1}^{r} \frac{1}{m_i}\right]. \tag{3.1}$$

Proof. In the simple cases ($r = 1$ and $r = 2 = m_1 = m_2$) $\rho(G) = 0$, and the formula holds. Now consider the remaining cases; we continue to use the notation in the proof of the theorem. Let g be the genus of the bordered Klein surface $X = U/K$. Then from (2.2)

$$\mu(\Gamma) = 2\pi\left[r - 1 - \sum_{i=1}^{r} \frac{1}{m_i}\right].$$

Applying (2.4) we obtain

$$g = 1 + o(G)\left[r - 1 - \sum_{i=1}^{r} \frac{1}{m_i}\right].$$

Since G acts on X, $\rho(G) \leq g$, and we have our upper bound.

4. Bordered Klein surfaces. Next we prove a result that allows us to apply work done on bordered Klein surfaces to arbitrary topological actions. Like Theorem 1, this result is probably a folk theorem. Indeed, it was stated without proof in the introduction of [**10**].

Theorem 2. *Let G be a finite group of homeomorphisms of a bordered surface X onto itself. Then X carries a Klein surface structure that makes G a group of dianalytic automorphisms of X.*

Proof. The quotient space $Y = X/G$ is a bordered surface, and the quotient mapping $\pi : X \to Y$ is a ramified covering with folding. Give the surface Y a dianalytic structure [**1**, p. 46]. It is not hard to see that, locally, the quotient mapping π acts like a morphism of Klein surfaces. Then, by one of the main results of [**1**], π is a morphism globally; there is a dianalytic structure on X such that π is a morphism of Klein surfaces [**1**, p. 28].

Now let $f \in G$. Then $\pi f = \pi$, of course. Since both π and πf are morphisms, so is f [**1**, p. 19]. Thus G is a group of dianalytic automorphisms of X. □

Each upper bound for the order of a group acting on a bordered Klein surface can now be interpreted in terms of the real genus.

Corollary 1 [**16**]. *Let G be a finite group with $\rho(G) \geq 2$. Then $o(G) \leq 12[\rho(G) - 1]$, and $o(G) = 12[\rho(G) - 1]$ if and only if G is an M^*-group. Furthermore, if $o(G) < 12[\rho(G) - 1]$, then $o(G) \leq 8[\rho(G) - 1]$.*

Corollary 1 has the following important consequence.

Corollary 2. *For each integer $\rho \geq 2$, the number of groups of real genus ρ is finite.*

Corollary 3 [**19**]. *Let G be a finite nilpotent group with $\rho(G) \geq 2$. Then $o(G) \leq 8[\rho(G) - 1]$. Moreover, if G is not a 2-group, then $o(G) < 8[\rho(G) - 1]$.*

Corollary 4 [**6**]. *Let p be an odd prime, and let G be a finite p-group with $\rho(G) \geq 2$. Then $o(G) \leq (p/(p-2))[\rho(G) - 1]$.*

Corollary 1 also yields a formula for the real genus of an M^*-group. Several families of groups (and lots of isolated groups) have been recognized as M^*-groups. For example, Singerman [**23**] has shown that most of the projective special linear groups $PSL(2, q)$ are M^*-groups.

Corollary 5 [**23**]. *Let q be a prime power other than $2, 7, 11$ or 3^n, where $n = 2$ or n is odd. Then*

$$\rho(PSL(2, q)) = 1 + (q+1)(q^2 - q)/12d,$$

where $d = (2, q-1)$.

Conder [**7**] obtained partial presentations for the symmetric group S_n and the alternating group A_n that show that both groups are M^*-groups for all $n > 167$ (each of these groups is also an M^*-group for many $n < 167$). For the large groups, we have the following formulas.

Corollary 6 [**7**]. *For each $n > 167$, $\rho(S_n) = 1 + n!/12$ and $\rho(A_n) = 1 + n!/24$.*

5. The symmetric genus. The real genus of a group is naturally related to the symmetric genus. The *symmetric genus* $\sigma(G)$ of a finite group G is the minimum genus of any Riemann surface on which G acts. The *strong symmetric genus* $\sigma^o(G)$ is the minimum genus of any Riemann surface on which G acts preserving orientation. This terminology was introduced by Tucker in [**24**].

