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SOME EXAMPLES OF MIXING RANDOM FIELDS

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ABSTRACT. Several classes of strictly stationary random fields are constructed, with various combinations of "strong mixing" properties. The purpose is to "separate" various mixing assumptions that are used in the literature on limit theory for random fields.

1. Introduction. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space. For any two σ-fields $\mathcal{A}$, $\mathcal{B} \subset \mathcal{F}$ define the following measures of dependence:

$$\alpha(\mathcal{A},\mathcal{B}) := \sup_{A\in\mathcal{A}, B\in\mathcal{B}} |P(A\cap B) - P(A)P(B)|,$$

$$\rho(\mathcal{A},\mathcal{B}) := \sup_{f\in L_2(\mathcal{A}), g\in L_2(\mathcal{B})} |\mathrm{Corr}\,(f,g)|,$$

$$\beta(\mathcal{A},\mathcal{B}) := \sup \frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_i\cap B_j) - P(A_i)P(B_j)|$$

where this last sup is taken over all pairs of partitions $\{A_1, \dots, A_I\}$ and $\{B_1, \dots, B_J\}$ of Ω such that $A_i \in \mathcal{A}$ for each i and $B_j \in \mathcal{B}$ for each j. The following inequalities are elementary:

$$\begin{aligned} &4\alpha(\mathcal{A},\mathcal{B}) \le \rho(\mathcal{A},\mathcal{B}) \le 1, \quad \text{and} \\ &2\alpha(\mathcal{A},\mathcal{B}) \le \beta(\mathcal{A},\mathcal{B}) \le 1. \end{aligned} \tag{1.1}$$

Suppose d is a positive integer. For each $l := (l_1, \dots, l_d) \in \mathbf{Z}^d$ denote the usual Euclidean norm $||l|| := (l_1^2 + \cdots + l_d^2)^{1/2}$. For any two nonempty disjoint subsets $S, T \subset \mathbf{Z}^d$, denote the distance between them by

$$\mathrm{dist}\,(S,T) := \inf_{s\in S, t\in T} ||s-t||.$$

Now suppose $X := (X_t, t \in \mathbf{Z}^d)$ is a strictly stationary random field on our probability space $(\Omega, \mathcal{F}, P)$. For each real number $r \ge 1$, and

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each pair of elements $j, k \in \{1, 2, 3, \dots\} \cup \{\infty\}$, define the following dependence coefficients:

$$\alpha_{j,k}(X, r) := \sup \alpha(\sigma(X_t, t \in S), \sigma(X_t, t \in T)),$$
$$\rho_{j,k}(X, r) := \sup \rho(\sigma(X_t, t \in S), \sigma(X_t, t \in T)),$$
$$\beta_{j,k}(X, r) := \sup \beta(\sigma(X_t, t \in S), \sigma(X_t, t \in T)),$$

where all three sups are taken over all pairs of disjoint nonempty subsets $S, T \subset \mathbf{Z}^d$ such that $\operatorname{dist}(S, T) \geq r$, $\operatorname{card} S \leq j$, and $\operatorname{card} T \leq k$. Here $\sigma(\cdots)$ denotes the σ-field generated by $(\cdots)$, and "card" denotes cardinality.

In the literature there are a large number of articles and books dealing with limit theory for strictly stationary random fields satisfying various "strong mixing" conditions. In some references, such as Guyon and Richardson [**8**] and Rosenblatt [**12**], the condition used was

$$\alpha_{\infty,\infty}(X, r) \to 0 \quad \text{as } r \to \infty. \tag{1.2}$$

Others, such as Neaderhouser [**10**] and Tran [**14**], used the weaker condition

$$\left[\sup_{k \geq 1} \frac{\alpha_{k,\infty}(X, r)}{k}\right] \to 0 \quad \text{as } r \to \infty. \tag{1.3}$$

This is equivalent to, for all $k = 1, 2, 3, \dots$, $\lim_{r \to \infty} \alpha_{k,\infty}(X, r) = 0$. However, the use of (1.3) in [**10, 14**] also involved an assumed rate of convergence of its left hand side as $r \to \infty$. Sherman [**13**] assumed (also with a rate of convergence) the similar but weaker condition

$$\left[\sup_{k \geq 1} \frac{\alpha_{k,k}(X, r)}{k}\right] \to 0 \quad \text{as } r \to \infty. \tag{1.4}$$

For further work on limit theory for mixing random fields, see [**3, 6, 7, 8, 9, 10, 12, 14, 16**] and the references therein.

