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HÖLDER ESTIMATES FOR LOCAL SOLUTIONS FOR $\bar{\partial}$ ON A CLASS OF NONPSEUDOCONVEX DOMAINS

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ABSTRACT. We prove in this paper that in a class of nonpseudoconvex domains we have the anticipated Hölder estimate for the local solution of the $\bar{\partial}$-equation. We then extend the class of domains where the theorem applies. It is also noted that the method can be applied to improve a theorem by Range and Diederich-Fornaess-Wiegerinck.

Introduction. We investigate in this paper the Hölder estimates for the local solutions of the $\bar{\partial}$-equation on domains of the form $\Omega = \{r(z) < 0 : r(z) = \sum_1^p |z_i|^{2m_i} + g(z_{p+1}, \dots, z_n)$, where g is C^∞ with $g(0) = 0$ and $dg(0) \neq 0\}$. Thus the domains we consider here are not necessarily pseudoconvex. We prove that the *right* Hölder estimate holds in a neighborhood of 0 for $(0, q)$ forms with $q \geq n - p$.

It is well known that for strictly pseudoconvex domains the (1/2) Hölder estimate holds. (See, for example, Grauert and Lieb [**6**], Henkin [**7**] and Kerzman [**10**].) In the weakly pseudoconvex domains there are the works of Range [**11**], Diederich-Fornaess-Wiegerinck [**3**] and Bruna and Castillo [**1**] on ellipsoid type domains. Recently, Fefferman and Kohn [**4**] and Range [**13**] proved the estimate in $\mathbf{C}^2$ on domains of finite type by using different methods.

In the case that the domain is not necessarily pseudoconvex Fischer and Lieb [**5**] proved the (1/2) Hölder estimate on q-convex domains with smooth boundary. Recently Schmalz [**14**] proved the same result on domains with nonsmooth boundary.

In this paper we study the problem on a certain class of nonpseudoconvex domains. In this class of domains the nonnegative directions of the Levi-form is of ellipsoid type as in Range [**11**] while in the other directions it is arbitrary. We prove (Main theorem) that the anticipated order of Hölder estimate holds in a neighborhood of the origin. This type of phenomenon is known in subelliptic estimates of

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the $\bar{\partial}$-Neumann problem (Ho [**8, 9**]). We then extend the result to a larger class of domains.

The technique used to prove the theorem here follows closely that of Range [**11**] and Diederich-Fornaess-Wiegerinck [**3**] while in the nonpseudoconvex domains we consider here we need to add some non-holomorphic terms to the support function. We also need to add a piece of boundary to apply the treatment of Range and Siu [**14**]. We will prove the following theorem:

Main theorem. *Let $\Omega \subset \mathbf{C}^n$ be defined by*

$$\{r(z) < 0 : r(z) = \sum_1^p |z_i|^{2m_i} + g(z_{p+1}, \dots, z_n),$$
$$g \text{ is a real } C^\infty \text{ function with } g(0) = 0 \text{ and } dg(0) \neq 0\}.$$

If $q \geq n-p$, then there exists a neighborhood U of 0 such that for every $(0,q)$ form f which has bounded coefficients, C^1 and $\bar{\partial}$-closed in $U \cap \Omega$, there is a solution u of the equation $\bar{\partial} u = f$ in $U \cap \Omega$ with the estimate

$$||u||_{\frac{1}{2m}} \lesssim ||f||_\infty$$

holds in a smaller set $B \cap \Omega$ where $B \subset\subset U$ is a neighborhood of 0.

In here $m = \max_{1 \leq i \leq p} m_i$, $1/2m$ denotes the Hölder norm and ∞ denotes the supremum norm.

Note. In fact we can strengthen this result to $B = U$ in the above theorem. (See the remark at the end of Section 3.)

1. Preliminaries. We write down the basic ingredients that lead to the solution of the $\bar{\partial}$-problem by integral kernels on domains with piecewise smooth boundary. We refer the reader to Range and Siu [**14**] for details.