Associated with each bordered surface X in a natural way is its *complex double* X_c, [**1**, pp. 37–41], a Riemann surface of the same algebraic genus. The surface X_c has an orientation-reversing involution $\sigma : X_c \to X_c$ such that $X_c/\sigma = X$. The automorphism groups of the two surfaces are intimately connected [**1**, p. 79]. Let H and G denote the automorphism groups of X and X_c, respectively. Also let G^+ be the subgroup of G consisting of the orientation-preserving automorphisms of X_c, and set $L = \langle \sigma \rangle$. then

$$H \cong \{f \in G^+ \mid f\sigma = \sigma f\},$$

and thus G contains a subgroup isomorphic to $L \times H \cong Z_2 \times H$.

Now let G be a group with real genus $\rho = \rho(G)$ so that G acts on a bordered surface X of algebraic genus ρ. Then $Z_2 \times G$ acts on the complex double X_c, and G acts on X_c preserving orientation. Thus we immediately obtain the following inequalities.

$$\sigma(Z_2 \times G) \le \rho(G) \tag{5.1}$$

$$\sigma^o(G) \le \rho(G). \tag{5.2}$$

Each inequality obviously implies $\sigma(G) \le \rho(G)$.

The group G also acts on the surface X^* obtained from X by attaching a disc to each boundary component [**10**, p. 268]. Let p denote the topological genus of the bordered surface X, and let k denote the number of components of ∂X.

Suppose first that X is orientable, so that $\rho(G) = \rho = 2p + k - 1$. Then X^* is an orientable surface without boundary of topological (and algebraic) genus p. In this case,

$$\sigma(G) \le (1/2)[\rho(G) - k + 1]. \tag{5.3}$$

Next suppose that X is nonorientable, so that $\rho(G) = \rho = p + k - 1$. Then X^* is nonorientable with topological genus p and algebraic genus $p - 1$. Let W be the complex double of X^*. Then the group G also acts on W, an orientable surface of genus $p - 1$. Thus, in this case

$$\sigma(G) \le \rho(G) - k. \tag{5.4}$$

From the inequalities (5.3) and (5.4), we easily obtain the following.

Proposition. *Let G be a finite group. If $\rho(G) > 0$, then $\sigma(G) < \rho(G)$.*

6. Groups of small real genus. Here we classify the groups with real genus $\rho \leq 3$. This is not difficult. We begin with a few observations about the automorphism groups of bordered surfaces.

Let X be a bordered Klein surface with k boundary components, and let G be a group of automorphisms of X. Then G acts as a group of permutations of the boundary ∂X. In other words, there is a representation $\theta : G \to S_k$, where $H = \text{kernel}\,\theta$ is the subgroup of G that fixes each component of ∂X. Then H is a normal subgroup with $[G : H] \leq k!$. This representation is useful if k is small.

Now let C be one of the boundary components, and let H_c be the subgroup of G that fixes C. Of course, $H \subset H_c$. Then H_c is a dianalytic group of automorphisms of C, which topologically is just a circle. If an automorphism f acts as the identity on C, then f must be the identity, by standard methods of analytic continuation. Hence, H_c is either a dihedral group or a cyclic group, and its subgroup H must also be a group of one of these two types.

The groups of real genus zero are not surprising. The only bordered surface of algebraic genus zero is the disc D. Clearly the groups Z_n and D_n act on D, and these are the only possible automorphism groups, since D has a single boundary component.

Theorem 3. *The finite group G has real genus zero if and only if G is Z_n or D_n.*

The bordered surfaces of algebraic genus one are the annulus and the Mobius strip.

Theorem 4. *The finite group G has real genus one if and only if G is $Z_2 \times D_n$ with n even or $Z_2 \times Z_n$ with n even, $n \geq 4$.*

Proof. Let $n \geq 2$. Then it is easy to see that there is an action of

$Z_2 \times D_n$ on the annulus. If n is odd, $Z_2 \times D_n \cong D_{2n}$ of course. But if n is even, then $\rho(Z_2 \times D_n) = 1$ and if also $n \geq 4$, $\rho(Z_2 \times Z_n) = 1$ as well.

Now suppose G is a finite group with $\rho(G) = 1$. A group acting on the Mobius strip must be cyclic or dihedral, since this surface has only one boundary component. Thus G must act on an annulus X, so that $\sigma(G) = 0$ by (5.3). Let H be the subgroup of G that fixes both components of ∂X. Then H is a cyclic or dihedral group with $[G : H] \leq 2$. Since $\rho(G) \neq 0$, we must have $[G : H] = 2$. Now an inspection of the list of groups of symmetric genus zero [**11**, pp. 287–291] shows that the only possibilities are $G \cong Z_2 \times D_n$ or $G \cong Z_2 \times Z_n$ for some even n. □

Neither Theorem 3 nor Theorem 4 is really new, of course. Group actions on spheres with holes are well known. Especially relevant here are [**2**] and [**12**].

