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MODIFICATIONS OF TOEPLITZ MATRICES: JUMP FUNCTIONS

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ABSTRACT. In [**1**] the authors have studied formal properties of the orthogonal polynomial sequences related to a modification of a measure on a Jordan curve by a finite number of jump functions in points of the curve. In this paper we analyze, for infinite Toeplitz matrices with nonsingular principal minors, the addition of jump functions in points placed outside (inside) the unit circle. Necessary and sufficient conditions for the regularity of the new Toeplitz moment matrix are provided and the relation with the polynomial modifications of the moments is established in a similar way to the Hankel case [**8**].

Introduction. Let μ be a Borel, finite and positive measure on the unit circle. In recent papers [**1, 2, 3**] we have studied formal properties of the monic orthogonal polynomial sequences (M.O.P.S.) corresponding to the measure u obtained by adding to μ a finite number of masses of Dirac on points of the circle. This problem appears in [**6**] and the generalization studied in [**1**] relates this type of modification with the polynomial ones.

In this paper we consider a situation as general as possible: Let M be an infinite, Hermitian and Toeplitz matrix with nonsingular principal minors. We consider a modification that represents an extension of the Dirac's delta in a point placed not necessarily on the circle. In Section 1 necessary and sufficient conditions for the new moment matrix $\mathcal{M}$ to be Hermitian, Toeplitz and with nonsingular principal minors are given. In Section 2 the connection between this type of modification and others introduced in [**5**] are provided. Some interesting examples are presented in Section 3 and, finally, if the moment functional L associated to M is semiclassical in the sense of [**9**], then it is shown in Section 4 that $\mathcal{L}$ (moment functional associated to $\mathcal{M}$) is semiclassical too. In this sense we extend a known result for Hankel matrices [**8**].

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1. Let $M = (c_n)_{n\in Z}$ be an infinite, Hermitian and Toeplitz matrix with nonsingular principal minors. We consider the space of the Laurent polynomials S, i.e., the linear closure of $(z^n)_{n\in Z}$, and the linear functional L defined in S by:

$$L(z^n) = c_n \qquad n > 0, \qquad L(z^{-n}) = \overline{c_n} = c_{-n} \qquad n > 0,$$
$$L(1) = c_0 \in \mathbf{R}\backslash\{0\}.$$

Definition 1.1. A linear functional satisfying the above conditions is called *regular*.

Let $(\phi_n(z))$ be the monic orthogonal polynomial sequence corresponding to L. If we denote by $k_n = L(\phi_n(z)\overline{\phi_n}(1/z))$, it is well known that $k_n/k_{n-1} = 1 - |\phi_n(0)|^2$ ([**6**]). We also define the sequence of n-kernels by $K_n(z,y) = \sum_{j=0}^{n} k_j^{-1}\phi_j(z)\overline{\phi_j(y)}$.

Given $a \in \mathbf{C}\backslash\{0\}$ and $\lambda \in \mathbf{R}\backslash\{0\}$, we consider the linear form $\mathcal{L} : S \to \mathbf{C}$ such that $\mathcal{L}(z^n) = L(z^n) + (\lambda^{-1}/2)(a^n + 1/\bar{a}^n)$. If $|a| = 1$, we are in the situation of [**2**] for L positive definite. The first question we try to solve is to study the properties of the functional $\mathcal{L}$, assuming $|a| \neq 1$. A similar question has been studied in [**4**] from another point of view.

Proposition 1.2. i) *$\mathcal{L}$ has a Hermitian and Toeplitz moment matrix $\mathcal{M}$.*

ii) *$\mathcal{M}$ has nonsingular principal minors if and only if*

$$\begin{vmatrix} K_n(a,a) & 2\lambda + K_n(a,1/\bar{a}) \\ 2\lambda + K_n(1/\bar{a},a) & K_n(1/\bar{a},1/\bar{a}) \end{vmatrix} \neq 0 \qquad \forall\, n \in \mathbf{N}.$$

Proof. The first part of the statement is obvious. In order to prove the second part, we use a known result ([**9**]) which says that the condition of nonsingularity of the principal minors is equivalent to the existence of a monic orthogonal polynomial sequence corresponding to that matrix.