There are trivial inequalities such as $\alpha_{k,k}(X, r) \leq \alpha_{k,\infty}(X, r)$, or $\alpha_{k,k}(X, r) \leq \alpha_{k,k}(X, s)$ for $s \leq r$. From such trivial inequalities and (1.1), one obtains "obvious relations" involving conditions (1.2), (1.3), (1.4), and similar mixing conditions based on $\beta_{j,k}(X, r)$ and $\rho_{j,k}(X, r)$. The main purpose of this paper is to construct some examples to indicate that there apparently are not too many connections between these mixing conditions aside from such "obvious relations." For example, Theorem 2 below will show that assumption (1.4) with (say) exponential mixing rate does not imply (1.3).

However, here are a couple of not so obvious relations for strictly stationary random fields. First, the condition $\lim_{r\to\infty} \rho_{\infty,\infty}(X,r) = 0$ is equivalent to (1.2), by [**4**, Remarks 1, 2, 3]. Second, the condition $\lim_{r\to\infty} \beta_{\infty,\infty}(X,r) = 0$ implies that there exists an $r \ge 1$ such that $\beta_{\infty,\infty}(X,r) = 0$ by [**2**, Remarks 2 and 3]; this fact was based on examples given by Dobrushin [**6**, p. 205] and Zhurbenko [**16**, p. 8, Example 2.1].

In the statement of our theorems, for a given positive integer d, the origin of $\mathbf{Z}^d$ will be denoted simply by 0 and a "nonzero element of $\mathbf{Z}^d$" will mean any element of $\mathbf{Z}^d$ except the origin. In context this should not cause any confusion.

Theorem 1. *Suppose that d is a positive integer. Suppose that $L_1, L_2, L_3, \dots$ is a sequence of nonzero elements of $\mathbf{Z}^d$ such that for all $n \ge 1$, $||L_n|| < ||L_{n+1}||$. Define the (strictly increasing) sequence of positive numbers $r_1, r_2, r_3, \dots$ by $r_n := ||L_n||$. Suppose that $c_1, c_2, c_3, \dots$ is a sequence of positive numbers such that $c_1 \le 1/8$ and for all $n \ge 1$, $c_{n+1} \le c_n/2$. Suppose $0 < q \le 1$.*

Then there exists a strictly stationary random field $X := (X_t, t \in \mathbf{Z}^d)$ with the following properties:

(1) *The distribution of X_0 does not have any atoms.*

(2) *For all $n \ge 1$, $\alpha(\sigma(X_0), \sigma(X_{L(n)})) \ge qc_n/2$ and $\rho(\sigma(X_0), \sigma(X_{L(n)})) = q$.*

(3) *For each $n \ge 1$, each integer k such that $1 \le k \le 1/(8c_n)$, one has that*

$$\alpha_{k,k}(X, r_n)/k \ge qc_n/2.$$

(4) *For each $n \ge 1$, each $k \ge 1$, one has that*

$$\beta_{k,\infty}(X, r_n)/k \le 8qc_n.$$

(5) *For all* $r \geq 1$, $\alpha_{\infty,\infty}(X,r) \geq q/(2\pi)$.

(*If* $q = 1$, *then, for all* $r \geq 1$, $\alpha_{\infty,\infty}(X,r) = 1/4$.)

(6) *For all* $r \geq 1$, $\rho_{1,1}(X,r) = \rho_{\infty,\infty}(X,r) = q$.

Theorem 2. *Suppose that* d, $(L_1, L_2, L_3, \dots)$, $(r_1, r_2, r_3, \dots)$, $(c_1, c_2, c_3, \dots)$, *and* q *are as in the hypothesis of Theorem 1. Then there exists a strictly stationary random field* $X := (X_t, t \in \mathbf{Z}^d)$ *with the following properties*:

(1) *The distribution of* X_0 *does not have any atoms.*

(2) *For all* $n \geq 1$, $\alpha(\sigma(X_0),\ \sigma(X_{L(n)})) \geq qc_n/2$ *and* $\rho(\sigma(X_0), \sigma(X_{L(n)})) = q$.

(3) *For each* $n \geq 1$, *each integer* k *such that* $1 \leq k \leq 1/(8c_n)$, *one has that*

$$\alpha_{k,k}(X, r_n)/k \geq qc_n/2.$$

(4) *For each* $n \geq 1$, *each* $k \geq 1$, *one has that*

$$\beta_{k,k}(X, r_n)/k \leq 8qc_n.$$

(5) *For all* $r \geq 1$, $\alpha_{1,\infty}(X,r) = q/4$.

(6) *For all* $r \geq 1$, $\rho_{1,1}(X,r) = \rho_{\infty,\infty}(X,r) = q$.

Theorem 3. *Suppose* d *is a positive integer. Suppose* $g : [1,\infty) \to (0,1]$ *is a nonincreasing function such that* $\lim_{r\to\infty} g(r) = 0$. *Then there exists a strictly stationary random field* $X := (X_t, t \in \mathbf{Z}^d)$ *with the following properties*:

(1) *The distribution of* X_0 *does not have any atoms.*

(2) *For all* $L \in \mathbf{Z}^d - \{0\}$, $\alpha(\sigma(X_0), \sigma(X_L)) = (1/4)g(||L||)$ *and* $\rho(\sigma(X_0), \sigma(X_L)) = g(||L||)$.