We will be studying domains that are intersections of two domains with smooth boundaries, hence we restrict our attention to the solution of $\bar{\partial}$ on such domains.

Definition 1.1 [**14**]. A bounded domain D in $\mathbf{C}^n$ is said to have a *piecewise smooth boundary* if there exist

(i) An open cover $\{U_1, U_2\}$ of an open neighborhood U of bD,

(ii) C^1 function $\rho_j : U_j \to \mathbf{R}$, $j = 1, 2$

such that

(a) $D \cap U = \{x \in U : \text{ either } x \notin U_j \text{ or } \rho_j(x) < 0,\ j = 1, 2\}$

(b) $d\rho_1, d\rho_2$ are linearly independent over $\mathbf{R}$ at every point of $U_1 \cap U_2$.

Let $S_1 = U_1 \cap D$, $S_2 = U_2 \cap D$ and $S_{12} = U_1 \cap U_2 \cap D$. The orientation is chosen so that

$$bD = S_1 + S_2 \quad \text{and} \quad bS_1 = S_{12}.$$

Let $\Delta = \{\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbf{R}^3 : \lambda_j \geq 0, \sum_{j=0}^{2} \lambda_j = 1\}$ be equipped with the canonical orientation and $\Delta_J = \{\lambda \in \Delta : \sum_{j \in J} \lambda_j = 1\}$.

A generating form for S_i is a form $W^{(i)}(\zeta, z) = \sum_1^n w_j^{(i)}(\zeta, z)\, d\zeta_j$ defined on $S_i \times D$ so that

$$\sum_1^n w_j^{(i)}(\zeta, z)(\zeta_j - z_j) = 1$$

for $\zeta \in S_i$ and $z \in D$. Denote

$$W^{(0)}(\zeta, z) = \sum_1^n \frac{\overline{(\zeta_i - z_i)}}{|\zeta - z|^2}\, d\zeta_i,$$

$$W^{(12)}(\zeta, \lambda, z) = \sum_{j=1}^{2} \lambda_j W^{(j)}(\zeta, z),$$

and

$$\hat{W}(\zeta, \lambda, z) = \sum_{j=0}^{2} \lambda_j W^{(j)}(\zeta, \lambda, z)$$

whenever these are defined.

The Cauchy-Fantappie kernel $\Omega_q(\hat{W})$ is defined as

$$\Omega_q(\hat{W}) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \binom{n-1}{q} \hat{W} \wedge (\bar{\partial}_{\zeta,\lambda} \hat{W})^{n-q-1} \wedge (\bar{\partial}_z \hat{W})^q$$

where $\bar\partial_{\zeta,\lambda} = \bar\partial_\zeta + d_\lambda$. $\Omega_q(W^{(i)})$ is given by the same formula as above except that $\hat W$ is replaced by $W^{(i)}$.

We have the following integral formula:

Theorem 1.2 [**14**]. *With the above notations, assume $W^{(i)}$ is a smooth generating form for S_i, $i = 1, 2$. Then for any $(0, q)$ form f with coefficients C^1 on Ω with $1 \le q \le n$, we have*

$$f = \int_{\sum_{0\notin I} S_I \times \Delta_I} f \wedge \Omega_q(W^{(I)}) + \bar\partial T_q f + T_{q+1}\bar\partial f \quad \text{on } \Omega$$

where

$$T_q f = \sum (-1)^{|I|+1} \int_{S_I \times \Delta_{0I}} f \wedge \Omega_{q-1}(\hat W) - \int_\Omega f \wedge W^{(0)}_{q-1}.$$

The following lemma is often used in proving the Hölder estimates for the solution. (See for example Range [**12**] and Range and Siu [**14**].)

Lemma 1.3. *If $f \in C^1(\Omega)$ and for some $0 < \alpha < 1$,*

$$|\mathrm{grad}\, f(z)| \lesssim (\mathrm{dist}\,(z, b\Omega))^{-\alpha}.$$

Then there exists $C > 0$ with

$$|f(z) - f(w)| \le C|z - w|^{1-\alpha} \qquad z, w \in \Omega.$$

The following computational lemma, which will be used in Section 3, is proved in Diedrich-Fornaess-Wiegerinck [**3**].