"$\Rightarrow$" Assume $\mathcal{L}$ is regular and $(\psi_n(z))$ is the corresponding M.O.P.S., then for $n \geq 1$

$$\psi_n(z) = \phi_n(z) + \sum_{j=0}^{n-1} \mu_{nj}\phi_j(z) \tag{1}$$

where

$$\begin{aligned}\mu_{nj} L(\phi_j(z)\bar{\phi}_j(1/z)) &= \mathcal{L}(\psi_n(z)\overline{\phi_j}(1/z)) \\ &\quad - (\lambda^{-1}/2)(\psi_n(a)\overline{\phi_j}(1/a) + \psi_n(1/\bar{a})\overline{\phi_j(a)}) \\ &= -(\lambda^{-1}/2)(\psi_n(a)\overline{\phi_j(1/\bar{a})} + \psi_n(1/\bar{a})\overline{\phi_j(a)}).\end{aligned}$$

Therefore, by substituting in (1), we have

$$\begin{aligned}(2)\quad & \psi_n(z) \\ &= \phi_n(z) - (\lambda^{-1}/2)(\psi_n(a)K_{n-1}(z,1/\bar{a}) + \psi_n(1/\bar{a})K_{n-1}(z,a))\end{aligned}$$

and taking $z = a$ and $z = 1/\bar{a}$ we get

$$\begin{aligned}\phi_n(a) &= \psi_n(a)(1 + (\lambda^{-1}/2)K_{n-1}(a,1/\bar{a})) \\ &\quad + \psi_n(1/\bar{a})(\lambda^{-1}/2)K_{n-1}(a,a) \\ \phi_n(1/\bar{a}) &= \psi_n(a)(\lambda^{-1}/2)K_{n-1}(1/\bar{a},1/\bar{a}) \\ &\quad + \psi_n(1/\bar{a})(1 + (\lambda^{-1}/2)K_{n-1}(1/\bar{a},a))\end{aligned}$$

or, equivalently, in matrix form, $\psi_n(a)$ and $\psi_n(1/\bar{a})$ are solutions of the system

$$(3)\quad \begin{aligned}&\begin{pmatrix} 1 + \frac{\lambda^{-1}}{2}K_{n-1}(a,1/\bar{a}) & \frac{\lambda^{-1}}{2}K_{n-1}(a,a) \\ \frac{\lambda^{-1}}{2}K_{n-1}(1/\bar{a},1/\bar{a}) & 1 + \frac{\lambda^{-1}}{2}K_{n-1}(1/\bar{a},a)\end{pmatrix}\begin{pmatrix}\psi_n(a) \\ \psi_n(1/\bar{a})\end{pmatrix} \\ &\qquad = \begin{pmatrix}\phi_n(a) \\ \phi_n(1/\bar{a})\end{pmatrix} \qquad n \geq 1.\end{aligned}$$

If we denote by A_{n-1} the determinant of the above matrix of coefficients, then

$$\begin{aligned}A_0 &= \begin{vmatrix} 1 + \frac{\lambda^{-1}}{2}k_0^{-1} & \frac{\lambda^{-1}}{2}k_0^{-1} \\ \frac{\lambda^{-1}}{2}k_0^{-1} & 1 + \frac{\lambda^{-1}}{2}k_0^{-1}\end{vmatrix} \\ &= 1 + \lambda^{-1}k_0^{-1} = k_0^{-1}(k_0 + \lambda^{-1}) \\ &= k_0^{-1}(c_0 + \lambda^{-1}) = k_0^{-1}\mathcal{L}(1) \neq 0.\end{aligned}$$

On the other hand, if $A_{m-1} \neq 0$ for some m, (2) becomes

$$(4)\quad \psi_m(z) = \frac{\begin{vmatrix}\phi_m(z) & K_{m-1}(z,a) & K_{m-1}(z,1/\bar{a}) \\ \phi_m(a) & K_{m-1}(a,a) & 2\lambda + K_{m-1}(a,1/\bar{a}) \\ \phi_m(1/\bar{a}) & 2\lambda + K_{m-1}(1/\bar{a},a) & K_{m-1}(1/\bar{a},1/\bar{a})\end{vmatrix}}{\begin{vmatrix}K_{m-1}(a,a) & 2\lambda + K_{m-1}(1,1/\bar{a}) \\ 2\lambda + K_{m-1}(1/\bar{a},a) & K_{m-1}(1/\bar{a},1/\bar{a})\end{vmatrix}}$$