(3) *For all* $r \in [1,\infty)$, $4\alpha_{\infty,\infty}(X,r) \leq \rho_{\infty,\infty}(X,r) \leq g(r)$.

(4) *For all* $L \in \mathbf{Z}^d$, $\beta(\sigma(X_0), \sigma(X_L)) = 1$. *Hence for all* $r \in [1,\infty)$, $\beta_{1,1}(X,r) = 1$.

Theorems 1, 2, and 3 will be proved in Sections 2, 3, and 4, respectively. In what follows, we shall use the following notations: For positive numbers a_n and b_n, $n = 1, 2, 3, \dots$, the notation $a_n \sim b_n$ (as $n \to \infty$) means $\lim_{n\to\infty} a_n/b_n = 1$, and the notation $a_n \asymp b_n$ means that, for the sequence a_n/b_n, $n = 1, 2, 3, \dots$, both the lim sup and the lim inf are finite and positive. The indicator function of an event A is denoted $I(A)$. Subscripts of the form a_b will be written $a(b)$. We shall conclude Section 1 with a series of remarks.

Remark 1. The hypothesis of Theorems 1 and 2 implies that $r_n := ||L_n|| \to \infty$ as $n \to \infty$. (There are only finitely many elements of $\mathbf{Z}^d$ within a given distance from the origin.)

Remark 2. Let us look at property (1) in Theorem 1. In the random field X constructed for Theorem 1, the random variables X_t will in fact be positive, with a positive (but not continuous) density on $(0, \infty)$. However, one can take X_0 to have any prescribed distribution that does not have any atoms; and, hence, X_0 can have prescribed moment properties such as, e.g., $E|X_0|^5 < \infty$ and $E|X_0|^{5+\varepsilon} = \infty$ for all $\varepsilon > 0$. One simply replaces each X_t by $h(X_t)$ where $h : (0, \infty) \to \mathbf{R}$ is an appropriate strictly increasing function. (By taking h strictly increasing, one has $\sigma(h(X_t)) = \sigma(X_t)$ for each t, and hence the mixing properties (2)–(6) in Theorem 1 are preserved.)

Remark 3. The point of property (2) in Theorem 1 is that the dependence properties of X can involve (essentially) arbitrary "directions" in $\mathbf{Z}^d$. For example, one can take the vectors L_n such that $||L_n|| \sim n$ as $n \to \infty$ and the unit vectors $L_n/||L_n||$ are dense in the unit sphere in $\mathbf{R}^d$.

Remark 4. Referring to properties (3) and (4) in Theorem 1, let us consider for each $r \geq 1$ the two quantities

$$\sup_{k\geq 1} \frac{\alpha_{k,k}(X,r)}{k} \quad \text{and} \quad \sup_{k\geq 1} \frac{\beta_{k,\infty}(X,r)}{k}. \tag{1.5}$$

Of course, the two related quantities $\sup_{k\geq 1} k^{-1}\alpha_{k,\infty}(X,r)$ and $\sup_{k\geq 1} k^{-1}\beta_{k,k}(X,r)$ are in between. By appropriate choices of the vectors L_n and numbers c_n (and regardless of the choice of q), one can get the two quantities in (1.5) to approach 0 together at essentially any of the usual rates as $r \to \infty$. For example, for $\theta > 0$ one can have

$$[\text{both quantities in (1.5)}] \asymp (\log r)^{-\theta} \quad \text{as } r \to \infty,$$

by choosing $c_n := (1/4) \cdot 2^{-n}$ and L_n such that $||L_n|| \sim \exp(2^{n/\theta})$ as $n \to \infty$. Similarly, for $\theta > 0$ one can have

$$[\text{both quantities in (1.5)}] \asymp r^{-\theta} \quad \text{as } r \to \infty,$$

by choosing $c_n := (1/4) \cdot 2^{-n}$ and L_n such that $||L_n|| \sim 2^{n/\theta}$ as $n \to \infty$. One can get the two quantities in (1.5) to decay together at an exponential (or even faster) rate by taking, e.g., $||L_n|| \sim n$ as $n \to \infty$ and letting c_n decrease at an appropriate rate faster than 2^{-n}.

Remark 5. The point of property (5) of Theorem 1 is that such conditions as $\sup_{k\geq 1} k^{-1}\beta_{k,\infty}(X,r) \to 0$ (as $r \to \infty$) do not imply $\alpha_{\infty,\infty}(X,r) \to 0$.

