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APPLICATION OF THE MONOTONE-ITERATIVE TECHNIQUES OF V. LAKSHMIKANTHAM FOR SOLVING THE INITIAL VALUE PROBLEM FOR IMPULSIVE DIFFERENTIAL-DIFFERENCE EQUATIONS

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ABSTRACT. In this paper a monotone-iterative technique is applied to the construction of extremal solutions of the initial value problem for an impulsive system of differential-difference equations.

1. Introduction. Impulsive differential equations find a wide application in the mathematical simulation of various phenomena and processes in the theory of optimal control, chemical technology, shock theory, impulse technology, population dynamics, etc., which during their evolution are subject to short-time perturbations. The presence of impulses in the system of differential equations affects essentially the character of the solutions and obstructs significantly the solving of the representative equations in quadratures. This requires the justification of effective methods for their approximate solution. One of these methods is based on the idea of combining the monotone-iterative method and the method of upper and lower solutions and has been justified by V. Lakshmikantham and his disciples for initial value and periodic problems for some classes of differential equations [**1, 3**–**10**].

In the present paper the monotone-iterative techniques of V. Lakshmikantham are applied to the approximate solution of the initial value problem for impulsive differential-difference equations.

We shall note that the question of approximate finding of a periodic solution of an impulsive differential-difference equation by means of another monotone method has been considered in [**2**].

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2. Statement of the problem. Consider the initial value problem for impulsive systems of differential-difference equations

$$\dot{x} = f(t, x(t), x(t-h)) \qquad \text{for } t \in [0, T],\ t \neq t_i,$$

$$\Delta x|_{t=t_i} = I_i(x(t_i)), \tag{1}$$

$$x(t) = \varphi(t) \qquad \text{for } t \in [-h, 0],$$

where $x = (x_1, x_2, \dots, x_n)$, $f : [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $f = (f_1, f_2, \dots, f_n)$, $I_i : \mathbf{R}^n \to \mathbf{R}^n$, $I_i = (I_{i1}, I_{i2}, \dots, I_{in})$, $\varphi : [-h, 0] \to \mathbf{R}^n$, $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$, $h = \text{const} > 0$, $0 < t_1 < t_2 < \cdots < t_d < T$, $\Delta x|_{t=t_i} = x(t_i + 0) - x(t_i - 0)$, $i = 1, 2, \dots, d$.

Let $x, y \in \mathbf{R}^n$, $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$. We shall say that $x \geq (\leq) y$ if for each $i = \overline{1, n}$ the inequalities $x_i \geq (\leq) y_i$ hold.

With each integer $j = \overline{1, n}$ we associate two nonnegative integers p_j and q_j such that $p_j + q_j = n - 1$ and for any element of $\mathbf{R}^n$ we denote

$$(x_j, [x]_{\rho_j}, [y]_{q_j}) = \begin{cases} (x_1, x_2, \dots, x_{\rho_j+1}, y_{\rho_j+2}, \dots, y_n) & \text{for } \rho_j \geq j, \\ (x_1, x_2, \dots, x_{\rho_j}, y_{\rho_j+1}, \dots, y_{j-1}, x_j, y_{j+1}, \dots, y_n) \\ \qquad\qquad \text{for } \rho_j < j. \end{cases}$$

According to the notation introduced, the initial value problem (1) can be written down in the form

$$\begin{aligned} \dot{x}_j &= f_j(t, x_j, [x]_{\rho_j}, [x]_{q_j}, x_j(t-h), [x(t-h)]_{\rho_j}, [x(t-h)]_{q_j}) \\ &\qquad\qquad \text{for } t \neq t_i,\ t \in [0, T], \\ \Delta x_j|_{t=t_i} &= I_{ij}(x_j(t_i), [x(t_i)]_{\rho_j}, [x(t_i)]_{q_j}), \\ x_j(t) &= \varphi_j(t) \qquad \text{for } t \in [-h, 0],\ j = \overline{1, n}. \end{aligned}$$

Consider the set $G([a, b], \mathbf{R}^n)$ of all functions $u : [a, b] \to \mathbf{R}^n$ which are piecewise continuous with points of discontinuity of the first kind at the points $t_i \in (a, b)$, $u(t_i) = u(t_i - 0)$, and the set $G^1([a, b], \mathbf{R}^n)$ of all functions $u \in G([a, b], \mathbf{R}^n)$ which are continuously differentiable for $t \in [a, b]$, $t \neq t_i$ and have continuous left derivatives at the points $t_i \in (a, b)$, $(i = \overline{1, d})$.