where the denominator is equal to $-4\lambda^2 A_{m-1}$. It follows from the hypothesis that $\mathcal{K}_m = \mathcal{L}(\psi_m(z)\bar{\psi}_m(1/z)) \neq 0$, $\mathcal{L}(\psi_m(z)\bar{\phi}_m(1/z)) \neq 0$ and taking into account (4), we obtain:

$$
\begin{aligned}
&\mathcal{L}(\psi_m(z)\bar{\phi}_m(1/z)) \\
&\quad = L(\psi_m(z)\bar{\phi}_m(1/z)) + (\lambda^{-1}/2)(\psi_m(a)\bar{\phi}_m(1/a) + \psi_m(1/\bar{a})\overline{\phi_m(a)}) \\
&\quad = k_m + \frac{1}{2\lambda A_{m-1}}\frac{\lambda^{-1}}{2}(\phi_m(a)\overline{\phi_m(1/\bar{a})}[2\lambda + K_{m-1}(1/\bar{a}, a)] \\
&\qquad\qquad - \phi_m(1/\bar{a})\overline{\phi_m(1/\bar{a})}K_{m-1}(a, a)) \\
&\quad + \frac{1}{2\lambda A_{m-1}}\frac{\lambda^{-1}}{2}(\overline{\phi_m(a)}\phi_m(1/\bar{a})[2\lambda + K_{m-1}(a, 1/\bar{a})] \\
&\qquad\qquad - \phi_m(a)\overline{\phi_m(a)}K_{m-1}(1/\bar{a}, 1/\bar{a})).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
4\lambda^2 A_m &= \begin{vmatrix} 2\lambda + K_m(a, 1/\bar{a}) & K_m(a, a) \\ K_m(1/\bar{a}, 1/\bar{a}) & 2\lambda + K_m(1/\bar{a}, a) \end{vmatrix} \\
&= \frac{1}{k_m}\begin{vmatrix} \phi_m(a)\overline{\phi_m(1/\bar{a})} & K_m(a, a) \\ \phi_m(1/\bar{a})\overline{\phi_m(1/\bar{a})} & 2\lambda + K_m(1/\bar{a}, a) \end{vmatrix} \\
&\quad + \begin{vmatrix} 2\lambda + K_{m-1}(a, 1/\bar{a}) & K_m(a, a) \\ K_{m-1}(1/\bar{a}, 1/\bar{a}) & 2\lambda + K_m(1/\bar{a}, a) \end{vmatrix} \\
&= \frac{1}{k_m}\begin{vmatrix} \phi_m(a)\overline{\phi_m(1/\bar{a})} & K_{m-1}(a, a) \\ \phi_m(1/\bar{a})\overline{\phi_m(1/\bar{a})} & 2\lambda + K_{m-1}(1/\bar{a}, a) \end{vmatrix} \\
&\quad + \begin{vmatrix} 2\lambda + K_{m-1}(a, 1/\bar{a}) & K_{m-1}(a, a) \\ K_{m-1}(1/\bar{a}, 1/\bar{a}) & 2\lambda + K_{m-1}(1/\bar{a}, a) \end{vmatrix} \\
&\quad + \frac{1}{k_m}\begin{vmatrix} 2\lambda + K_{m-1}(a, 1/\bar{a}) & \phi_m(a)\overline{\phi_m(a)} \\ K_{m-1}(1/\bar{a}, 1/\bar{a}) & \phi_m(1/\bar{a})\overline{\phi_m(a)} \end{vmatrix} \\
&= 4\lambda^2 A_{m-1} + \frac{1}{k_m}(\phi_m(a)\overline{\phi_m(1/\bar{a})}[2\lambda + K_{m-1}(1/\bar{a}, a)] \\
&\qquad - \phi_m(1/\bar{a})\overline{\phi_m(1/\bar{a})}K_{m-1}(a, a) \\
&\qquad + \overline{\phi_m(a)}\phi_m(1/\bar{a})[2\lambda + K_{m-1}(a, 1/\bar{a})] \\
&\qquad - \phi_m(a)\overline{\phi_m(a)}K_{m-1}(1/\bar{a}, 1/\bar{a})) \\
&= 4\lambda^2 A_{m-1} + \frac{1}{k_m}[\mathcal{L}(\psi_m(z)\bar{\phi}_m(1/z)) - k_m] \cdot 4\lambda^2 A_{m-1} \\
&= 4\lambda^2 A_{m-1}\mathcal{L}(\psi_m(z)\bar{\phi}_m(1/z))k_m^{-1}.
\end{aligned} \tag{5}
$$

In the hypothesis $A_{m-1} \neq 0$, from (5) $A_m \neq 0$ holds. Then, as $A_0 \neq 0$ we can deduce by induction $A_n \neq 0$ for $n \geq 1$ using the regularity of the linear functional $\mathcal{L}$, i.e., $\mathcal{L}(\psi_n(z)\bar{\phi}_n(1/z)) \neq 0$ for $n \geq 1$.