Thus there are infinitely many groups with $\rho = 0$ and infinitely many with $\rho = 1$. For each larger value of ρ, we already know that there are only finitely many groups. Interestingly, for $\rho = 2$, there are none at all.

Theorem 5. *There are no groups of real genus two.*

Proof. Suppose G is a group with $\rho(G) = 2$. Then $o(G) \leq 12$ by Corollary 1 to Theorem 2. But the only M^*-group of order 12 is D_6 [**10**, p. 278] and $\rho(D_6) = 0$. Hence, $o(G) \leq 8$. The only group with order 8 or less that does not have real genus zero or one is the quaternion group Q, and we shall show in Section 7 that $\rho(Q) = 5$. □

We must mention here that in [**5**] Bujalance and Gamboa determine all possible automorphism groups of bordered Klein surfaces of genus two. Their results together with Theorem 3 yield Theorem 5.

The classification for real genus three is messier but still not difficult with the ideas of this section. There are five topological types of surfaces to consider. We shall not provide a proof, however. Bujalance, Etayo and Gamboa have already classified the full automorphism groups of bordered Klein surfaces of genus three in [**3**], although they provide a proof for only one topological type. Their results, combined with Theorems 3–5, yield Theorem 6.

Theorem 6. *The finite group G has real genus three if and only if G is A_4 or S_4.*

There are at least three groups of real genus four. The M^*-group $D_3 \times D_3$ acts on a torus with three holes [**10**, Section 2]. Hence, $D_3 \times D_3$ and its subgroups $Z_3 \times Z_3$ and $Z_3 \times D_3$ have real genus four.

7. Genus formulas. We have already seen formulas for the real genus of several important families of groups. Next we find the real genus of the elementary abelian 2-groups and 3-groups.

Theorem 7. $\rho(Z_2)^n = 1 + 2^{n-2}(n-3)$ *for* $n \geq 1$.

Proof. The formula holds for $1 \leq n \leq 3$ by Theorems 3 and 4. Let $n \geq 4$ and write $G = (Z_2)^n$. Let Δ be the NEC group with signature

$$(0; +; [\ ]; \{(2, 2, \overset{n+1}{\cdots}, 2)\}).$$

(The single period cycle has $n+1$ periods equal to 2.) Then $\mu(\Delta) = \pi(n-3)/2$ by (2.2), and Δ has a presentation with generators $c_1, c_2, \dots, c_{n+1}$ and relations $(c_i)^2 = 1$, $(c_1c_2)^2 = (c_2c_3)^2 = \cdots = (c_nc_{n+1})^2 = (c_{n+1}c_1)^2 = 1$. Let $z_1, \dots, z_n$ be a set of n generators for G. There is a homomorphism ϕ of Δ onto G defined by $\phi(c_i) = z_i$ for $i = 1, \dots, n$ and $\phi(c_{n+1}) = 1$. Then $L = \operatorname{kernel} \phi$ is a bordered surface group, and G acts on the bordered surface $Y = U/L$. If g denotes the algebraic genus of Y, then from (2.4) we obtain $g = 1 + 2^{n-2}(n-3)$. Consequently, $\rho(G) \leq 1 + 2^{n-2}(n-3)$ (this improves (3.1)).

Now suppose G acts on the bordered surface X of algebraic genus g. We know $g \geq 4$. Then represent X as U/K where K is a bordered surface group. Obtain an NEC group Γ with signature (2.1) and a homomorphism $\alpha : \Gamma \to G$ onto G such that kernel $\alpha = K$. Also $Z_2 \times G \cong (Z_2)^{n+1}$ acts on the complex double X_c, and in fact $\Gamma/K^+ \cong (Z_2)^{n+1}$ [**20**, Section 3]. Therefore,

$$\operatorname{rank}(\Gamma) \geq n + 1. \tag{7.1}$$

Each element of G has order two, of course. Since the bordered surface group $K \cong \operatorname{kernel} \alpha$ contains reflections but no other elements of finite order, each period m_i and each link period n_{ij} in the signature of Γ must be two. Let γ denote the algebraic genus of U/Γ. From (2.2), we obtain

$$\mu(\Gamma) = (1/2)\pi(4\gamma - 4 + 2r + S),$$

where $S = s_1 + \cdots + s_k$. Again, let l denote the number of empty period cycles in the signature of Γ, so that $0 \le l \le k$.