Remark 6. Referring to property (6) in Theorem 1, a sufficiently small positive value of $\lim_{r\to\infty} \rho_{\infty,\infty}(X,r)$ can imply some nice moment properties and thereby facilitate the proofs of some limit theorems. (For an example of this in a related context, see Peligrad [**11**]). In particular, one has the following proposition:

Proposition. *Suppose d is a positive integer and $q_1, q_2, q_3, \ldots$ is a nonincreasing sequence of numbers in $[0,1]$ such that $\lim_{n\to\infty} q_n < 1/128$. Then there exists a positive constant $C := C(d, q_1, q_2, q_3, \ldots)$ with the following property: Suppose $X := (X_t, t \in \mathbf{Z}^d)$ is a strictly stationary random field such that $EX_0 = 0$, $EX_0^4 < \infty$, and for all $n \geq 1$, $\rho_{\infty,\infty}(X,n) \leq q_n$; then, for every nonempty finite set $S \subset \mathbf{Z}^d$, one has that*

$$E\left(\sum_{t\in S} X_t\right)^4 \leq C[(\text{card}\, S)(EX_0^4) + (\text{card}\, S)^2(EX_0^2)^2].$$

To prove this proposition, first let $m \geq 1$ be such that $q_m \leq 1/128$, and let $C := 24m^{4d}$. Now, if the random field X is as in the proposition, then by simple arithmetic, the random field $X^* := (X_{mt}, t \in \mathbf{Z}^d)$ satisfies (in essence) the hypothesis of [**3**, Lemma 5]. Applying that lemma to X^* and then imitating the proof of [**3**, Lemmas 2, 6] one gets the proposition. With a better proof, the $1/128$ in the proposition can be relaxed.

Remark 7. Let us say that a set $S \subset \mathbf{Z}^d$ is a "coordinate half-space" if it is of one of the forms $\{l := (l_1, \dots, l_d) \in \mathbf{Z}^d : l_j \le m\}$ or $\{l := (l_1, \dots, l_d) \in \mathbf{Z}^d : l_j \ge m\}$ where $j \in \{1, \dots, d\}$ and $m \in \mathbf{Z}$. Suppose that for each $r \ge 1$ one defines

$$\alpha_{chs}(X, r) := \sup \alpha(\sigma(X_t, t \in S), \sigma(X_t, t \in T))$$

and

$$\rho_{chs}(X, r) := \sup \rho(\sigma(X_t, t \in S), \sigma(X_t, t \in T))$$

where each sup is taken over all pairs of coordinate half-spaces $S, T \subset \mathbf{Z}^d$ such that $\operatorname{dist}(S, T) \ge r$. One can verify that for our random field X in Theorem 1, one has $\rho_{chs}(X, r) = q$ for all $r \ge 1$. (This is just an elementary consequence of properties (2) and (6).) In the case $d \ge 2$ one can use [**4**, Theorem 1(A) and Remark 2] to show that X consequently also satisfies $\alpha_{chs}(X, r) \ge q/(2\pi)$ for all $r \ge 1$.

Remark 8. For concreteness, Remarks 2–7 were focused on Theorem 1; but parts of Remarks 2–7 carry over to Theorems 2 and 3.

Remark 9. For a given $l := (l_1, \dots, l_d) \in \mathbf{Z}^d$, we took $||l||$ to be the Euclidean norm. Suppose that we had instead taken the norm $||l||_\infty := \max_{1 \le j \le d} |l_j|$, and suppose we had used that norm in the definitions of $\alpha_{m,n}(X, r)$, $\beta_{m,n}(X, r)$, and $\rho_{m,n}(X, r)$ and in the statements of Theorems 1, 2, and 3. Then these theorems would still hold verbatim, with the same proofs. The above remarks would also carry over with respect to this norm. These comments are also true for other "reasonable" norms, such as $||l||_p := (\sum_{j=1}^d |l_j|^p)^{1/p}$ where p is any fixed number in $[1, \infty)$. Of course, throughout the rest of this paper, we shall take $|| \cdot ||$ to mean our original Euclidean norm.

Remark 10. The random fields X constructed here for Theorems 1, 2, and 3 each have the property that the tail σ-field

$$\bigcap_{S \subset \mathbf{Z}^d, (\operatorname{card} S) < \infty} \sigma(X_t, t \in \mathbf{Z}^d - S)$$

is trivial, i.e., contains only events of probability 0 or 1. The argument is elementary—essentially just that of Kolmogorov's 0-1 law. (Note that if $d = 1$, it is the "double tail" σ-field that is discussed here.)

2. Proof of Theorem 1. Let us start with a lemma.

Lemma 2.1. *Suppose $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots$ and $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \dots$ are σ-fields and the σ-fields $\mathcal{A}_n \vee \mathcal{B}_n$, $n = 1, 2, 3, \dots$, are independent. Then*

(i) $\rho(\vee_{n=1}^{\infty} \mathcal{A}_n, \vee_{n=1}^{\infty} \mathcal{B}_n) = \sup_{n \ge 1} \rho(\mathcal{A}_n, \mathcal{B}_n)$, *and*

(ii) $\beta(\vee_{n=1}^{\infty} \mathcal{A}_n, \vee_{n=1}^{\infty} \mathcal{B}_n) \le \sum_{n=1}^{\infty} \beta(\mathcal{A}_n, \mathcal{B}_n)$.

