ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 2, Spring 1993

NON-HOMEOMORPHIC DISJOINT SPACES WHOSE UNION IS ω^*

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ABSTRACT. For certain pairs $\langle \alpha, \beta \rangle$ of cardinals we show that the Stone-Čech remainder $\omega^* = \beta(\omega)\backslash\omega$ can be written in the form $\omega^* = \cup_{\xi<\alpha} C_\xi$ where the spaces C_ξ are pairwise disjoint, pairwise non-homeomorphic, countably compact, and dense in ω^*, with each $|C_\xi| = \beta$. In specific cases the condition that the spaces $\{C_\xi : \xi < \alpha\}$ are non-homeomorphic may be strengthened, as follows:

(i) $\alpha = 2^{\mathbf{c}}$, $c \leq \beta = \beta^\omega < 2^{\mathbf{c}}$: for $\xi < \alpha$ there is no one-to-one continuous function from C_ξ into $\cup_{\eta<\xi} C_\eta$.

(ii) $\omega \leq \alpha \leq 2^{\mathbf{c}}$, $\beta = 2^{\mathbf{c}}$: for $\eta < \xi < \alpha$ there is no continuous function from C_η onto C_ξ.

(iii) $1 \leq \alpha \leq 2^{\mathbf{c}}$, $\beta = 2^{\mathbf{c}}$: for $\xi < \alpha$ there is no one-to-one continuous function from C_ξ into $\omega^*\backslash C_\xi$.

1. Preliminaries. The symbol ω denotes the least infinite cardinal number and the countably infinite discrete topological space, and ω^* is the Stone-Čech remainder $\beta(\omega)\backslash\omega$. We consider only Tychonoff spaces, and we write $X \approx Y$ if X and Y are homeomorphic. The expression $X \subseteq_h Y$ means that X embeds into Y, i.e., there is $X' \subseteq Y$ such that $X \approx X'$.

For spaces X, Y and K with K compact and continuous $f : X \to Y \subseteq K$, the symbol $\bar{f}$ denotes the continuous function $\bar{f} : \beta X \to K$ such that $f \subseteq \bar{f}$. In this context we will consider repeatedly the question whether or not a point $p \in \beta X\backslash X$ satisfies $\bar{f}(p) \in Y$. We note in this connection that the choice of the enveloping compact space K is irrelevant. That is, if K and L are compact spaces containing Y and if f is continuous from X into Y, then for each $p \in \beta X$ the function $f_K = f : X \to Y \subseteq K$ satisfies $\bar{f}_K(p) \in Y$ if and only if the function $f_L = f : X \to Y \subseteq L$ satisfies $\bar{f}_L(p) \in Y$.

Received by the editors on December 16, 1990, and in revised form on April 30, 1991.
AMS-MOS *Subject Classification.* Primary 54A25, 54D40, Secondary 54D30, 04A20.
Key words and phrases. Stone-Čech remainder, disjoint union, Rudin-Frolík order, countably compact, extra countably compact, disjoint refinement lemma.

The set of embeddings (that is, homeomorphisms) of ω into the space X is denoted $\mathbf{H}(X)$, and

$$E(X) = \{p \in \omega^* : \text{ there is } h \in \mathbf{H}(X) \text{ such that } \bar{h}(p) \in X\}.$$

For $p, q \in \omega^*$, we write $p \sim q$ if there is a permutation h of ω (equivalently, $h \in \mathbf{H}(\omega)$) such that $\bar{h}(p) = q$. We write

$$T(p) = \{q \in \omega^* : p \sim q\} \qquad \text{for } p \in \omega^*,$$

and

$$T(\omega^*) = \{T(p) : p \in \omega^*\}.$$

The set $T(p)$ is called the (Frolík) *type* of p, and $T(\omega^*)$ is the set of all types of ω^*. A subset S of ω^* is said to be *T-saturated* if $T(p) \subseteq S$ whenever $p \in S$; that is, if $S = \cup\{T(p) : p \in S\}$.