Lemma 1.4. *For $q > 1$, $j \ge 0$ and A positive close to 0, we have*

$$\int_{|z|<R} \frac{|z + w|^j \quad dx\, dy}{(A + |z + w|^j |z|^2)^q} = O(A^{1-q})$$

independent of w, $|w| < R$.

2. The $\bar{\partial}$-solution.

Lemma 2.1. *Let Ω be defined as in the main theorem and $q \geq n-p$. Then there exists a small neighborhood U of 0 such that for all $(0,q)$ forms f which is C^1 in $U \cap \overline{\Omega}$ and $\bar{\partial}$-closed, there is an operator $T_q : C^k_{0,q}(U \cap \overline{\Omega}) \to C^k_{0,q-1}(U \cap \Omega)$ such that $f = \bar{\partial} T_q f$.*

Proof. By a change of variables we may assume that

$$g(z_{p+1}, \dots, z_n) = 2\mathrm{Re}\, z_n + O(|z'|^2)$$

where we denote $z' = (z_{p+1}, \dots, z_n)$. It is not difficult to see that the Levi-form has at least p nonnegative eigenvalues in a neighborhood of 0.

Let us first construct a generating form for a piece of the boundary $r = 0$ near the origin. Define

$$P_i(\zeta, z) = \frac{\partial r}{\partial \zeta_i}(\zeta) \qquad i = 1, 2, \dots, p, n$$
$$P_i(\zeta, z) = \frac{\partial r}{\partial \zeta_i}(\zeta) + C\overline{(\zeta_i - z_i)} \qquad i = p+1, \dots, n-1$$

where C is some positive constant to be determined. Define

$$\Phi_1(\zeta, z) = \sum_1^n P_i(\zeta, z)(z_i - \zeta_i).$$

We have

$$2\mathrm{Re}\, \Phi_1(\zeta, z) = 2\mathrm{Re} \sum_1^n \frac{\partial r}{\partial \zeta_i}(\zeta)(z_i - \zeta_i) - 2C \sum_{p+1}^{n-1} |z_i - \zeta_i|^2.$$

From Lemma 5.5 of Range [**11**] we know that for $z, \zeta \in \mathbf{C}$

(2.1)
$$|z|^{2l} - |\zeta|^{2l} - 2\mathrm{Re}\left(\left(\frac{\partial}{\partial \zeta}|\zeta|^{2l}\right)(z - \zeta)\right) \gtrsim \left(\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}}|\zeta|^{2l}\right)|z-\zeta|^2 + |z-\zeta|^{2l}$$

Applying (3.1) to each of the variables $z_1, \dots, z_p$ we get

$$\begin{aligned} r(z) - r(\zeta) - 2\mathrm{Re} \sum_1^n \frac{\partial r}{\partial \zeta_i}(\zeta)(z_i - \zeta_i) \\ \gtrsim \sum_1^p \left(\frac{\partial^2 r}{\partial \zeta_i \partial \bar{\zeta}_i}(\zeta)|z_i - \zeta_i|^2 + |z_i - \zeta_i|^{2m_i} \right) \\ - C' \left(\sum_{p+1}^n |z_i - \zeta_i|^2 \right) \end{aligned}$$

Hence if we choose C large enough we have for $\zeta \in b\Omega$

$$\begin{aligned} |\mathrm{Re}\, \Phi_1(\zeta, z)| \gtrsim\ - r(z) + \sum_1^p \left(\frac{\partial^2 r}{\partial \zeta_i \partial \bar{\zeta}_i}(\zeta)|z_i - \zeta_i|^2 + |z_i - \zeta_i|^{2m_i} \right) \\ + \sum_{p+1}^{n-1} |z_i - \zeta_i|^2 - C'|z_n - \zeta_n|^2. \end{aligned}$$