Definition 1. The couples functions $v, w \in G^1([0,T], \mathbf{R}^n)$, $v, w \in G([-h,T], \mathbf{R}^n)$ are said to be *couples lower* and *upper quasisolutions* of the initial value problem (1) if the following inequalities hold

(2)
$$\dot{v}_j(t) \leq f_j(t, v_j(t), [v(t)]_{\rho_j}, [w(t)]_{q_j}, v_j(t-h), [v(t-h)]_{\rho_j}, [w(t-h)]_{q_j}) \quad \text{for } t \neq t_i,\ t \in [0,T],$$

(3)
$$\dot{w}_j(t) \geq f_j(t, w_j(t), [w(t)]_{\rho_j}, [v(t)]_{q_j}, w_j(t-h), [w(t-h)]_{\rho_j}, [v(t-h)]_{q_j}),$$
$$\Delta v_j|_{t=t_i} \leq I_{ij}(v_j(t_i), [v(t_i)]_{\rho_j}, [w(t_i)]_{q_j}),$$
$$\Delta w_j|_{t=t_i} \geq I_{ij}(w_j(t_i), [w(t_i)]_{\rho_j}, [v(t_i)]_{q_j}),$$

(4)
$$\begin{matrix} v_j(t) \leq \varphi_j(t) \\ w_j(t) \geq \varphi_j(t) \end{matrix} \qquad \text{for } t \in [-h, 0].$$

Definition 2. In the case when (1) is an initial value problem for a scalar impulsive differential-difference equation, i.e., $n = 1$, $p_1 = q_1 = 0$, the couples lower and upper quasisolutions of the initial value problem (1) are called *lower* and *upper solutions* of the same problem.

Definition 3. The couples functions $v, w \in G^1([0,T], \mathbf{R}^n)$, $v, w \in G([-h,T], \mathbf{R}^n)$ are said to be *couples quasisolutions* of the initial value problem (1) if (2), (3) and (4) are satisfied only as equalities.

Definition 4. The couples functions $v, w \in G^1([0,T], \mathbf{R}^n)$, $v, w \in G([-h,T], \mathbf{R}^n)$ are said to be *couples minimal* and *maximal quasisolutions* of the initial value problem (1) if they are couples quasisolutions of the same problem and for any couples quasisolutions $u(t), z(t)$ of (1) the inequalities $v(t) \leq u(t) \leq w(t)$ and $v(t) \leq z(t) \leq w(t)$ hold for $t \in [-h, T]$.

Remark 1. Note that in general for the couples minimal and maximal quasisolutions $v(t), w(t)$ of (1) the inequality $v(t) \leq w(t)$ holds for $t \in [-h, T]$ while for arbitrary couples quasisolutions $u(t), z(t)$ of (1) analogous inequalities relating the functions $u(t)$ and $z(t)$ need not hold.

Remark 2. If for each $j = \overline{1,n}$ the equalities $\rho_j = n-1$, $q_j = 0$ hold and the functions $v(t), w(t)$ are couples quasisolutions of the initial value problem (1), then they are solutions of the same problem. If in this case problem (1) has a unique solution $u(t)$, then the couples functions (u,u) are couples minimal and maximal quasisolutions of (1).