"$\Leftarrow$" It suffices to show that the sequence defined by (4) satisfies the following conditions:

$$\mathcal{L}(\psi_n(z)\bar{\psi}_m(1/z)) = 0 \qquad m \neq n$$
$$\mathcal{L}(\psi_n(z)\bar{\psi}_n(1/z)) \neq 0,$$

and it follows easily from (4) and (5). □

Remarks. 1. From (5) we deduce that $\mathcal{K}_n = k_n(A_n/A_{n-1})$ for $n \geq 1$. Moreover, taking into account the expressions of $\mathcal{M}_0$, M_0 and A_0, we get:

$$\det \mathcal{M}_n = A_n \det M_n.$$

(We denote by $\mathcal{M}_n$ and M_n the principal minors of $\mathcal{M}$ and M.)

2. From

$$A_{n-1} = \frac{\lambda^{-2}}{4}(|K_{n-1}(1/\bar{a}, a)|^2 - K_{n-1}(a,a)K_{n-1}(1/\bar{a}, 1/\bar{a})) + \frac{\lambda^{-1}}{2}(K_{n-1}(1/\bar{a}, a) + K_{n-1}(a, 1/\bar{a})) + 1$$

it follows that the functional $\mathcal{L}$ has associated a M.O.P.S. for every $\lambda \in \mathbf{C}\backslash\{0\}$ except, at most, a countable set.

3. (4) has a similar form to that obtained in [**1**] for the case of two masses placed on the curve.

4. If $|a| \to 1$ it is easy to get that the limit functional $\mathcal{L}$ is regular if and only if $\lambda + K_n(a,a) \neq 0$ for all $n \in \mathbf{N}$. In this case,

$$\psi_n(z) = \phi_n(z) - \frac{\phi_n(a)}{\lambda + K_{n-1}(a,a)} K_{n-1}(z,a) \qquad n \geq 1.$$

2. Let L be a functional and φ the associated indefinite inner product, and consider the functional L_1 such that the corresponding indefinite inner product is φ_1 defined as follows:

$$\begin{aligned}\phi_1(P(z),Q(z)) &= L_1(P(z)\bar{Q}(1/z))\\ &= \varphi((z-a)(z-1/\bar{a})P(z),(z-a)(z-1/\bar{a})Q(z)).\end{aligned}$$

In particular, if L is regular, then L_1 is regular if and only if

$$B_n = \begin{vmatrix} K_n(a,a) & K_n(a,1/\bar{a}) \\ K_n(1/\bar{a},a) & K_n(1/\bar{a},1/\bar{a}) \end{vmatrix} \neq 0 \qquad \forall\, n \geq 1.$$

Furthermore, if we denote by $Q_n(z)$ the M.O.P.S. corresponding to L_1, it is easy to show that:

$$(6) \quad (z-a)(z-1/\bar{a})Q_{n-2}(z) = \frac{\begin{vmatrix} \phi_n(z) & K_{n-1}(z,a) & K_{n-1}(z,1/\bar{a}) \\ \phi_n(a) & K_{n-1}(a,a) & K_{n-1}(a,1/\bar{a}) \\ \phi_n(1/\bar{a}) & K_{n-1}(1/\bar{a},a) & K_{n-1}(1/\bar{a},1/\bar{a}) \end{vmatrix}}{\begin{vmatrix} K_{n-1}(a,a) & K_{n-1}(a,1/\bar{a}) \\ K_{n-1}(1/\bar{a},a) & K_{n-1}(1/\bar{a},1/\bar{a}) \end{vmatrix}}.$$

Professor Garcia-Lazaro pointed out the above result to the authors [**5**]. (Compare this result with that obtained in [**6**] for L positive definite). Now then, by operating on (4), we get