The Rudin-Frolík pre-order $\sqsubset$ on ω^* is defined as follows:

$$p \sqsubset q \text{ if there is } h \in \mathbf{H}(\omega^*) \text{ such that } \bar{h}(p) = q.$$

It is clear that if $p' \sim p \sqsubset q \sim q'$ then $p' \sqsubset q'$. Thus the relation $\sqsubset$ extends to a relation (also denoted $\sqsubset$) on $T(\omega^*) = \omega^*/\sim$ as follows:

$$T(p) \sqsubset T(q) \qquad \text{if } p \sqsubset q.$$

It is a theorem of Z. Frolík [**8**] and M.E. Rudin [**17**] (see also [**5**] for an expository account) that the relation $\sqsubseteq$ defined on $T(\omega^*)$ by the rule

$$T(p) \sqsubseteq T(q) \qquad \text{if } \ T(p) \sqsubset T(q) \quad \text{or} \quad T(p) = T(q)$$

is reflexive, anti-symmetric, and transitive. For $p \in \omega^*$, we write

$$A(p) = \{q \in \omega^* : p \sqsubset q\} \quad \text{and} \quad B(p) = \{q \in \omega^* : q \sqsubset p\}$$

(the symbols are suggested by the words "above" and "below"), and from the three sources just cited we collect the following facts.

Theorem 1.1 (Frolík [**8**], Rudin [**17**]). *Let* $p \in \omega^*$. *Then*

(a) $p \sqsubset p$ *is false* (*and hence* $T(p) \cap A(p) = T(p) \cap B(p) = \varnothing$);

(b) $|T(p)| = \mathbf{c}$;

(c) $|A(p)| = 2^{\mathbf{c}}$;

(d) $|B(p)| \leq \mathbf{c}$;

(e) $\{T(q) : q \in B(p)\}$ *is linearly ordered under* $\sqsubset$; *and*

(f) $T(p)$ *is* $\sqsubseteq$*-minimal in* $T(\omega^*)$ *if and only if there is no countable discrete subset* D *of* ω^* *such that* $p \in \overline{D} \backslash D$.

For $X \subseteq \omega^*$, we write

$$A(X) = \bigcup_{p \in X} A(p) \quad \text{and} \quad B(X) = \bigcup_{p \in X} B(p),$$

and we note that the sets $A(X)$ and $B(X)$ are T-saturated. The symbol $E(X)$ is defined for every (Tychonoff) space X, while $A(X)$ and $B(X)$ are defined only for $X \subseteq \omega^*$. We will use the following simple result.

Theorem 1.2. *If* $X \subseteq \omega^*$, *then* $E(X) \subseteq B(X)$.

Proof. For $p \in E(X)$, there is an $h \in \mathbf{H}(X)$ such that $\bar{h}(p) = x \in X$. From $h \in \mathbf{H}(\omega^*)$ follows $p \sqsubset x \in X$ and hence $p \in B(X)$. □

Remark 1.3. It is well known (see, for example, [**9, 5**]) that the space ω^* contains both (i) a family of $\mathbf{c}$-many pairwise disjoint nonempty open subsets, and (ii) a family of $2^{\mathbf{c}}$-many pairwise disjoint homeomorphs of $\beta(\omega)$; further, every nonempty open subset of ω^* contains an open-and-closed copy of ω^* (and hence a copy of $\beta(\omega)$). From these facts the following statements are nearly immediate. (Here, as in [**18, 7**], we say that a subspace C of ω^* is *extra countably compact* in ω^* if and only if every infinite subset of ω^* has an accumulation point in C.)

Theorem 1.4. (a) *Every set of the form* $T(p)$ (*and* a fortiori, *every nonempty* T*-saturated subset of* ω^*) *is dense in* ω^*.

(b) *Every dense subset* C *of* ω^* *satisfies* $|C| \geq \mathbf{c}$.

(c) *Every extra countably compact subset* C *of* ω^* *satisfies* $|C| = 2^{\mathbf{c}}$.

(d) *Every extra countably compact subset of* ω^* *is dense in* ω^*.

(e) *Every set of the form $A(p)$ (with $p \in \omega^*$) is extra countably compact in ω^*.*

(To prove (e) it is enough to note that if $h \in \mathbf{H}(\omega^*)$ then $p \sqsubset \bar{h}(p)$ (so $\bar{h}(p) \in A(p)$) and $\bar{h}(p)$ is an accumulation point of the discrete set $h[\omega]$.)

In the three remaining sections of this paper (Sections 2, 3 and 4) we achieve three decompositions of the form $\omega^* = \cup_{\xi<\alpha} C_\xi$ with C_ξ pairwise disjoint, pairwise non-homeomorphic, countably compact, dense in ω^* and of cardinality prescribed in advance. As our Abstract indicates, the sets C_ξ may be chosen to satisfy certain additional constraints. The method of Section 2 depends on what we call p-closed sets and the p-closure; Section 3 uses the Disjoint Refinement Lemma; and Section 4 is based on the existence of $2^{\mathfrak{c}}$-many $\sqsubset$-minimal types in $T(\omega^*)$.