For any couples functions $v, w \in G([-h,T], \mathbf{R}^n)$, $v, w \in G^1([0,T], \mathbf{R}^n)$ such that $v(t) \le w(t)$ for $t \in [-h,T]$ we define the set of functions

$$S(v,w) = \{u \in G([-h,T], \mathbf{R}^n), u \in G^1([0,T], \mathbf{R}^n) : \\ v(t) \le u(t) \le w(t) \text{ for } t \in [-h,T]\}.$$

3. Main results.

Lemma 1. *Let the functions* $g \in G([-h,T], \mathbf{R})$, $g \in G^1([0,T], \mathbf{R})$ *satisfy the inequalities*

$$\dot{g}(t) \le -Mg(t) - Ng(t-h) \qquad \textit{for } t \ne t_i,\ t \in [0,T], \tag{6}$$

$$\Delta g|_{t=t_i} \le -L_i g(t_i), \tag{7}$$

$$g(0) \le g(t) \le 0 \qquad \textit{for } t \in [-h,0],$$

where $M, N, L_i < 1$ $(i = \overline{1,d})$ *are positive constants such that*

$$\tau(M+N) \le (1-L)^{d+1}, \tag{9}$$

$$\tau = \max\{t_1, T-t_d, \max[(t_{i+1}-t_i) : i = 1,2,\dots,d-1]\}, \\ L = \max\{L_i : i = 1,2,\dots,d\}.$$

Then the inequality $g(t) \le 0$ *for* $t \in [-h,T]$ *holds.*

Proof. Suppose that this is not true, i.e., there exists a point $\xi \in [0,T]$ such that $g(\xi) > 0$. The following three cases are possible:

Case 1. Let $g(0) = 0$ and $g(t) \ge 0$, $g(t) \not\equiv 0$ for $t \in [0,b]$, where $b > 0$ is a sufficiently small constant. From inequality (8) it follows that $g(t) \equiv 0$ for $t \in [-h, 0]$. Then by assumption there exist points $\xi_1, \xi_2 \in [0, T]$, $\xi_1 < \xi_2$ such that $g(t) = 0$ for $t \in [-h, \xi_1]$ and $g(t) > 0$ for $t \in (\xi_1, \xi_2)$. From inequality (6) it follows that $\dot{g}(t) \leq 0$ for $t \in (\xi_1, \xi_2]$, $t \neq t_i$, which together with inequality (7) shows that the function $g(t)$ is monotone nonincreasing in the interval $[\xi_1, \xi_2]$, i.e., $g(t) \leq g(\xi_1) = 0$ for $t \in [\xi_1, \xi_2]$. Last inequality contradicts the assumption.

Case 2. Let $g(0) < 0$. By the assumption and inequality (7) it follows that there exists a point $\eta \in (0, T]$, $\eta \neq t_i$ $(i = \overline{1, d})$ such that $g(t) \leq 0$ for $t \in [-h, \eta)$, $g(\eta) = 0$ and $g(t) > 0$ for $t \in (\eta, \eta + \varepsilon)$ where $\varepsilon > 0$ is small enough. Introduce the notation $\inf\{g(t) : t \in [-h, \eta]\} = -\lambda$, $\lambda = \text{const} > 0$. Then there are two possibilities:

Case 2.1. Let a point $\zeta \in [0, \eta]$ exist, $\zeta \neq t_i$ $(i = \overline{1, d})$ such that $g(\zeta) = -\lambda$. For definiteness, let $\zeta \in (t_k, t_{k+1}]$ and $\eta \in (t_{k+m}, t_{k+m+1}]$, $m \geq 0$. Choose a point $\eta_1 \in (t_{k+m}, t_{k+m+1}]$, $\eta_1 > \eta$ so that $g(\eta_1) > 0$. By the mean value theorem the equalities

$$\begin{gathered} g(\eta_1) - g(t_{k+m} + 0) = \dot{g}(\xi_m)(\eta_1 - t_{k+m}), \\ g(t_{k+m} + 0) - g(t_{k+m-1} - 0) = \dot{g}(\xi_{m-1})(t_{k+m} - t_{k+m-1}), \\ g(t_{k+1} + 0) - g(\zeta) = \dot{g}(\xi_0)(\zeta - t_{k+1}) \end{gathered} \tag{10}$$

hold, where $\xi_0 \in (\zeta, t_{k+1})$, $\xi_m \in (t_{k+m}, \eta_1)$, $\xi_i \in (t_{k+i}, t_{k+i+1})$, $i = \overline{1, m-1}$.