If we put $d = r(\zeta) - r(z)$ and $\lambda = \mathrm{Im}\, \Phi_1(\zeta, z)$, then using the implicit function theorem we may see that

$$\begin{aligned} |z_n - \zeta_n|^2 &\lesssim |\lambda|^2 + |d|^2 \\ &\quad + c\left(\sum_1^p |\zeta_i|^{2m_i - 2}|z_i - \zeta_i|^2 + |z_i - \zeta_i|^{2m_i} + \sum_{p+1}^{n-1} |z_i - \zeta_i|^2 \right) \\ &\lesssim c\Bigg(|\lambda| + |d| \\ &\quad + \sum_1^p |\zeta_i|^{2m_i - 2}|z_i - \zeta_i|^2 + |z_i - \zeta_i|^{2m_i} + \sum_{p+1}^{n-1} |z_i - \zeta_i|^2 \Bigg) \end{aligned}$$

where c is as small as we please if $\zeta \in b\Omega$ and z are close enough to 0. Hence,
(2.2)

$$\begin{aligned} |\Phi_1(\zeta, z)| \gtrsim\ |\mathrm{Im}\, \Phi_1| + |r(z)| + \sum_1^p (|\zeta_i|^{2m_i - 2}|z_i - \zeta_i|^2 + |z_i - \zeta_i|^{2m_i}) \\ + \sum_{p+1}^{n-1} |z_i - \zeta_i|^2 \end{aligned}$$

when $\zeta \in b\Omega$ and z are close enough to 0.

Clearly,

$$W^{(1)}(\zeta, z) = \sum_1^n \frac{P_i(\zeta, z)}{\Phi_1}\, d\zeta_i$$

is a generating form on $(\{r = 0\} \cap U) \times (\Omega \cap U)$ where U is a small neighborhood of 0.

Let $\rho = \sum_1^p |z_i|^{2m_i} + \sum_{p+1}^n |z_i|^2 - \varepsilon^2$ be the defining function of the piece of boundary S_2. We choose ε small enough so that the estimate (2.2) holds in $D = \Omega \cap U \cap \{\sum_1^p |z_i|^{2m_i} + \sum_{p+1}^n |z_i|^2 < \varepsilon^2\}$ and that dr and $d\rho$ are linearly independent over $\mathbf{R}$ on $r = \rho = 0$. We will see that ε needs to satisfy some additional conditions later on. Thus, D has a piecewise smooth boundary as defined in Section 1.

Denote

$$\Phi_2(\zeta, z) = \sum_1^n \frac{\partial \rho}{\partial \zeta_i}(\zeta_i - z_i).$$

Clearly, (2.2) still holds if Φ_1 is replaced by Φ_2 and

$$W^{(2)}(\zeta, z) = \sum_1^n \frac{\partial \rho / \partial \zeta_i}{\Phi_2}\, d\zeta_i$$

is a generating form for $S_2 \times D$.

We can now apply Theorem 1.2, namely, if f is a $\bar\partial$-closed $(0, q)$ form with C^1 coefficients in D, we have

$$f = \int_{\sum_{0 \notin I} S_I \times \Delta_I} f \wedge \Omega_q(W^{(I)}) + \bar\partial T_q f$$

where

$$T_q f = \sum (-1)^{|I|+1} \int_{S_I \times \Delta_{0I}} f \wedge \Omega_{q-1}(\hat W) - \int_\Omega f \wedge W^{(0)}_{q-1} = J_1 + J_2.$$

Note that $\Omega_q(W^{(I)}) = 0$ on $S_I \times \Delta_{0I}$. This follows from the fact that Ω_q involves q times differentiation of $W^{(I)}$ in $\bar z$, the P_i's in $W^{(1)}$ have nonholomorphic terms in $z_{p+1}, \dots, z_{n-1}$ only, with the number of variables being $n-p-1<q$ and all the coefficients in $W^{(2)}$ are holomorphic in z. Hence, $u=T_q f$ is a solution for $\bar{\partial}u=f$. The regularity property of T_q is standard. This completes the proof of Lemma 2.1. □