We use frequently and without explicit mention the fact that every infinite (Hausdorff) space X contains a copy of the countably infinite discrete space ω; that is, $\mathbf{H}(X) \neq \varnothing$.

Remarks 1.5. (a) The methods and results of the present paper are considerably stronger than those of [**18, 7**], where it was shown nevertheless that the space ω^* contains large families of pairwise non-homeomorphic extra countably compact subspaces.

(b) In work to appear [**12**], the second-listed author will recapture some of the results of the present paper and other facts about the space ω^*, using the methods of elementary sub-models.

(c) The reader conversant with topological Ramsey theory will notice that our results and our methods can be used to find a number of spaces Y for which the relation $\omega^* \to (Y)^1_2$ holds, and others for which it fails. We offer several results in this connection in [**4**].

(d) We announced several of the results of this paper and of [**4**] in [**3**].

2. The method of p-closure. Given $p \in \omega^*$, we say that a space Y is *p-closed* if every $h \in \mathbf{H}(Y)$ satisfies $\bar{h}(p) \in Y$. (It is not difficult to see that there is a (Hausdorff, nonregular) topology on ω^* with respect to which the closed sets are exactly what we here call the p-closed sets. We explore and exploit this topology in [**4**]; we do not describe its properties here because they are not needed for our present purposes. The terminology is chosen as a suggestive weakening of the concept of a p-compact space introduced by A. Bernstein [**1**]: Y is p-compact if each $h : \omega \to Y$ satisfies $\bar{h}(p) \in Y$. That not every p-closed space is p-compact is shown by a theorem of van Mill [**15**, (3.3), **16**, (4.4.1)]; there are $q \in \omega^*$ such that $q \in \bar{A} \backslash A$ for some countable $A \subseteq \omega^*$ but for no countable discrete $A \subseteq \omega^*$.)

Clearly the intersection of p-closed subspaces of a fixed space is p-closed, so for every p-closed space X and $Y \subseteq X$ there is a smallest p-closed subspace of X containing Y; this we denote $\text{p-cl}_X Y$, or $\text{p-cl}\, Y$ if ambiguity is impossible. The following lemma allows us to construct $\text{p-cl}\, Y$ "from the inside out" and to bound its cardinality in terms of $|Y|$. The construction parallels that of a countably compact extension given by Comfort and Saks [**6**, (1.1)] and of a minimal p-compact extension given by Ginsburg and Saks [**10**, (2.12)].

Lemma 2.1. *Let X be p-closed and let Y be an infinite subset of X, and for $\xi \leq \omega^+$ define $Y_0 = Y$,*

$$Y_{\xi+1} = Y_\xi \cup \{\bar{h}(p) : h \in \mathbf{H}(Y_\xi)\},$$

and

$$Y_\xi = \bigcup_{n<\xi} Y_\eta \quad \textit{for limit ordinals} \quad \xi \leq \omega^+.$$

Let $C = Y_{\omega^+}$. Then

(a) $C = \text{p-cl}\, Y$;

(b) *C is countably compact*;

(c) $|C| \leq |Y|^\omega$;

(d) $p \in E(C)$; *and*

(e) *if $Y \subseteq \omega^*$ and Y is T-saturated, then C is T-saturated.*

Proof. (a) Clearly $C \subseteq \text{p-cl}\, Y$, and C is p-closed.

(b) For every infinite subset A of C there is an $h \in \mathbf{H}(A)$, and then $\bar{h}(p)$ is an accumulation point of A in C.

(c) This is the case $\xi = \omega^+$ of the statement, easily proved by induction, that $Y_\xi \leq |Y|^\omega$ for all $\xi \leq \omega^+$.

(d) There is an $h \in \mathbf{H}(Y) \subseteq \mathbf{H}(C)$, and from $\bar{h}(p) \in C$ follows $p \in E(C)$.

(e) A routine inductive argument shows that Y_ξ is T-saturated for all $\xi \leq \omega^+$. □

Lemma 2.2. *Let X and Y be disjoint, T-saturated subspaces of ω^* with $Y \neq \varnothing$, let $p \in \omega^* \backslash B(X)$, and let $C = p\text{-cl}Y$. Then $C \cap X = \varnothing$.*

Proof. There is an $h \in \mathbf{H}(Y) \subseteq \mathbf{H}(C)$ and from $\bar{h}(p) \in C$ follows $p \in E(C)$. If there is a $q \in C \cap X$, then in the notation of 2.1 there are $\xi < \omega^+$ and $h \in \mathbf{H}(Y_\xi)$ such that $\bar{h}(p) = q \in X$, and then from $p \sqsubset q$ follows $p \in B(X)$, a contradiction. □