(7)

$$\begin{aligned}\psi_m(z) &= \frac{1}{4\lambda A_{m-1}}\Bigg(-B_{m-1}(z-a)(z-1/\bar{a})Q_{m-2}(z) + 4\lambda^2\phi_m(z)\\ &\quad +2\lambda\begin{vmatrix} \phi_m(z) & K_{m-1}(z,1/\bar{a}) \\ \phi_m(a) & K_{m-1}(a,1/\bar{a}) \end{vmatrix} + 2\lambda\begin{vmatrix} \phi_m(z) & K_{m-1}(z,a) \\ \phi_m(1/\bar{a}) & K_{m-1}(1/\bar{a},a) \end{vmatrix}\Bigg)\\ &= \frac{1}{4\lambda A_{m-1}}(-B_{m-1}(z-a)(z-1/\bar{a})Q_{m-2}(z) + 4\lambda^2\phi_m(z)\\ &\qquad + 2\lambda Q_m^{(1)}(z) + 2\lambda Q_m^{(2)}(z))\end{aligned}$$

where

$$Q_m^{(1)}(z) \in ((z-1/\bar{a})\mathbf{P}_{m-2})^{\perp_m} \cap (z-a)\mathbf{P}_{m-1}$$

and

$$Q_m^{(2)}(z) \in ((z-a)\mathbf{P}_{m-2})^{\perp_m} \cap (z-1/\bar{a})\mathbf{P}_{m-1}.$$

Note that in the case $|a| = 1$, the result has been obtained in [**1**]. The fact that $Q_m^{(1)}(z) = Q_m^{(2)}(z) \in ((z-a)\mathbf{P}_{m-2})^{\perp_m} \cap (z-a)\mathbf{P}_{m-1}$ means that the sequence $(R_n(z))$, with $Q_m^{(1)}(z) = (z-a)K_{m-1}(a,a)R_{m-1}(z)$, is M.O.P.S. corresponding to the functional $L^{(1)}$ such that

$$\varphi^{(1)}(P(z),Q(z)) = L^{(1)}(P(z)\bar{Q}(1/z)) = \varphi((z-a)P(z),(z-a)Q(z)).$$

3. Next we give some applications of the previous sections. Consider for $|b| \neq 1$ the regular functional L such that the associated M.O.P.S. is $\phi_n(z) = z^{n-1}(z-b)$, $n \geq 1$. It is well known that L has the following moments:

$$c_n = b^n \qquad n \in \mathbf{N} \quad \text{and} \quad c_{-n} = \bar{b}^n \qquad n \in \mathbf{N}.$$

The corresponding functional $\mathcal{L}$ introduced in Section 1 will be regular if and only if

$$\begin{vmatrix} K_n(a,a) & 2\lambda + K_n(a,1/\bar{a}) \\ 2\lambda + K_n(1/\bar{a},a) & K_n(1/\bar{a},1/\bar{a}) \end{vmatrix} \neq 0.$$

Now then, assuming $|a| \neq 1$, we have:

$$\begin{aligned} K_n(a,a) &= 1 + \frac{|a-b|^2}{1-|b|^2}\frac{|a|^{2n}-1}{|a|^2-1} \\ K_n(1/\bar{a},1/\bar{a}) &= 1 + \frac{|1/\bar{a}-b|^2}{1-|b|^2}\frac{|a|^{2n}-1}{|a|^2-1}\frac{1}{|a|^{2n-2}} \\ K_n(a,1/\bar{a}) &= 1 + \frac{n}{1-|b|^2}(a-b)(1/a-\bar{b}). \end{aligned}$$

Therefore, for $\lambda \neq \lambda_n^{\pm}$, $n \in \mathbf{N}$, with

$$\lambda_n^{\pm} = \frac{\operatorname{Re} K_n(a,1/\bar{a}) \pm \sqrt{K_n(a,a)K_n(1/\bar{a},1/\bar{a}) - \operatorname{Im}^2 K_n(a,1/\bar{a})}}{2}$$

we have the regularity of $\mathcal{L}$.