From (7) and (10) we obtain the inequalities

$$\begin{gathered} g(\eta_1) - (1 - L_{k+m})g(t_{k+m}) \leq \dot{g}(\xi_m)\tau, \\ g(t_{k+m}) - (1 - L_{k+m-1})g(t_{k+m-1}) \leq \dot{g}(\xi_{m-1})\tau, \\ g(t_{k+1}) - g(\zeta) \leq \dot{g}(\xi_0)\tau. \end{gathered} \tag{11}$$

From inequalities (11) by elementary transformations we obtain the inequality

$$\begin{aligned} & g(\eta_1) - (1 - L_{k+1})(1 - L_{k+2}) \cdots (1 - L_{k+m})g(\zeta) \\ & \leq [\dot{g}(\xi_m) + (1 - L_{k+m})\dot{g}(\xi_{m-1}) + (1 - L_{k+m})(1 - L_{k+m-1})\dot{g}(\xi_{m-2}) \\ & \qquad + \cdots + (1 - L_{k+m})(1 - L_{k+m-1}) \cdots (1 - L_{k+1})\dot{g}(\xi_0)]\tau. \end{aligned} \tag{12}$$

Inequalities (6) and (12) and the choice of the points η_1 and ζ imply the inequality

$$-(1-L)^m(-\lambda) < -[1+(1-l)+(1-l)^2+\cdots+(1-l)^m](M+N)\tau(-\lambda)$$

where $l = \min\{L_i : i = 1, 2, \dots, d\}$.

Then the following inequality holds

$$(1-L)^m \leq \frac{(M+N)\tau}{1-l}. \tag{13}$$

Inequality (13) contradicts inequality (9).

Case 2.2. Let a point $t_k \in [0, \eta)$ exist such that $g(t_k + 0) < g(t)$ for $t \in [0, \eta)$, i.e., $g(t_k + 0) = -\lambda$. By arguments analogous to those in Case 2.1 where $\zeta = t_k + 0$, we again get a contradiction.

Case 3. Let $g(0) = 0$ and $g(t) \leq 0$, $g(t) \not\equiv 0$ for $t \in [0, b]$, where $b > 0$ is a sufficiently small constant. By arguments analogous to those in Case 2, we again get a contradiction.

Thus, Lemma 1 is proved. $\square$

Theorem 1. *Let the following conditions hold*:

1. *The functions $v, w \in G([-h, T], \mathbf{R}^n)$, $v, w \in G^1([0, T], \mathbf{R}^n)$ are couples lower and upper quasisolutions of the initial value problem with impulses* (1) *and satisfy the inequalities $v(t) \leq w(t)$ for $t \in [-h, T]$ and $v(0) - \varphi(0) \leq v(t) - \varphi(t)$, $w(0) - \varphi(0) \geq w(t) - \varphi(t)$ for $t \in [-h, 0]$.*

2. *The function $f \in C([0, T] \times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n)$, $f = (f_1, f_2, \dots, f_n)$, $f_j(t, x, y) = f_j(t, x_j, [x]_{\rho_j}, [x]_{q_j}, y_j, [y]_{\rho_j}, [y]_{q_j})$ is monotone nondecreasing with respect to $[x]_{\rho_j}$ and $[y]_{\rho_j}$ and monotone nonincreasing with respect to $[x]_{q_j}$ and $[y]_{q_j}$, and for $r, z \in S(v, w)$, $r(t) \leq z(t)$ the inequalities $f_j(t, z_j(t), [z(t)]_{\rho_j}, [z(t)]_{q_j}, z_j(t-h), [z(t-h)]_{\rho_j}, [z(t-h)]_{q_j}) - f_j(t, r_j(t), [z(t)]_{\rho_j}, [z(t)]_{q_j}, r_j(t-h), [z(t-h)]_{\rho_j}, [z(t-h)]_{q_j}) \geq -M_j(z_j(t) - r_j(t)) - N_j(z_j(t-h) - r_j(t-h))$, $j = \overline{1, n}$, $t \in [0, T]$, hold, where M_j, N_j $(j = \overline{1, n})$ are positive constants.*