Part (i) is due to Csaki and Fischer [**5**]; a short proof of it can be found in Witsenhausen [**15**, p. 105, Theorem 1]. Part (ii) can be found in [**1**, p. 1318, Lemma 2.1].

Now let us turn our attention to a class of random sequences that will be used as "building blocks" for the random field in Theorem 1.

Definition 2.2. Suppose $0 < c \le 1/2$ and $0 < q \le 1$. A random sequence (indexed by $\mathbf{Z}$) is said to have the *$\mathcal{S}(c,q)$-distribution* if it has the same distribution as the random sequence $Y := (Y_t, t \in \mathbf{Z})$ defined below: First let $((V_t, W_t), t \in \mathbf{Z})$ be a sequence of i.i.d. random vectors with the following marginal distribution:

$$P((V_0, W_0) = (0,0)) = (1-q)(1-c)^2 + q(1-c),$$
$$P((V_0, W_0) = (0,1)) = P((V_0, W_0) = (1,0)) = (1-q)c(1-c), \quad \text{and}$$
$$P((V_0, W_0) = (1,1)) = (1-q)c^2 + qc.$$

Now let the random sequence $Y := (Y_t, t \in \mathbf{Z})$ be defined as follows: for all $t \in \mathbf{Z}$, $Y_t := 2V_t + W_{t-1}$.

Lemma 2.3. *Suppose $0 < c \le 1/2$ and $0 < q \le 1$. Suppose $Y := (Y_t, t \in \mathbf{Z})$ is a random sequence with the $\mathcal{S}(c,q)$-distribution. Then this sequence Y has the following properties*:

(1) *Y is strictly stationary and 1-dependent.*

(2) *The distribution of the random variable Y_0 is as follows*:

$$P(Y_0 = 0) = (1-c)^2,$$
$$P(Y_0 = 1) = P(Y_0 = 2) = c(1-c),$$
$$P(Y_0 = 3) = c^2, \quad \textit{and}$$
$$P(Y_0 \notin \{0,1,2,3\}) = 0.$$

(3) $\alpha(\sigma(Y_0), \sigma(Y_1)) \geq qc/2$ *and* $\rho(\sigma(Y_0), \sigma(Y_1)) = q$.

(4) *For every integer k such that $1 \leq k \leq \max\{1, 1/(8c)\}$, one has that $k^{-1}\alpha_{k,k}(Y,1) \geq qc/2$.*

(5) *For all $k = 1, 2, 3, \ldots$, $k^{-1}\beta_{k,\infty}(Y,1) \leq 4qc$.*

(6) $\rho_{\infty,\infty}(Y,1) = q$.

Proof. Without loss of generality, we assume that the sequence Y is accompanied by a sequence $((V_t, W_t), t \in \mathbf{Z})$ such that all conditions in Definition 2.2 are fulfilled. For technical convenience, deleting a null set from our probability space if necessary, we assume further that the random variables V_t and W_t take only the values 0 and 1, and hence the random variables Y_t take only the values 0, 1, 2, and 3. We thus have (by Definition 2.2), for all $t \in \mathbf{Z}$,

$$\sigma(Y_t) = \sigma(V_t, W_{t-1}). \tag{2.1}$$

By trivial arithmetic, one has, for all $t \in \mathbf{Z}$,

$$\begin{gathered} P(V_t = 0) = P(W_t = 0) = 1 - c \quad \text{and} \\ P(V_t = 1) = P(W_t = 1) = c. \end{gathered} \tag{2.2}$$

Properties (1) and (2) in Lemma 2.3 are elementary consequences of (2.2) and Definition 2.2.

In order to facilitate the proof of properties (3)–(6), we need the following facts:

$$\begin{gathered} \alpha(\sigma(V_0), \sigma(W_0)) = qc(1-c), \\ \beta(\sigma(V_0), \sigma(W_0)) = 2qc(1-c), \quad \text{and} \\ \rho(\sigma(V_0), \sigma(W_0)) = q. \end{gathered} \tag{2.3}$$

The first two of these equations are elementary consequences of Definition 2.2. To prove the third, first note that, since V_0 can take only two values (0 and 1), any $\sigma(V_0)$-measurable random variable can be represented in the form $aV_0 + b$ with a, b real. The same holds for W_0. Hence $\rho(\sigma(V_0), \sigma(W_0)) = |\text{Corr}\,(V_0, W_0)|$, which equals q by simple arithmetic. □

Proof of (3). By (2.1) and (2.3),

$$\alpha(\sigma(Y_0), \sigma(Y_1)) \ge \alpha(\sigma(V_0), \sigma(W_0)) = qc(1-c) \ge qc/2.$$

By (2.1), Definition 2.2, Lemma 2.1, and (2.3),

$$\begin{aligned}\rho(\sigma(Y_0), \sigma(Y_1)) &= \sup\{\rho(\sigma(V_0), \sigma(W_0)), \rho(\sigma(W_{-1}), \sigma(V_1))\} \\ &= \rho(\sigma(V_0), \sigma(W_0)) = q. \qquad \square\end{aligned}$$

Proof of (4). For $k = 1$, this just follows from (3).