Lemma 2.3. *Let $p \in \omega^*$, and let X and Y be infinite spaces such that Y is p-closed. If there is a one-to-one continuous function f from Y into X, then $p \in E(X)$.*

Proof. Choose $h \in \mathbf{H}(Y)$. Since $|f \circ h[\omega]|| = \omega$, there is an $A \subseteq \omega$ such that $|A| = \omega$ and $f \circ h|A$ is a homeomorphism from A into X. Let i be a one-to-one function from ω onto A. Then $h \circ i \in \mathbf{H}(Y)$ and $f \circ h \circ i \in \mathbf{H}(X)$, and from $(h \circ i)^-(p) \in Y$ follows $f \circ (h \circ i)^-(p) \in f[Y] \subseteq X$ and hence $(f \circ h \circ i)^-(p) \in X$; thus $p \in E(X)$. □

Theorem 2.4. *Let β be a cardinal number such that $\mathbf{c} \leq \beta = \beta^\omega < 2^{\mathbf{c}}$. The space ω^* can be partitioned in the form $\omega^* = \cup_{\xi < 2^{\mathbf{c}}} C_\xi$ where the spaces C_ξ are pairwise disjoint, countably compact, dense in ω^*, and of cardinality β, and for $\xi < 2^{\mathbf{c}}$ there is no one-to-one continuous function from C_ξ into $\cup_{\eta < \xi} C_\eta$ (in particular, the spaces C_ξ are pairwise non-homeomorphic).*

Proof. Let $\{q_\xi : \xi < 2^{\mathbf{c}}\}$ be a faithful indexing of ω^*. For $\xi < 2^{\mathbf{c}}$ we will define X_ξ, p_ξ, Y_ξ and C_ξ so that

(i) $X_0 = \varnothing$;

(ii) $p_0 \in \omega^*$;

(iii) $|X_\xi| \leq \beta \cdot |\xi|$;

(iv) $|Y_\xi| = \beta$ and Y_ξ is T-saturated;

(v) $p_\xi \in \omega^* \backslash B(X_\xi)$;

(vi) $X_\xi \cap Y_\xi = \varnothing$ and $q_\xi \in X_\xi \cup Y_\xi$; and

(vii) $C_\xi = p_\xi\text{-cl}Y_\xi$.

Indeed, define X_0 and p_0 by (i) and (ii), choose $Y_0 \subseteq \omega^*$ such that $q_0 \in Y_0$, $|Y_0| = \beta$ and Y_0 is T-saturated, and set $C_0 = p_0\text{-cl}\, Y_0$. Now let $\zeta < 2^{\mathbf{c}}$ and suppose that X_ξ, p_ξ, Y_ξ and C_ξ have been defined for all $\xi < \zeta$ so that (iii)–(vii) hold for $\xi < \zeta$. Let $X_\zeta = \cup_{\xi<\zeta} C_\xi$, and note from (iv), (vii) and 2.1(c) that X_ζ is T-saturated. From (iv) and (vii) it follows that $|X_\zeta| < 2^{\mathbf{c}}$, so $|B(X_\zeta)| < 2^{\mathbf{c}}$ by 1.1(d). Hence there is $p_\zeta \in \omega^* \backslash B(X_\zeta)$ and there is $Y_\zeta \subseteq \omega^* \backslash X_\zeta$ such that $|Y_\zeta| = \beta$ and Y_ζ is T-saturated; if $q_\zeta \notin X_\zeta$ we choose Y_ζ so that $q_\zeta \in Y_\zeta$. Finally, we set $C_\zeta = p_\zeta\text{-cl}\, Y_\zeta$.

The definition of X_ξ, p_ξ, Y_ξ and C_ξ is complete for all $\xi < 2^{\mathbf{c}}$. The relation $\omega^* = \cup_{\xi<2^{\mathbf{c}}} C_\xi$ is immediate from (vi). That $C_\xi \cap C_\eta = \varnothing$ for $\eta < \xi < 2^{\mathbf{c}}$ follows from 2.2 and the relation $C_\eta \subseteq X_\xi$. From (vii) and 2.2(b) it follows that each of the spaces C_ξ is countably compact. That each C_ξ satisfies $|C_\xi| = \beta$ follows from (iv), 2.2(c) and the hypothesis $\beta = \beta^\omega$; the sets C_ξ are dense in ω^* by 1.4(a) and 2.1(e). The fact that for $\xi < 2^{\mathbf{c}}$ there is no one-to-one continuous function from C_ξ into $X_\xi = \cup_{\eta<\xi} C_\eta$ follows from (v), (vii) and 2.3. □