3. *The functions $I_i : \mathbf{R}^n \to \mathbf{R}^n$, $I_i = (I_{i1}, I_{i2}, \dots, I_{in})$, $(i = \overline{1, d})$, $I_{ij}(x) = I_{ij}(x_j, [x]_{\rho_j}, [x]_{q_j})$ are monotone nondecreasing with respect*

to $[x]_{\rho_j}$ *and monotone nonincreasing with respect to* $[x]_{q_j}$ *and for* $x, y \in \mathbf{R}^n$, $v(t_i) \le y \le x \le w(t_i)$ $(i = \overline{1,d})$ *satisfy the inequalities*

$$I_{ij}(x_j, [x]_{\rho_j}, [x]_{q_j}) - I_{ij}(y_j, [x]_{\rho_j}, [x]_{q_j}) \ge -L_{ij}(x_j - y_j), j = \overline{1,n}, i = \overline{1,d},$$

where L_{ij} *are positive constants,* $L_{ij} < 1$ $(i = \overline{1,d}, j = \overline{1,n})$.

4. *The inequalities*

$$(M_i + N_i)\tau \le (1 - L_i)^{d+1}, i = \overline{1,n}$$

hold, where

$$\tau = \max\{t_1, (T - t_d), \max\{(t_{i+1} - t_i) : i = 1, 2, \dots, d-1\}\},$$
$$L_i = \max\{L_{ij} : j = 1, 2, \dots, d\}.$$

Then there exist two monotone sequences $\{v^{(k)}(t)\}_0^\infty$ *and* $\{w^{(k)}(t)\}_0^\infty$, $v^{(0)}(t) \equiv v(t)$, $w^{(0)}(t) \equiv w(t)$, *which are uniformly convergent in the interval* $[-h, T]$ *and their limits* $\bar{v}(t) = \lim_{k\to\infty} v^{(k)}(t)$ *and* $\bar{w}(t) = \lim_{k\to\infty} w^{(k)}(t)$ *are couples minimal and maximal quasisolutions of the initial value problem* (1). *Moreover, if* $u(t)$ *is a solution of the initial value problem* (1) *for which* $u \in S(v, w)$ *for* $t \in [-h, T]$, *then the inequalities* $\bar{v}(t) \le u(t) \le \bar{w}(t)$ *hold for* $t \in [-h, T]$.

Proof. Let us fix two functions $\eta, \mu \in S(v, w)$, $\eta = (\eta_1, \eta_2, \dots, \eta_n)$, $\mu = (\mu_1, \mu_2, \dots, \mu_n)$. Consider the initial value problems for the scalar impulsive linear differential-difference equations

(14)
$$\dot{x}_j(t) + M_j x_j(t) + N_j x_j(t-h) = \sigma_j(t, \eta, \mu) \qquad \text{for } t \ne t_i,\ t \in [0, T],$$

(15) $$\Delta x_j|_{t=t_i} = -L_{ij} x_j(t_i) + J_{ij}(\eta, \mu),$$

(16) $$x_j(t) = \varphi_j(t) \qquad \text{for } t \in [-h, T],\ j = \overline{1,n},$$

where

$$\sigma_j(t, \eta, \mu) = f_j(t, \eta_j(t), [\eta(t)]_{\rho_j}, [\mu(t)]_{q_j}, \eta_j(t-h), [\eta(t-h)]_{\rho_j}, [\mu(t-h)]_{q_j}) + M_j \eta_j(t) + N_j \eta_j(t-h), \qquad j = \overline{1,n},$$
$$J_{ij}(\eta, \mu) = I_{ij}(\eta_j(t_i), [\eta(t_i)]_{\rho_j}, [\mu(t_i)]_{q_j}) + L_{ij}\eta_j(t_i), \qquad j = \overline{1,n},\ i = \overline{1,d}.$$

The initial value problems (14), (15), (16) have a unique solution for each fixed couple of functions $\eta, \mu \in S(v,w)$.