Suppose instead that $2 \le k \le 1/(8c)$. Define the events

$$A := \{Y_1 = Y_3 = Y_5 = \cdots = Y_{2k-1} = 0\}$$

and

$$B := \{Y_2 = Y_4 = Y_6 = \cdots = Y_{2k} = 0\}.$$

By (2.2) and Definition 2.2 and an elementary calculation,

$$P(A) = P(B) = (1-c)^{2k}.$$

Also,

$$\begin{aligned}P(A \cap B) &= P(W_0 = 0, V_{2k} = 0, \text{ and } (V_j, W_j) = (0,0) \\ &\qquad\qquad \forall\, j = 1, 2, \dots, 2k-1) \\ &= (1-c)^2 \cdot [(1-q)(1-c)^2 + q(1-c)]^{2k-1} \\ &= (1-c)^{2k+1} \cdot [(1-q)(1-c) + q]^{2k-1}.\end{aligned}$$

Now for $0 < y < x$ one has the arithmetic $x^{2k-1} - y^{2k-1} = (x-y) \cdot \prod_{j=0}^{2k-2} x^j y^{2k-2-j} \geq (x-y)(2k-1)y^{2k-2}$. Hence

$$\begin{aligned}\alpha_{k,k}(Y,1) &\geq P(A \cap B) - P(A)P(B) \\ &= (1-c)^{2k+1}[[(1-q)(1-c)+q]^{2k-1} - (1-c)^{2k-1}] \\ &\geq (1-c)^{2k+1}[[(1-q)(1-c)+q] - (1-c)] \\ &\quad \cdot (2k-1)(1-c)^{2k-2} \\ &= (1-c)^{4k-1}[cq](2k-1) \\ &\geq (1-4kc)(cq)k \\ &\geq cqk/2\end{aligned}$$

(since $4ck \leq 1/2$). Thus, (4) holds. □

Proof of (5). Suppose k is a positive integer. Suppose S is a subset of $\mathbf{Z}$ such that $(\operatorname{card} S) \leq k$. Denote its complement $T := \mathbf{Z} - S$. To prove (5) it suffices to prove that

$$\beta(\sigma(Y_t, t \in S), \sigma(Y_t, t \in T)) \leq 4qck. \tag{2.4}$$

Define the sets $S^* := \{t \in \mathbf{Z} : t+1 \in S\}$ and $T^* := \{t \in \mathbf{Z} : t+1 \in T\}$. The sets S^* and T^* are disjoint, and their union is $\mathbf{Z}$. The sets $S \cap S^*$, $S \cap T^*$, $T \cap S^*$, $T \cap T^*$ form a partition of $\mathbf{Z}$. By (2.1), Definition 2.2, Lemma 2.1, and (2.3),

$$\begin{aligned}\beta(\sigma(Y_t, t \in S), \sigma(Y_t, t \in T)) &\leq \sum_{t \in S \cap T^*} \beta(\sigma(V_t), \sigma(W_t)) \\ &\quad + \sum_{t \in T \cap S^*} \beta(\sigma(W_t), \sigma(V_t)) \\ &\quad + \sum_{t \in S \cap S^*} \beta(\sigma(V_t, W_t), \{\Omega, \phi\}) \\ &\quad + \sum_{t \in T \cap T^*} \beta(\{\Omega, \phi\}, \sigma(V_t, W_t)) \\ &= [\operatorname{card}(S \cap T^*) + \operatorname{card}(T \cap S^*)] \\ &\quad \cdot 2qc(1-c) + 0 \\ &\leq 2k \cdot 2qc(1-c).\end{aligned}$$

Thus, (2.4) holds. This completes the proof of (5). □

Proof of (6). Suppose S is an arbitrary subset of $\mathbf{Z}$ such that $S \neq \phi$ and $S \neq \mathbf{Z}$. Denote its complement $T := \mathbf{Z} - S$. Defining the sets S^* and T^* as in the proof of (5), we have by (2.1), Definition 2.2, Lemma 2.1, and (2.3),

$$\begin{aligned}\rho(\sigma(Y_t, t \in S), \sigma(Y_t, t \in T)) &= \sup \begin{cases} \rho(\sigma(V_t), \sigma(W_t)), & t \in S \cap T^* \\ \rho(\sigma(W_t), \sigma(V_t)), & t \in T \cap S^* \\ \rho(\sigma(V_t, W_t), \{\Omega, \phi\}), & t \in S \cap S^* \\ \rho(\{\Omega, \phi\}, \sigma(V_t, W_t)), & t \in T \cap T^* \end{cases} \\ &= q\end{aligned}$$