3. Estimation of kernel. By an argument as in Range and Siu [**14**] we may assume that $f \in C^1(U \cap \bar{\Omega})$. It is well known that the integral J_2 in Section 2 satisfies $||J_2||_{I/2} \leq C||f||_\infty$. Hence we only need to prove the desired estimate for J_1. Since we only want to prove the Hölder estimate in $B \cap \Omega$ where $B \subset\subset U$ we only need to show that

$$\left| d_z \int_{S_1 \times \Delta_{01}} f \wedge \Omega_{q-1}(\hat{W}) \right| \lesssim (\delta_{S_1}(z))^{(1/2m)-1}.$$

Now

$$\int_{S_1 \times \Delta_{01}} f \wedge \Omega_{q-1}(\hat{W}) = \int_{S_1} f \wedge A_{q-1}(W^{(1)})$$

where

$$A_{q-1}(W^{(1)}) = \sum_{j=0}^{n-q-1} \sum_{k=0}^{q-1} a_{q-1}^{j,k} A_{q-1}^{j,k}(W^{(1)})$$

for some constants $a_{q-1}^{j,k}$ and

$$\begin{aligned}(3.1)\quad A_{q-1}^{j,k}(W^{(1)}) = W^{(1)} \wedge W^{(0)} \wedge (\bar{\partial}_\zeta W^{(1)})^j \wedge (\bar{\partial}_\zeta W^{(0)})^{n-q-1-j} \\ \wedge (\bar{\partial}_z W^{(1)})^k \wedge (\bar{\partial}_z W^{(0)})^{q-1-k}.\end{aligned}$$

In view of Lemma 1.3 and that $\delta_{S_1} \geq \delta_D$ to obtain the desired Hölder estimate, we need to show that

$$(3.2)\qquad \int_{S_1} |d_z A_{q-1}^{j,k}(W^{(1)})(\cdot, z)| \lesssim (\delta_{S_1}(z))^{(1/2m)-1}$$

where $m = \max_{1 \leq i \leq p} m_i$.

Computing $W^{(1)}$ and putting this into (3.1) we see that
(3.3)

$$A^{j,k}_{q-1}(W^{(1)}) = O\left(\sum_{i=1}^{p} |\zeta_i|^{2m_i-1} + \text{ other terms}\right) \wedge \sum_{i=1}^{n} |\zeta - z|\, d\zeta_i \wedge \left(\sum_{1}^{p} |\zeta_i|^{2m_i-2}\, d\bar{\zeta}_i \wedge d\zeta_i + \text{ other terms}\right)^{j} \wedge \left(\sum_{i=1}^{n} d\bar{\zeta}_i \wedge d\zeta_i\right)^{n-q-1-j} \frac{\wedge\left(\sum_{i=p+1}^{n-1} d\bar{z}_i \wedge d\zeta_i\right)^{k} \wedge \left(\sum_{i=1}^{n} d\bar{z}_i \wedge d\zeta_i\right)^{q-1-k}}{\Phi_1^{j+k+1}|\zeta - z|^{2(n-j-k-1)}}$$

Expanding the above expression we may see that the terms of $A^{j,k}_{q-1}(W^{(1)})$ have coefficients of the type (total order of the form in ζ is $2n-1$):

$$\frac{\prod_{\substack{h=\{1,\dots,j+1\}\\ 1\le i_h\le p}} |\zeta_{i_h}|^{2m_{i_h}-2}}{\Phi_1^{j+k+1}|\zeta - z|^{2n-2j-2k-3}} \times \text{ other terms}$$

where i_h denotes a distinct sequence of integers between 1 to n and ... in the above expression. We should note here that every term $|\zeta_i|^{2m_i-2}$ is associated to a form $d\zeta_i \wedge \overline{d\zeta}_i$ or $d\zeta_i$. Now we apply d_z to the above expression. If we differentiate the 'other terms', we get terms that are stronger and we may omit these terms. Hence we only differentiate the denominator and we get terms of the following two types:

$$\frac{\prod_{\substack{h=\{1,\dots,j+1\}\\ 1\le i_h\le p}} |\zeta_{i_h}|^{2m_{i_h}-2}}{\Phi_1^{j+k+2}|\zeta - z|^{2n-2j-2k-3}} \times \text{ other terms} \tag{3.4}$$

$$\frac{\prod_{\substack{h=\{1,\dots,j+1\}\\ 1\le i_h\le p}} |\zeta_{i_h}|^{2m_{i_h}-2}}{\Phi_1^{j+k+1}|\zeta - z|^{2n-2j-2k-2}} \times \text{ other terms} \tag{3.5}$$