The following are some interesting cases:

1. The situation corresponding to $b = 0$ has been studied in [**4**].

2. If $a = b$, we get $K_n(a,a) = K_n(a,1/\bar{a}) = 1$, $K_n(1/\bar{a},1/\bar{a}) = 1/|a|^{2n}$, that is, $\lambda_n^{\pm} = (-1 \pm 1/|a|^n)/2$. Furthermore, $K_{n-1}(z,a) = 1$, $K_{n-1}(a,1/\bar{a}) = z^{n-1}/a^{n-1}$ and therefore

$$\psi_n(z) = \frac{\begin{vmatrix} z^{n-1}(z-a) & 1 & \frac{z^{n-1}}{a^{n-1}} \\ 0 & 1 & 1+2\lambda \\ \frac{1-|a|^2}{(\bar{a})^n} & 1+2\lambda & \frac{1}{|a|^{2n-2}} \end{vmatrix}}{\begin{vmatrix} 1 & 1+2\lambda \\ 1+2\lambda & \frac{1}{|a|^{2n-2}} \end{vmatrix}}$$
$$= z^{n-1}(z-a) + \frac{(1-|a|^2)(1+2\lambda - \frac{z^{n-1}}{a^{n-1}})}{\bar{a}^n(1/|a|^{2n-2} - 1 - 4\lambda - 4\lambda^2)}$$
$$= z^{n-1}\left(z - a - \frac{1-|a|^2}{\bar{a}[1-|a|^{2n-2}(1+2\lambda)^2]}\right)$$
$$+ \frac{(1+2\lambda)(1-|a|^2)}{\bar{a}^n(1/|a|^{2n-2} - (1+2\lambda)^2)}$$
$$= z^{n-1}\left(z - \frac{|a|^{2n}(1+2\lambda)^2 - 1}{\bar{a}[|a|^{2n-2}(1+2\lambda)^2 - 1]}\right)$$
$$+ \frac{(1+2\lambda)(1-|a|^2)|a|^{2n-2}}{\bar{a}^n[1-(1+2\lambda)^2|a|^{2n-2}]}.$$

We remark that $\mathcal{L}$ is regular for $\lambda = -1/2$ and $\psi_n(z) = z^{n-1}(z - 1/\bar{a})$.

3. If $b = 1/\bar{a}$, then $K_n(a,a) = |a|^{2n}$, $K_n(a,1/\bar{a}) = K_n(1/\bar{a},1/\bar{a}) = 1$, that is, $\lambda_n^{\pm} = (-1 \pm |a|^n)/2$. Furthermore, $K_{n-1}(z,a) = z^{n-1}(\bar{a})^{n-1}$, $K_{n-1}(z,1/\bar{a}) = 1$, and therefore

$$\psi_n(z) = \frac{\begin{vmatrix} z^{n-1}(z-1/\bar{a}) & z^{n-1}(\bar{a})^{n-1} & 1 \\ a^{n-1}(a-1/\bar{a}) & |a|^{2n-2} & 2\lambda+1 \\ 0 & 2\lambda+1 & 1 \end{vmatrix}}{\begin{vmatrix} |a|^{2n-2} & 2\lambda+1 \\ 2\lambda+1 & 1 \end{vmatrix}}$$
$$= z^{n-1}(z-1/\bar{a}) - \frac{\frac{a^{n-1}(|a|^2-1)}{\bar{a}}(z^{n-1}(\bar{a})^{n-1} - (2\lambda+1))}{|a|^{2n-2} - (2\lambda+1)^2}$$
$$= z^{n-1}\left(z - 1/\bar{a} - \frac{|a|^{2n-2}(|a|^2-1)}{\bar{a}[|a|^{2n-2} - (2\lambda+1)^2]}\right)$$

$$+\frac{(2\lambda+1)a^{n-1}(|a|^2-1)}{\bar{a}[|a|^{2n-2}-(2\lambda+1)^2]}$$
$$=z^{n-1}\left(z-\frac{|a|^{2n-2}-(2\lambda+1)^2+|a|^{2n-2}(|a|^2-1)}{\bar{a}[|a|^{2n-2}-(2\lambda+1)^2]}\right)$$
$$+\frac{(2\lambda+1)a^{n-1}(|a|^2-1)}{\bar{a}[|a|^{2n-2}-(2\lambda+1)^2]}.$$

We remark that $\mathcal{L}$ is regular for $\lambda=-1/2$ and $\psi_n(z)=z^{n-1}(z-a)$.