Define a map $\mathcal{A} : S(v,w) \times S(v,w) \to S(v,w)$ by the equality $\mathcal{A}(\eta,\mu) = x$, where $x = (x_1, x_2, \dots, x_n)$, $x_j(t)$ is the unique solution of the initial value problems (14), (15), (16) for the couple of functions $\eta, \mu \in S(v,w)$.

We shall show that $v \le \mathcal{A}(v,w)$. Introduce the notation

$$v^{(1)} = \mathcal{A}(v,w), \qquad g(t) = v(t) - v^{(1)}(t), \qquad g = (g_1, g_2, \dots, g_n).$$

Then the following inequalities hold

$$\begin{aligned}
\dot{g}_j(t) &= \dot{v}_j(t) - \dot{v}_j^{(1)}(t) \\
&\le f_j(t, v_j, [v(t)]_{\rho_j}, [w(t)]_{q_j}, v_j(t-h), [v(t-h)]_{\rho_j}, [w(t-h)]_{q_j}) \\
&\qquad - M_j v_j^{(1)}(t) - N_j v_j^{(1)}(t-h) - \sigma_j(t, v, w) \\
&= -M_j g_j(t) - N_j g_j(t-h) \qquad \text{for } t \ne t_i,\ t \in [0,T],
\end{aligned}$$

$$g_j(t_i + 0) - g_j(t_i - 0) \le -L_{ij} g_j(t_i),$$

$$g_j(0) = v_j(0) - v_j^{(1)}(0) = v_j(0) - \varphi_j^{(1)}(0) \le g_j(t) \le 0 \qquad \text{for } t \in [-h, 0].$$

By Lemma 1 the functions $g_j(t)$, $j = \overline{1,n}$ are nonpositive for $t \in [-h,T]$, i.e., the inequality $v \le \mathcal{A}(v,w)$ holds.

In an analogous way it is proved that $w \ge \mathcal{A}(w,v)$.

Let $\eta, \mu \in S(v,w)$ be such that $\eta(t) \le \mu(t)$ for $t \in [-h,T]$. Set $x^{(1)} = \mathcal{A}(\eta,\mu), x^{(2)} = \mathcal{A}(\mu,\eta)$, $g(t) = x^{(1)}(t) - x^{(2)}(t)$. Then the functions $g_j(t)$, $j = \overline{1,n}$ satisfy the inequalities

$$\begin{aligned}
\dot{g}_j(t) &= \dot{x}_j^{(1)}(t) - \dot{x}_j^{(2)}(t) = -M_j(x_j^{(1)}(t) - x_j^{(2)}(t)) \\
&\quad - N_j(x_j^{(1)}(t-h) - x_j^{(2)}(t-h)) + M_j(\eta_j(t) - \mu_j(t)) \\
&\quad + N_j(\eta_j(t-h) - \mu_j(t-h)) \\
&\quad + f_j(t, \eta_j, [\eta(t)]_{\rho_j}, [\mu(t)]_{q_j}, \eta_j(t-h), [\eta(t-h)]_{\rho_j}, [\mu(t-h)]_{q_j}) \\
&\quad - f_j(t, \mu_j, [\mu(t)]_{\rho_j}, [\eta(t)]_{q_j}, \mu_j(t-h), [\mu(t-h)]_{\rho_j}, [\eta(t-h)]_{q_j}) \\
&\le -M_j g_j(t) - N_j g_j(t-h) \qquad \text{for } t \in [0,T],
\end{aligned}$$

$$g_j(t_i + 0) - g_j(t_i - 0) \le -L_{ij} g_j(t_i),$$

$$g_j(0) \le g_j(t) \le 0 \qquad \text{for } t \in [-h, 0].$$

By Lemma 1, the functions $g_j(t)$, $j = \overline{1,n}$ are nonpositive, i.e., $\mathcal{A}(\eta,\mu) \le \mathcal{A}(\mu,\eta)$.