We will integrate the terms (3.4) and (3.5) over a small neighborhood of the boundary. Let $\delta = \operatorname{dist}(z, S_1)$. Let $\alpha \ll \varepsilon$ be a small fixed number. We only consider points z that $\delta(z) \leq \alpha$ given any such point z. Consider a piece of the boundary $S = \{\zeta \in S_1 : |z-\zeta| \geq 2\alpha\}$. Since the integrands (3.4) and (3.5) are all smooth except at $\zeta = z$, the integrals over S are bounded independent of z. Pick any point $t(z)$ on S_1 such that $|z-t| \leq \alpha$. We need to estimate the integral on $S_1 \cap B_{3\alpha}(t)$. From (2.2) we get

$$
\begin{aligned}
|\Phi_1(\zeta,z)| \gtrsim\ & \delta + |\mathrm{Im}\,\Phi_1| + \sum_2^p (|\zeta_i|^{2m_i-2}|z_i-\zeta_i|^2 + |z_i-\zeta_i|^{2m_i}) \\
& + \sum_{p+1}^{n-1} |z_i-\zeta_i|^2.
\end{aligned} \tag{3.6}
$$

If α is small enough, then we can always use the coordinates

$$
\begin{aligned}
\tau_j &= \mathrm{Re}\,(z_j-\zeta_j) \qquad j = 1,\dots,n-1 \\
\sigma_j &= \mathrm{Im}\,(z_j-\zeta_j) \qquad j = 1,\dots,n-1 \\
\lambda &= \mathrm{Im}\,\Phi_1 \\
\delta &= \delta_{S_1}.
\end{aligned}
$$

Consider the integral of (3.4) over $S_1 \cap B_{3\alpha}(t)$, calling it I. We note that in the product

$$
\prod_{\substack{h=\{1,\dots,j+1\} \\ 1\leq i_h\leq p}} |\zeta_{i_h}|^{2m_{i_h}-2}
$$

of (3.4) there is at least $j-(n-p-1-k)$ number of terms. ($d\zeta_i$'s in the term $(\sum_{i=p+1}^{n-1} d\bar{z}_i \wedge d\zeta_i)^k$ occupies k spots from $d\zeta_{p+1}$ to $d\zeta_{n-1}$. Hence this left only $n-p-1-k$ spots to be filled. In the worst case all the spots are filled by the forms associated to the product above.) We assume that the product is from ζ_1 to $\zeta_{j-n+p+1+k}$ and also that $m_p = m$. Then we have