First suppose $\gamma \ge 1$. Since $\gamma + 1 \ge k$, $4\gamma + 2r > \gamma + k + r$ and $\mu(\Gamma) > (1/2)\pi(\gamma + r + S + k - 4) \ge (1/2)\pi[\operatorname{rank}(\Gamma) - 4]$, using (2.5). Next let $\gamma = 0$, so that $k = 1$. If $l = 0$, then easily $\mu(\Gamma) = (1/2)\pi(-4+2r+s_1) \ge (1/2)\pi[\operatorname{rank}(\Gamma)-4]$, again using (2.5). If $l = 1$, then $s_1 = 0$ and $\operatorname{rank}(\Gamma) \le r+1$. Then $r \ge 4$ by (7.1) so that in this case $\mu(\Gamma) = (1/2)\pi(2r-4) > (1/2)\pi(r-3) \ge (1/2)\pi[\operatorname{rank}(\Gamma)-4]$.

Therefore, in any case, by applying (7.1) we obtain $\mu(\Gamma) \ge (1/2)\pi(n-3)$. A simple calculation using (2.4) now gives $g \ge 1 + 2^{n-2}(n-3)$. Hence, $\rho(G) \ge 1 + 2^{n-2}(n-3)$. This completes the proof. □

Maclachlan has computed the strong symmetric genus for each finite abelian group [**15**]. Applying his main result [**15**, p. 711], we see that $\sigma^o(Z_2)^n = \rho(Z_2)^n$ for $n \ge 4$. Thus we have an infinite family of groups for which the strong symmetric genus and the real genus are the same.

The inequality (5.2) could have been applied, of course, to shorten the proof of Theorem 7. We preferred instead to give a proof indicating a general technique that should be useful in studying the real genus parameter.

An interesting consequence of Theorem 7 is a formula for the symmetric genus of an elementary abelian 2-group. We have (5.1) and always the graph-theoretical genus $\gamma(G) \le \sigma(G)$ [**24**, p. 90].

Corollary. $\sigma(Z_2)^n = 1 + 2^{n-3}(n-4)$ *for* $n \ge 2$.

Proof. This follows immediately from Theorem 7 and the formula for $\gamma(Z_2)^n$ [**25**, p. 88], since $\gamma(Z_2)^n$ and $\rho(Z_2)^{n-1}$ agree. □

The proof of the following is similar to that of Theorem 7 but easier because 3-groups have no involutions. Also, the upper bound (3.1) is attained. We omit this proof.

Theorem 8. *$\rho(Z_3)^n = 1 + 3^{n-1}(2n-3)$ for $n \geq 1$.*

Finally we consider the infinite family of dicyclic groups. For $n \geq 2$, let G_n be the group with generators x, y and defining relations

$$x^{2n} = 1, \qquad x^n = y^2, \qquad y^{-1}xy = x^{-1}.$$

Then G_n is called the *dicyclic* group of order $4n$ [**8**, p. 7]. Each element outside the big cyclic subgroup $\langle x \rangle$ has order 4, and there is a unique element (x^n) of order 2. Also, the subgroup $J = \langle x^n \rangle$ is normal in G_n, and clearly $G_n/J \cong D_n$. Thus x^n is not part of a two-element generating set for G_n, and it follows that there are at least two generators of order larger than two in any generating set for G_n. The group G_n is also generated by the two elements $w = xy$ and y of the order 4 [**8**, p. 8] with defining relations

$$w^2 = y^2 = (w^{-1}y)^n.$$

The smallest dicyclic group G_2 is isomorphic to the quaternion group Q, and the group G_3 is isomorphic to the nonabelian group T of order 12 that is not A_4 and not D_6. Also, the group T is a semidirect product of Z_3 by Z_4 [**21**, p. 138].

Theorem 9. *If $n \neq 3$, $\rho(G_n) = 1 + 2n$. Furthermore, $\rho(G_3) = 6$.*

Proof. Since G_n is generated by two elements of order 4, (3.1) gives $\rho(G_n) \leq 1 + 2n$. We also have $\rho(G_3) \leq 6$, since $G_3 = T$ is generated by an element of order 3 and another of order 4.

Now let G_n act on a bordered surface X of algebraic genus g. We shall show that $g \geq 1 + 2n$ in general. We know $g \geq 2$. Obtain an NEC group Γ with signature (2.1) and a homomorphism $\phi : \Gamma \to G_n$ onto G_n such that $X = U/K$, where $K = \operatorname{kernel} \phi$ is a bordered surface group with $\mu(K) = 2\pi(g-1)$. Let γ denote the algebraic genus of the quotient space $Y = U/\Gamma$, and simplify the canonical presentation for Γ

as in Section 2. In this simplified presentation, there must be at least two elements with order larger than two, since $\Gamma/K \cong G_n$. Therefore,

$$\gamma + r \geq 2. \tag{7.2}$$

Applying (2.3), we have

$$(g-1)/4n = \mu(\Gamma)/2\pi,$$

which is given by (2.2). We obtain a lower bound for this expression. If $\gamma \geq 2$, then easily $(g-1)/4n \geq 1$ and $g \geq 4n+1$. Suppose $\gamma = 1$. Then from (7.2) we have $r \geq 1$ with at least one ordinary period larger than two. Thus, $(g-1)/4n \geq 2/3$ and $g \geq 1 + 8n/3$.