Define the sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ by the equalities

$$v^{(0)} \equiv v, \qquad w^{(0)} \equiv w,$$
$$v^{(k+1)} = \mathcal{A}(v^{(k)}, w^{(k)}), \qquad w^{(k+1)} = \mathcal{A}(w^{(k)}, v^{(k)}).$$

The functions $v^{(k)}(t)$ and $w^{(k)}(t)$ for $t \in [-h,T]$ and $k \ge 0$ satisfy the inequalities
(17)
$$v^{(0)}(t) \le v^{(1)}(t) \le \cdots \le v^{(k)}(t) \le \cdots \le w^{(k)}(t) \le \cdots w^{(1)}(t) \le w^{(0)}(t).$$

Hence the sequences $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ are uniformly convergent for $t \in [-h,T]$. Introduce the notation $\bar{v}(t) = \lim_{k\to\infty} v^{(k)}(t)$ and $\bar{w}(t) = \lim_{k\to\infty} w^{(k)}(t)$. We shall show that the couples functions $(\bar{v},\bar{w})$ are couples minimal and maximal quasisolutions of the initial value problem (1). From inequalities (17) it follows that the inequality $\bar{v}(t) \le \bar{w}(t)$ holds for $t \in [-h,T]$. From the definition of the functions $v^{(k)}(t)$ and $w^{(k)}(t)$ it follows that the functions $(\bar{v},\bar{w})$ are couples quasisolutions of the initial value problem (1). Let $r, z \in S(v,w)$ be couples quasisolutions of problem (1). From inequalities (17) it follows that there exists a positive integer k such that $v^{(k-1)}(t) \le r(t) \le w^{(k-1)}(t)$ and $v^{(k-1)}(t) \le z(t) \le w^{(k-1)}(t)$ for $t \in [-h,T]$. Introduce the notations $g_j = v_j^{(k)} - r_j$, $j = \overline{1,n}$. By Lemma 1 the inequalities $g_j(t) \le 0$ hold for $t \in [-h,T]$, i.e., $v^{(k)}(t) \le r(t)$.

In an analogous way it is proved that the inequalities $r(t) \le w^{(k)}(t)$ and $v^{(k)}(t) \le z(t) \le w^{(k)}(t)$ hold for $t \in [-h,T]$ which shows that the couples functions $(\bar{v},\bar{w})$ are couples minimal and maximal quasisolutions of (1).

Let $u(t)$ be any solution of the problem (1) such that $v(t) \le u(t) \le w(t)$. Consider the couples (u,u) which are couples quasisolutions of (1). From the fact that the couples functions $(\bar{v},\bar{w})$ are couples minimal and maximal quasisolutions of the initial value problem (1), it follows that the inequalities $\bar{v}(t) \le u(t) \le \bar{w}(t)$ hold for $t \in [-h,T]$. $\square$

Theorem 2. *Let the following conditions be fulfilled*:

1. *For any* $j = \overline{1,n}$ *the equalities* $p_j = n-1$, $q_j = 0$ *hold.*

2. *The conditions of Theorem* 1 *hold.*

3. *The initial value problem* (1) *has a unique solution* $u(t)$ *for* $t \in [-h, T]$ *for which* $v(t) \leq u(t) \leq w(t)$.

Then there exist two monotone sequences of functions $\{v^{(k)}(t)\}_0^\infty$ *and* $\{w^{(k)}(t)\}_0^\infty$, $v^{(0)}(t) \equiv v(t)$, $w^{(0)}(t) \equiv w(t)$, *which are uniformly convergent and tend to the unique solution* $u(t)$ *of the initial value problem* (1).

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