Remarks 2.5. (a) Since every dense subset D of ω^* satisfies $|D| \geq \mathbf{c}$, the condition $\beta \geq \mathbf{c}$ in the statement of Theorem 2.4 cannot be relaxed. We do not know whether 2.4 remains true for $\mathbf{c} \leq \beta < 2^{\mathbf{c}}$ if the condition $\beta = \beta^\omega$ is omitted; we note that for $\mathbf{c} \leq \beta < 2^{\mathbf{c}}$ the condition $\beta = \beta^\omega$ is satisfied if GCH is assumed (for then $\beta = \mathbf{c}$) or if there is a positive integer n such that β is the nth successor of $\mathbf{c}$.

(b) The condition in 2.4 that for $\xi < 2^{\mathbf{c}}$ there is no one-to-one continuous function from C_ξ into $\cup_{\eta<\xi} C_\eta$ cannot be strengthened to assert that C_ξ admits no one-to-one continuous function into $\omega^* \backslash C_\xi$: we have noted above that the space ω^* contains $2^{\mathbf{c}}$-many pairwise disjoint homeomorphs of ω^*, and clearly no set of cardinality less than $2^{\mathbf{c}}$ can meet each of these. This simple reasoning proves the following general result.

Theorem. *Every subspace C of ω^* such that $|C| < 2^{\mathbf{c}}$ satisfies $C \subseteq_h \omega^* \backslash C$.*

3. The disjoint refinement lemma. The disjoint refinement lemma asserts that if κ is an infinite cardinal and $\{S_\eta : \eta < \kappa\}$ is a (not necessarily faithfully indexed) family of sets with each $|S_\xi| = \kappa$, then there is a family $\{T_\eta : \eta < \kappa\}$ such that $|T_\eta| = \kappa$ and $T_\eta \subseteq S_\eta$ for all $\eta < \kappa$, and $T_\xi \cap T_\eta = \varnothing$ for $\xi < \eta < \kappa$. (For a proof of this result and for references to the literature, the reader might consult [**5**, (7.5)].) In the following theorem we let $\{D_\eta : \eta < 2^{\mathbf{c}}\}$ be a listing of all countably infinite subsets of ω^*, for $\eta < 2^{\mathbf{c}}$ we set $S_\eta = \overline{D}_\eta \backslash D_\eta$, and we choose $T_\eta \subseteq S_\eta$ as given by the disjoint refinement lemma.

Theorem 3.1. *Let $\omega \leq \alpha \leq 2^{\mathbf{c}}$. The space ω^* can be partitioned in the form $\omega^* = \cup_{\xi<\alpha} C_\xi$ where the spaces C_ξ are pairwise disjoint, extra countably compact in ω^* (hence dense in ω^* and of cardinality $2^{\mathbf{c}}$), and for $\xi < \alpha$ there is no continuous function from $\cup_{\eta<\xi} C_\eta$, nor from any of the spaces C_η with $\eta < \xi$, onto C_ξ (in particular, the spaces C_ξ are pairwise non-homeomorphic).*

Proof. Choose $\{T_\eta : \eta < 2^{\mathbf{c}}\}$ as above; use the inequality $2^{\mathbf{c}} \geq |\alpha \times 2|$ to find a subset $\{p(\eta, \xi, \varepsilon) : \xi < \alpha, \varepsilon \in \{0, 1\}\}$ of T_η faithfully indexed by $\alpha \times 2$, and for $\xi < \alpha$, set

$$\begin{aligned} E_\xi^0 &= \{p(\eta, \xi, 0) : \eta < 2^{\mathbf{c}}\}, \\ E_\xi^1 &= \{p(\eta, \xi, 1) : \eta < 2^{\mathbf{c}}\}, \quad \text{and} \\ E_\xi &= E_\xi^0 \cup E_\xi^1. \end{aligned}$$

We note that each of the sets E_ξ^0 (with $\xi < \alpha$) is extra countably compact in ω^*; indeed, for $\eta < 2^{\mathbf{c}}$ the point $p(\eta, \xi, 0) \in E_\xi^0$ is an accumulation point of D_η.