$$
\begin{aligned}
I \le & \int_{\substack{|\operatorname{Im}\Phi_1|<R \\ |\tau_i|,|\sigma_i|<R}} \frac{\prod_1^{j-n+p+1+k} |(x_h-\tau_h)^2+(y_h-\sigma_h)^2|^{m_h-1} d\operatorname{Im}\Phi_1 \wedge d\tau_1 \wedge d\sigma_1 \wedge \cdot \wedge d\tau_{n-1} \wedge d\sigma_{n-1}}{\Phi_1^{j+k+2} \left|\delta^2+\sum_1^{n-1}(\tau_h^2+\sigma_h^2)\right|^{n-j-k-3/2}} \\
\le & \int_{|\tau_i|,|\sigma_i|<R} \frac{\prod_1^{j-n+p+1+k} |(x_h-\tau_h)^2+(y_h-\sigma_h)^2|^{m_h-1} d\tau_1 \wedge d\sigma_1 \wedge \cdots \wedge d\tau_{n-1} \wedge d\sigma_{n-1}}{\left(\delta+\sum_1^p(|(x_h-\tau_h)^2+(y_h-\sigma_h)^2|^{m_h-1}(\tau_h^2+\sigma_h^2)+(\tau_h^2+\sigma_h^2)^{m_h})+\sum_{p+1}^{n-1}(\tau_h^2+\sigma_h^2)\right)^{j+k+1}} \\
& \cdot \frac{1}{\left(\delta^2+\sum_1^{n-1}(\tau_h^2+\sigma_h^2)\right)^{n-j-k-3/2}} \\
\le & \int_{|\tau_i|,|\sigma_i|<R} \frac{d\tau_{j-n+p+2+k} \wedge d\sigma_{j-n+p+2+k} \wedge \cdots \wedge d\tau_{n-1} \wedge d\sigma_{n-1}}{\left(\delta+\sum_{j-n+p+2+k}^{p}(\tau_h^2+\sigma_h^2)^{2m_i}+\sum_{p+1}^{n-1}(\tau_h^2+\sigma_h^2)\right)^{n-p}\left(\delta^2+\sum_{j-n+p+2+k}^{n-1}(\tau_h^2+\sigma_h^2)\right)^{n-j-k-3/2}} \\
\le & \int_{|\tau_i|,|\sigma_i|<R} \frac{d\tau_{j-n+p+2+k} \wedge d\sigma_{j-n+p+2+k} \wedge \cdots \wedge d\tau_p \wedge d\sigma_p}{\left(\delta+\sum_{j-n+p+2+k}^{p}(\tau_h^2+\sigma_h^2)^{m_h}\right)\left(\delta^2+\sum_{j-n+p+2+k}^{p}(\tau_h^2+\sigma_h^2)\right)^{n-j-k-3/2}} \\
\lesssim & \int_0^{nR} \frac{r^{2n-2j-2k-3}\,dr}{(\delta+r^{2m_p})(\delta^2+r^2)^{n-j-k-3/2}} \\
\le & \int_0^{nR} \frac{dr}{(\delta+r^{2m_p})} \\
\lesssim & \ \delta^{(1/2m)-1}
\end{aligned}
$$

where in the second line we substitute (3.6) into Φ_1 and integrate with respect to $\operatorname{Im}\Phi_1$. In the third line we integrate with respect to $d\tau_1, d\sigma_1, \ldots, d\tau_{j-n+p+1+k}, d\sigma_{j-n+p+1+k}$ and apply Lemma (1.4). In the fourth line we integrate with respect to $d\tau_{p+1}, d\sigma_{p+1}, \ldots, d\tau_{n-1}, d\sigma_{n-1}$. We use polar coordinates in the fifth line.

The integral of (3.5) was dealt with in a similar way. This completes the proof of the main theorem. □

Remark. In fact we can prove the main theorem with the set $B = U$. For this, we need to prove the analog of (3.2) for S_I where $|I| = 1, 2$. The estimate on S_{12} is more tedious. We omit the details here.

4. Hölder estimates on some other domains.

Theorem 4.1. *Let $\Omega \subset \mathbf{C}^n$ be defined by $\Omega = \{z : r(z) < 0 : r(z) = \sum_1^p s_i(|z_i|^2) + g(z_{p+1}, \ldots, z_n)\}$ that satisfies*

(i) *g is a real C^∞ function and $g(0) = 0$*

(ii) *s_i's are real C^∞ functions, $s_i(0) = 0$, and $s_i(t) > 0$ for $0 < t < \delta$ for some $\delta > 0$.*

Then the same conclusion holds as in the main theorem with

$$||u||_{1/2m} \lesssim ||f||_\infty$$

where m is some positive integer.