(since $S \cap T^*$ or $T \cap S^*$ is nonempty). Thus (6) holds. □

Definition 2.4. Suppose d is a positive integer. Suppose $L \neq 0$ is an element of $\mathbf{Z}^d$. The *L-partition* of $\mathbf{Z}^d$ is the (unique) partition of $\mathbf{Z}^d$ whose members are sets of the form $\{\dots, l - 2L, l - L, l, l + L, \dots\}$ with $l \in \mathbf{Z}^d$.

Of course, if $d = 1$ then the number of members of this partition is finite. If, instead, $d \geq 2$, then the partition is countably infinite.

Definition 2.5. Suppose $0 < c \leq 1/2$, $0 < q \leq 1$, d is a positive integer, and $L \neq 0$ is an element of $\mathbf{Z}^d$. A random field $Z := (Z_t, t \in \mathbf{Z}^d)$ is said to have the $\mathcal{T}(c, q, d, L)$-*distribution* if it has the following two properties:

(1) For each $l \in \mathbf{Z}^d$, the random sequence $(Z_{l+jL}, j \in \mathbf{Z})$ has the $\mathcal{S}(c, q)$-distribution.

(2) Letting $S_1, S_2, S_3, \dots$ denote the members of the L-partition of $\mathbf{Z}^d$, one has that the σ-fields $\sigma(Z_t, t \in S_1)$, $\sigma(Z_t, t \in S_2)$, $\sigma(Z_t, t \in S_3), \dots$ are independent.

In what follows, if d is a positive integer and $r \geq 1$ is a real number, a random field $Y := (Y_t, t \in \mathbf{Z}^d)$ is said to be *r-dependent* if it has the following property: For any two nonempty subsets $S, T \subset \mathbf{Z}^d$ with dist $(S,T) > r$, one has that the σ-fields $\sigma(Y_t, t \in S)$ and $\sigma(Y_t, t \in T)$ are independent.

Lemma 2.6. *Suppose* $0 < c \le 1/2$, $0 < q \le 1$, *d is a positive integer, $L \ne 0$ is an element of $\mathbf{Z}^d$, and $Z := (Z_t, t \in \mathbf{Z}^d)$ is a random field with the $\mathcal{T}(c,q,d,L)$-distribution. Define the positive number $r := ||L||$. Then the random field Z has the following properties*:

(1) *Z is strictly stationary and r-dependent.*

(2) *The distribution of the random variable Z_0 is as follows*:

$$P(Z_0 = 0) = (1-c)^2,$$
$$P(Z_0 = 1) = P(Z_0 = 2) = c(1-c),$$
$$P(Z_0 = 3) = c^2, \quad \textit{and}$$
$$P(Z_0 \notin \{0,1,2,3\}) = 0.$$

(3) $\alpha(\sigma(Z_0), \sigma(Z_L)) \ge qc/2$ *and* $\rho(\sigma(Z_0), \sigma(Z_L)) = q$.

(4) *For every integer k such that $1 \le k \le \max\{1, 1/(8c)\}$, one has that $k^{-1}\alpha_{k,k}(Z,r) \ge qc/2$.*

(5) *For all $k = 1,2,3,\dots$, $k^{-1}\beta_{k,\infty}(Z,1) \le 4qc$.*

(6) $\rho_{\infty,\infty}(Z,1) = q$.

Proof. Properties (1)–(4) are elementary consequences of Definition 2.5 and Lemma 2.3. The proofs of properties (5) and (6) are similar to each other, each using Lemma 2.1; we shall just give the proof of (5). □

Proof of (5). Suppose k is a positive integer. Suppose S and T are any two nonempty disjoint subsets of $\mathbf{Z}^d$ such that $(\operatorname{card} S) \le k$. It suffices to prove that

$$\beta(\sigma(Z_t, t \in S), \sigma(Z_t, t \in T)) \le 4qck. \tag{2.5}$$

Let $Q_1, Q_2, Q_3, \dots$ denote the members of the L-partition of $\mathbf{Z}^d$. For each i, define $k(i) := \operatorname{card}(S \cap Q_i)$. It will be helpful to define the random sequence $Y := (Y_j, j \in \mathbf{Z})$ by $Y_j := Z_{jL}$; this sequence Y has the $\mathcal{S}(c,q)$-distribution. We have

$$\begin{aligned}\beta(\sigma(Z_t, t\in S), \sigma(Z_t, t \in T)) \\ &\le \sum_i \beta(\sigma(Z_t, t\in S\cap Q_i), \sigma(Z_t, t\in T\cap Q_i)) \\ &\le \sum_i \beta_{k(i),\infty}(Y,1) \\ &\le \sum_i 4qc\cdot k(i) \\ &\le 4qck\end{aligned}$$

by Lemma 2.1, Definition 2.5, and Lemma 2.3. Thus (2.5) holds. This completes the proof of (5) and of Lemma 2.6. □