We will define the sets C_ξ by recursion so that

(i) $C_0 = E_0^0 \cup (\omega^* \backslash \cup_{\xi<\alpha} E_\xi)$,

(ii) $E_\xi^0 \subseteq C_\xi \subseteq E_\xi$ for non-zero limit ordinals $\xi < \alpha$, and

(iii) $E_\xi^0 \cup (E_\eta \backslash C_\eta) \subseteq C_\xi \subseteq E_\xi \cup (E_\eta \backslash C_\eta)$ for $\xi = \eta + 1 < \alpha$.

We proceed by recursion. Use (i) to define C_0. Now let $\zeta < \alpha$ and suppose that C_ξ has been defined for all $\xi < \zeta$. If ζ is a limit ordinal set $H_\zeta = E^0_\zeta$, and if ζ is the successor ordinal $\zeta = \eta + 1$ set $H_\zeta = E^0_\zeta \cup (E^1_\eta \backslash C_\eta)$. Since $H_\zeta \cap E^1_\zeta = \varnothing$ and $|E^1_\zeta| = 2^{\mathbf{c}}$, the family

$$\mathbf{A} = \{H_\zeta \cup A : A \subseteq E^1_\zeta\}$$

is a family of subsets of ω^* such that $|\mathbf{A}| = 2^{2^{\mathbf{c}}} > 2^{\mathbf{c}}$.

Now for $\xi < \zeta$, we have

$$d(C_\xi) \leq w(C_\xi) \leq w(\omega^*) = \mathbf{c},$$

so the number of continuous functions from C_ξ into ω^* does not exceed $|\omega^*|^{\mathbf{c}} = (2^{\mathbf{c}})^{\mathbf{c}} = 2^{\mathbf{c}}$. Thus since $|\mathbf{A}| > 2^{\mathbf{c}}$ there is a set in $\mathbf{A}$ (call it C_ζ) such that none of the spaces C_ξ, $\xi < \zeta$, maps continuously onto C_ζ and $\cup_{\xi<\zeta} C_\xi$ does not map continuously onto C_ζ.

The definition of C_ξ for all $\xi < \alpha$ is complete.

We show that the family $\{C_\xi : \xi < \alpha\}$ is as required.

It is clear from the construction that for $\xi < \alpha$ there is no continuous function from $\cup_{n<\xi} C_\eta$, nor from any of the spaces C_η with $\eta < \xi$, onto C_ξ.

The family $\{E_\xi : \xi < \alpha\}$ is pairwise disjoint, and $C_\xi \subseteq E_\xi$ for nonzero limit ordinals ξ, and $C_\xi \subseteq E_\xi \cup (E_\eta \backslash C_\eta)$ for ordinals $\xi = \eta + 1$. It follows that the family $\{C_\xi : \xi < \alpha\}$ is pairwise disjoint.

Since $C_\xi \supseteq E^0_\xi$ and E^0_ξ is extra countably compact in ω^*, each set C_ξ is extra countably compact in ω^*.

Since

$$C_0 \supseteq \omega^* \backslash \cup_{\xi<\alpha} E_\xi$$

and

$$E_\eta = E^0_\eta \cup E^1_\eta \subseteq C_\eta \cup C_{\eta+1} \qquad \text{for all } \eta < \alpha,$$

we have

$$\omega^* \supseteq \cup_{\xi<\alpha} C_\xi \supseteq (\omega^* \backslash \cup_{\xi<\alpha} E_\xi) \cup (\cup_{\eta<\alpha} E_\eta) = \omega^*,$$

as required. $\square$

Remarks 3.2. (a) The technique of counting the number of continuous functions from subspaces of ω^* into ω^*, and of using these estimates to select $C_\zeta \in \mathbf{A}$ with the appropriate properties, came to our attention through the work of Hodel [**11**, (3.3)].

(b) In the recursive construction of C_ζ, with C_ξ having been defined for all $\xi < \zeta$, we chose a family $\mathbf{F}$ of subsets of $\cup_{\xi<\zeta} C_\xi$ such that $|\mathbf{F}| = |\zeta| + 1$, namely, $\mathbf{F} = \{C_\xi : \xi < \zeta\} \cup \{\cup_{\xi<\zeta} C_\xi\}$, and we chose C_ζ so that no $F \in \mathbf{F}$ maps continuously onto C_ζ. It should be clear to the reader that the argument used allows for a much larger family $\mathbf{F}$. Indeed, if at stage ζ any family $\mathbf{F} = \mathbf{F}(\zeta)$ of subsets of ω^* is chosen with $|\mathbf{F}| < 2^{2^{\mathbf{c}}}$, then C_ζ may be chosen so that no $F \in \mathbf{F}$ maps continuously onto C_ζ.