4. In the case $|a|=1$, according to Remark 4 in Section 1, we get:

$$\psi_n(z)=\phi_n(z)-\frac{\phi_n(a)}{\lambda+K_{n-1}(a,a)}K_{n-1}(z,a)$$

with

$$K_{n-1}(z,a)=1+\frac{(z-b)(\bar{a}-\bar{b})}{1-|b|^2}\,\frac{z^{n-1}(\bar{a})^{n-1}-1}{z\bar{a}-1}$$

and

$$K_{n-1}(a,a)=1+(n-1)\frac{|a-b|^2}{1-|b|^2}.$$

Now then, for $\lambda\neq\lambda_n$ with $\lambda_n=-(1+(n-1)|a-b|^2/(1-|b|^2))$, we have

$$\psi_n(z)=z^{n-1}(z-b)-\frac{a^{n-1}|a-b|^2}{\lambda+1+(n-1)\frac{|a-b|^2}{1-|b|^2}}(z-b)\frac{z^{n-1}-a^{n-1}}{z-a}\,\frac{(\bar{a})^{n-2}}{1-|b|^2}$$
$$-\frac{a^{n-1}(a-b)}{\lambda+1+(n-1)\frac{|a-b|^2}{1-|b|^2}}.$$

4. In order to study the semiclassical character of the new functional, we give the following definitions [**9**]:

Definition 4.1. Given a linear functional L in S, we define the derivative DL as the linear functional such that $DL(P(z))=-iL(zP'(z))$ for all $P(z)\in S$.

Remark. $DL(z^n) = -i\,n\,c_n$ for all $n \in \mathbf{Z}\backslash\{0\}$ and $DL(1) = 0$. Then DL is not regular in the sense of Definition 1.1.

Definition 4.2. A regular Toeplitz functional L is semiclassical if there exist polynomials $A(z)$ and $B(z)$, $A(z) \neq 0$, such that the following equation holds: $D(A(z)L) = B(z)L$, where D is the derivative.

Proposition 4.1. *If L is semiclassical, then $\mathcal{L}$ is semiclassical too.*

Proof. Our aim is to prove that $\mathcal{L}$ satisfies $D((z-a)^2(z-1/\bar{a})^2A(z)\mathcal{L}) = (iz[(z-a)^2(z-1/\bar{a})^2]'A(z)+(z-a)^2(z-1/\bar{a})^2B(z))\mathcal{L}$, and therefore $\mathcal{L}$ is semiclassical. Indeed:

$$
\begin{aligned}
D((z-a)^2&(z-1/\bar{a})^2A(z)\mathcal{L}) \\
&= D((z-a)^2(z-1/\bar{a})^2A(z)L) \\
&\quad + D\Big((z-a)^2(z-1/\bar{a})^2A(z)\frac{\lambda}{2}(\delta_a+\delta_{1/\bar{a}})\Big) \\
&= (z-a)^2(z-1/\bar{a})^2D(A(z)L) \\
&\quad + iz((z-a)^2(z-1/\bar{a})^2)'A(z)L \\
&\quad + (z-a)^2(z-1/\bar{a})^2D\Big(A(z)\frac{\lambda}{2}(\delta_a+\delta_{1/\bar{a}})\Big) \\
&\quad + iz((z-a)^2(z-1/\bar{a})^2)'A(z)\frac{\lambda}{2}(\delta_a+\delta_{1/\bar{a}}) \\
&= (z-a)^2(z-1/\bar{a})^2B(z)L \\
&\quad + (z-a)^2(z-1/\bar{a})^2D\Big(A(z)\frac{\lambda}{2}(\delta_a+\delta_{1/\bar{a}})\Big) \\
&\quad + iz((z-a)^2(z-1/\bar{a})^2)'A(z)\mathcal{L}.
\end{aligned}
$$

Since $(z-a)^2(z-1/\bar{a})^2D(A(z)(\lambda/2)(\delta_a+\delta_{1/\bar{a}}))=0$,

$$(z-a)^2(z-1/\bar{a})^2B(z)\frac{\lambda}{2}(\delta_a+\delta_{1/\bar{a}})=0,$$

and taking into account the definition of $\mathcal{L}$, we have

$$D((z-a)^2(z-1/\bar{a})^2A(z)\mathcal{L})$$
$$= (z-a)^2(z-1/\bar{a})^2B(z)\mathcal{L} - (z-a)^2(z-1/\bar{a})^2B(z)\frac{\lambda}{2}(\delta_a+\delta_{1/\bar{a}})$$
$$+ iz((z-a)^2(z-1/\bar{a})^2)'A(z)\mathcal{L}$$
$$= ((z-a)^2(z-1/\bar{a})^2B(z) + iz[(z-a)^2(z-1/\bar{a})^2]'A(z))\mathcal{L}.$$

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