Now assume $\gamma = 0$ so that Y is the disc D. Using (7.2) again, we have $r \geq 2$ and at least two of the ordinary periods are greater than two. If $r \geq 3$, then $(g-1)/4n \geq -1 + 2(2/3) + 1/2 = 5/6$ and $g \geq 1 + 10n/3$. Suppose $r = 2$. Assume first that the quotient mapping $\pi : U \to D$ is ramified above the boundary of D. In this case there are at least two link periods equal to 2 [**16**, pp. 204, 205]. Thus we have $(g-1)/4n \geq -1 + 2(2/3) + 2 \cdot (1/4)$ and $g \geq 1 + 10n/3$.

Finally, assume $r = 2$ and there is no ramification above ∂D. Then Γ has signature $(0; +; [m_1, m_2]; \{()\})$ and is generated by x_1, x_2, e and the reflection c. But e is clearly redundant and $c \in K = \operatorname{kernel} \phi$, since the bordered surface group K contains reflections and any reflection in Γ is conjugate to c [**14**, p. 1198]. Thus G_n is generated by the two images $\phi(x_1), \phi(x_2)$ with orders m_1, m_2.

Suppose G_n has elements of order 3. Then 3 divides n, of course. So write $n = 3l$. But the two elements of G_n of order 3 are contained in the normal subgroup $N = \langle x^{2l} \rangle$, and it is not hard to see that the quotient group G_n/N is the dicyclic group G_l if $l > 1$. Hence, if $l > 1$, an element of order 3 cannot be part of a two-element generating set for G_n. Also, G_3 is not generated by two elements of order 3. For the group G_3, then, if $m_1 = 3$, $m_2 \geq 4$ and we have $(g-1)/12 \geq -1 + (2/3) + 3/4$ and $g \geq 6$.

Assume $n \neq 3$. Whether G_n has elements of order 3 or not, we must have $m_1 \geq 4$, $m_2 \geq 4$. Then we obtain $(g-1)/4n \geq -1 + 2(3/4)$. Hence, $g \geq 2n+1$ in the hard case. Therefore, $\rho(G_n) \geq 1 + 2n$. $\square$

Theorem 9 completes the calculation of the real genus of each group of order less than 16. The following table gives $\rho(G)$ and also $\sigma(G)$ for each group G with $o(G) < 16$ and $\rho(G) > 0$.

TABLE 1. Groups of small order with positive real genus.

order	group G	$\rho(G)$	$\sigma(G)$
8	$Z_2 \times Z_4$	1	0
8	$(Z_2)^3$	1	0
8	Q	5	1
9	$Z_3 \times Z_3$	4	1
12	$Z_2 \times Z_6$	1	0
12	A_4	3	0
12	T	6	1

8. Open problems. There are many unsolved problems about the real genus parameter. We mention some of the more natural and accessible ones.

Problem 1. For each $\rho \geq 4$, classify the groups with real genus ρ.

Problem 2. Determine $\rho(A)$ for each finite abelian group A.

Problem 3. Determine $\rho(H)$ for each finite Hamiltonian group H.

Of course, there is a problem of this type for each class of finite groups.

Problem 4. Find $\rho(G)$ for each group G of order less than 32.

This is already an interesting problem for order 16. There are 14 groups of order 16. Our results here give the real genus of 3 of the 5 abelian groups and just 3 of the 9 nonabelian ones.

For each integer $n \geq 2$, define $f(n)$ to be the number of groups with real genus n. We know that $f(2) = 0$ and $f(3) = 2$. From Theorem 9 it follows that $f(n)$ is positive for all odd n, except perhaps $n = 7$. Also, there are quite a few sequences of values of g for which there is a bordered surface of genus g with maximal symmetry and associated M^*-group [**10, 17**]. Some of the sequences contain only even values (for example, see [**17**, p. 9]). Hence, $f(n)$ is positive for infinitely many even n. Thus there is the following intriguing problem.

Problem 5. Determine whether $f(n) = 0$ for any $n > 2$.

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Department of Mathematics, Towson State University, Baltimore, Maryland 21204