4. The method of $\sqsubset$-minimal ultrafilters. The method of this section furnishes a decomposition of ω^* which has features in common with that of Section 2 (the absence of one-to-one continuous functions) and with that of Section 3 (the constituent sets C_ξ are extra countably compact in ω^*).

Theorem 4.1. *Let $1 \le \alpha \le 2^{\mathbf{c}}$. The space ω^* can be partitioned in the form $\omega^* = \cup_{\xi<\alpha} C_\xi$ where the spaces C_ξ are pairwise disjoint, extra countably compact in ω^* (hence dense in ω^* and of cardinality $2^{\mathbf{c}}$), and for $\xi < \alpha$ there is no one-to-one continuous function from C_ξ into $\omega^* \backslash C_\xi$ (in particular, the spaces C_ξ are pairwise non-homeomorphic).*

Proof. We consider first the case $\alpha = 2^{\mathbf{c}}$.

It is a theorem of Kunen, first proved assuming Martin's Axiom [**13**] and later in ZFC alone without additional assumptions [**14**], that there are $p \in \omega^*$ such that $T(p)$ is $\sqsubset$-minimal in $T(\omega^*)$; indeed, the number of such minimal types is $2^{\mathbf{c}}$ [**14**]. Let $\{T(p_\xi) : 1 \le \xi < 2^{\mathbf{c}}\}$ be a faithful enumeration of these minimal types, and define

$$C_\xi = T(p_\xi) \cup A(p_\xi) \qquad \text{for } 1 \le \xi < 2^{\mathbf{c}}$$

and

$$C_0 = \omega^* \backslash \bigcup_{1 \le \xi < 2^{\mathbf{c}}} C_\xi.$$

(It is known that $C_0 \neq \varnothing$. For example, Bukovský and Butkovičová [**2**] have shown the existence of $p \in \omega^*$ such that $\{T(q) : q \sqsubset p\}$ is order-isomorphic to the negative integers. Clearly, $p_\xi \sqsubseteq p$ fails for all $\xi < 2^{\mathbf{c}}$ for such p, so $p \in C_0$.)

The key to the verification that the relation $\omega^* = \cup_{\xi<2^{\mathbf{c}}} C_\xi$ expresses ω^* in the required form is Theorem 1.1 (e): For each $p \in \omega^*$ the set $\{T(q) : q \sqsubset p\}$ is linearly ordered under $\sqsubset$. This shows that if $1 \leq \eta < \xi < 2^{\mathbf{c}}$ then $C_\eta \cap C_\xi = \varnothing$. Indeed, if some $q \in \omega^*$ satisfies $q \in C_\eta \cap C_\xi$ then from $T(p_\xi) \sqsubseteq T(q)$ and $T(p_\eta) \sqsubseteq T(q)$ will follow $T(p_\xi) \sqsubseteq T(p_\eta)$ or $T(p_\eta) \sqsubseteq T(p_\xi)$, contrary to the definition of the family $\{T(p_\xi) : 1 \leq \xi < 2^{\mathbf{c}}\}$. That $C_0 \cap C_\xi = \varnothing$ for $1 \leq \xi < 2^{\mathbf{c}}$ is clear, since $T(p_\xi) \sqsubseteq T(q)$ holds for every $q \in C_\xi$ and is false for every $q \in C_0$.

We note next for $\xi < 2^{\mathbf{c}}$ that $A(C_\xi) \subseteq C_\xi$. (For $\xi > 0$ this is immediate from the definition, and for $\xi = 0$ it follows from 1.1 (e).) This observation has two consequences: Each of the sets C_ξ is extra countably compact in ω^* (use (1.4 (e)) and each C_ξ is p-closed for every $p \in C_\xi$ (since for every $h \in \mathbf{H}(C_\xi)$ from $p \sqsubset \bar{h}(p)$ follows $\bar{h}(p) \in A(C_\xi) \subseteq C_\xi$.)

A straightforward appeal to 1.1 (e) shows also that $B(q) \subseteq C_\xi$ whenever $q \in C_\xi$.

It remains to show that if $\xi < 2^{\mathbf{c}}$ then there is no one-to-one continuous function from C_ξ into $\omega^* \backslash C_\xi$. It is enough to choose $p \in C_\xi$ and to apply 2.3 with $Y = C_\xi$, $X = \omega^* \backslash C_\xi$; the space C_ξ is p-closed, so if such a function exists then

$$p \in E(\omega^* \backslash C_\xi) \subseteq B(\omega^* \backslash C_\xi) = \cup\{B(q) : q \in \omega^* \backslash C_\xi\} \subseteq \omega^* \backslash C_\xi,$$

a contradiction.