Proof. If s_i satisfies the required condition, then clearly

$$s_i(t) = b_k t^k + \text{ higher order terms}$$

with $b_k > 0$. With the same functions P_i and Φ_1 as defined in Lemma 2.1 it is easily seen that (2.1) can be replaced by

(4.1)

$$s_i(|z|^2) - s_i(|\zeta|^2) - 2\mathrm{Re}\left(\frac{\partial}{\partial \zeta} s_i(|\zeta|^2)(z-\zeta)\right) \gtrsim |\zeta|^{2k-2}|z-\zeta|^2 + |z-\zeta|^{2k}$$

in a neighborhood of 0. Hence (2.1) holds with some m_i, $i = 1, \ldots, p$ and we obtain the inequality (2.2). For that set of m_i we use the same defining function ρ for the boundary S_2 with ε small enough. We then define the same Φ_2. All the estimates and the proof of the main theorem then go through without change. □

Remark. Same as in the remark at the end of Section 3, the Hölder estimate in the above theorem can be shown to hold in the whole set $U \cap \Omega$.

On complex ellipsoids

$$E = \left\{ r(z) < 0 : \sum_{i=1}^{n} |z_i|^{2m_i} - 1 < 0 \right\}.$$

Range [**11**] showed that the Hölder estimate holds for $\alpha < 1/2m$, $m = \max_{1 \le i \le n} m_i$ for $(0,q)$ forms. In Diederich-Fornaess-Wiegerinck [**3**] it was improved to $\alpha = 1/2m$ for $(0,q)$ forms. It is noted here that we have

Theorem 4.2. *On the complex ellipsoid E if f is some $(0,q)$ form such that $\bar\partial u = 0$, then Range's solution $u = Tf$ of $\bar\partial u = f$ satisfies*

$$||u||_{1/2\alpha} \lesssim ||f||_\infty$$

where $\alpha = (n-q+1)^{\text{th}}$ largest number out of $\{m_1, \dots, m_n\}$.

Note that the $(n-q+1)^{\text{th}}$ largest number of $\{2m_1, \dots, 2m_n\}$ is the maximum of the q-type of the points on E as defined in D'Angelo [**2**]. We just outline the proof of this theorem since it follows from that of Range [**11**] while in here we adopt the generating form $W^{(1)}$ and estimate the kernels as in the proof of the main theorem.

Proof. Assume that $m_i \le m_{i+1}$, $i = 1, \dots, n-1$. Let

$$P_i(\zeta, z) = \frac{\partial r}{\partial \zeta_i}(\zeta) \qquad i = 1, 2, \dots, n-q+1$$
$$P_i(\zeta, z) = \frac{\partial r}{\partial \zeta_i}(\zeta) + C\overline{(\zeta_i - z_i)} \qquad i = n-q+2, \dots, n$$

and

$$\Phi = \sum_{i=1}^{n} P_i(\zeta, z)(z_i - \zeta_i).$$

Then with C large enough we have

$$|\operatorname{Re}\Phi(\zeta, z)| \gtrsim -r(z) + \sum_{1}^{n-q+1} (|\zeta_i|^{2m_i-2}|z_i - \zeta_i|^2 + |z_i - \zeta_i|^{2m_i}) + \sum_{n-q+2}^{n} |z_i - \zeta_i|^2$$

for all $z \in E$ and $\zeta \in bE$. Hence

$$|\Phi(\zeta, z)| \gtrsim |\mathrm{Im}\,\Phi| + |r(z)| + \sum_{1}^{n-q+1} (|\zeta_i|^{2m_i-2}|z_i - \zeta_i|^2 + |z_i - \zeta_i|^{2m_i}) + \sum_{n-q+2}^{n} |z_i - \zeta_i|^2$$

and $\sum_1^n (P_i(\zeta, z)/\Phi)\, d\zeta_i$ is a generating form on $bE \times E$. Now $\Omega_q(W) = 0$ since Ω_q involves q times differentiation in $\bar{z}$ of $\sum_1^n (P_i(\zeta, z)/\Phi)\, d\zeta_i$ and there are only $q-1$ antiholomorphic variables in z. Hence $f = \bar{\partial} T_q f$ as before. The integral kernel and the estimate of the kernel is the same as the estimate of $\int_{S_i} f \wedge A_{q-1}(W^{(i)})$ in Section 3. Hence, it follows that

$$||T_q f||_{1/2m} \lesssim ||f||_\infty$$

where $m = m_{n-q+1}$. □

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