The proof for the case $\alpha = 2^{\mathbf{c}}$ is complete. In case $\alpha < 2^{\mathbf{c}}$ it is enough to write

$$C'_\xi = C_\xi \quad \text{for } 1 \leq \xi < \alpha \quad \text{and} \quad C'_0 = \omega^* \backslash \bigcup_{1 \leq \xi < \alpha} C'_\xi.$$

The family $\{C'_\xi : \xi < \alpha\}$ is then as required. □

Remark 4.2. The reasoning given in the last paragraph of the foregoing proof concerning the decomposition $\omega^* = \cup_{\xi<2^{\mathbf{c}}} C_\xi$ shows this: if $C(A)$ is defined for $A \subseteq 2^{\mathbf{c}}$ by the relation

$$C(A) = \bigcup_{\xi \in A} C_\xi,$$

then for subsets A and B of $2^{\mathbf{c}}$ there is a one-to-one continuous function from $C(A)$ into $C(B)$ if and only if $A \subseteq B$. Indeed, if $A \subseteq B$, the inclusion function from $C(A)$ into $C(B)$ is as required, and if there is $\xi \in A \backslash B$ then there is no one-to-one continuous function from C_ξ into $\omega^* \backslash C_\xi$, hence none from $C(A)$ into $C(B)$. Thus the family $\{C(A) : \varnothing \neq A \subseteq 2^{\mathbf{c}}\}$ is a family of $2^{2^{\mathbf{c}}}$-many dense, extra countably compact, pairwise non-homeomorphic subspaces of ω^*.

REFERENCES

1. Allen R. Bernstein, *A new kind of compactness for topological spaces*, Fund. Math. **66** (1970), 185–193.

2. L. Bukovský and E. Butkovičová, *Ultrafilters with $\aleph_0$ predecessors in Rudin-Frolík order*, Comm. Math. Univ. Carolin. **22** (1981), 429–447.

3. W.W. Comfort and Akio Kato, *Non-homeomorphic disjoint spaces whose union is ω^**, Abstracts Amer. Math. Soc. **11** (1990), p. 332 (Abstract).

4. W.W. Comfort, Akio Kato and Saharon Shelah, *Topological partition relations of the form $\omega^* \to (Y)^1_2$*, Proc. Seventh (1991) Summer Conference on General Topology and Applications, in honor of Mary Ellen Rudin and Her Work, Ann. New York Acad. Sci., 1993, to appear.

5. W.W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Grundlehren der math. Wissenschaften Band **211**, Springer-Verlag, Berlin, Heidelberg and New York, 1974.

6. W.W. Comfort and Victor Saks, *Countably compact groups and finest totally bounded topologies*, Pacific J. Math. **49** (1973), 33–44.

7. W.W. Comfort and Charles Waiveris, *Intersections of countably compact subspaces of Stone-Čech compactifications*, Uspekhi Mat. Nauk **35** (1980), 67–77 (in Russian). English translation: Russian Math. Surveys **35** (1980), 79–89.

8. Zdeněk Frolík, *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 87–91.

9. Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, D. Van Nostrand Co., Inc., Princeton, Toronto, New York and London, 1960.

10. John Ginsburg and Victor Saks, *Some applications of ultrafilters to topology*, Pacific J. Math. **57** (1975), 403–418.

11. R.E. Hodel, *The number of metrizable spaces*, Fund. Math. **115** (1983), 127–141.

12. Akio Kato, *Elementary submodels for a partition of $\beta\omega\backslash\omega$*, submitted.

13. Kenneth Kunen, *Some points in $\beta\mathbf{N}$*, Math. Proc. Cambridge Philos. Soc. **80** (1976), 385–398.

14. ———, *Weak P-points in N^**, Colloquia Mathematica Societatis János Bolyai **23** (1978), 741–749.

15. Jan van Mill, *Sixteen topological types in $\beta\omega$-ω*, Topol. Appl. **13** (1982), 43–57.

16. ———, *An introduction to $\beta\omega$*, in *Handbook of set-theoretic topology*, K. Kunen and J.E. Vaughan, ed., North-Holland Publ. Co., Amsterdam, New York, Oxford, 1984.

17. Mary Ellen Rudin, *Partial orders on the types of $\beta\mathbf{N}$*, Trans. Amer. Math. Soc. **155** (1971), 353–362.

18. Charles Waiveris, *Intersections of countably compact subspaces of βX*, Topol. Proc. **4** (1979), 177–191.

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