Proof of Theorem 1. Let $U := (U_t, t \in \mathbf{Z}^d)$ be a random field consisting of i.i.d. random variables uniformly distributed on $[0,1]$. For each $n \ge 1$, let $Z^{(n)} := (Z_t^{(n)}, t \in \mathbf{Z}^d)$ be a random field with the $\mathcal{T}(c_n, q, d, L_n)$-distribution. Assume that these random fields $U, Z^{(1)}, Z^{(2)}, Z^{(3)}, \dots$ are independent of each other.

By the hypothesis of Theorem 1, we have that $\sum_{n=1}^{\infty} c_n < \infty$ and also that for all $n \ge 1$, $c_n \le 1/8$. By Lemma 2.6, we have that for each $n \ge 1$, $P(Z_0^{(n)} = 0) = (1-c_n)^2 \ge 1 - 2c_n$. Hence by the Borel-Cantelli Lemma (and the stationarity of each random field $Z^{(n)}$), one has that for each $t \in \mathbf{Z}^d$,

$$P(Z_t^{(n)} \ne 0 \text{ for infinitely many } n \ge 1) = 0.$$

Deleting a null-set from our probability space, if necessary, we assume (just for technical convenience) that for all $t \in \mathbf{Z}^d$, the events $\{U_t \notin (0,1)\}$ and $\{Z_t^{(n)} \notin \{0,1,2,3\}$ for some $n\}$ and $\{Z_t^{(n)} \ne 0$ for infinitely many $n\}$ are all empty. (See Lemma 2.6 (2)).

Define the random field $X := (X_t, t \in \mathbf{Z}^d)$ as follows:

$$\forall\, t \in \mathbf{Z}^d, \quad X_t := U_t + \sum_{n=1}^{\infty} 4^{n-1} Z_t^{(n)}.$$

Note that, by the above assumptions, one has that

$$\forall\, t \in \mathbf{Z}^d, \quad \sigma(X_t) = \sigma(U_t, Z_t^{(1)}, Z_t^{(2)}, Z_t^{(3)}, \dots). \tag{2.6}$$

Now let us verify the properties of X listed in Theorem 1. Both the strict stationarity of X and property (1) are elementary consequences of Lemma 2.6 and the construction of X. □

It will be convenient to prove property (6) next.

Proof of (6). For each $n \geq 1$,

$$\rho(\sigma(X_0), \sigma(X_{L(n)})) \geq \rho(\sigma(Z_0^{(n)}), \sigma(Z_{L(n)}^{(n)})) = q \tag{2.7}$$

by (2.6) and Lemma 2.6 (3). From the hypothesis of Theorem 1 (see Remark 1) one has that $r_n := ||L_n|| \to \infty$ as $n \to \infty$. Hence for all $r \geq 1$, $\rho_{1,1}(X, r) \geq q$. However, by (2.6), Lemma 2.1 and Lemma 2.6 (6), one has that $\rho_{\infty,\infty}(X, 1) = q$. Property (6) follows. □

Proof of (2). The second part of (2) follows from (2.7) and property (6). The first part of (2) holds since, by (2.6) and Lemma 2.6, one has for each $n \geq 1$,

$$\alpha(\sigma(X_0), \sigma(X_{L(n)})) \geq \alpha(\sigma(Z_0^{(n)}), \sigma(Z_{L(n)}^{(n)})) \geq qc_n/2. \qquad \square$$

The proof of property (3) is like that of the first part of (2). Also, property (5) follows from (6) and [**4**, Theorem 1 (C), (D), and Remarks 2 and 3]. All that remains is to prove property (4).

Proof of (4). Suppose $n \geq 1$ and $k \geq 1$. By Lemma 2.1,

$$\beta_{k,\infty}(X, r_n) \leq \sum_{m=1}^{\infty} \beta_{k,\infty}(Z^{(m)}, r_n). \tag{2.8}$$

For each $m < n$ (if $n \geq 2$), one has $r_m < r_n$ by the hypothesis of Theorem 1, and hence $\beta_{k,\infty}(Z^{(m)}, r_n) = 0$ by Lemma 2.6 (1). For each $m \geq n$, $\beta_{k,\infty}(Z^{(m)}, r_n) \leq \beta_{k,\infty}(Z^{(m)}, 1) \leq 4kqc_m$ by Lemma 2.6 (5). Hence, by (2.8) and the hypothesis $c_{j+1} \leq c_j/2$ in Theorem 1, one has that $\beta_{k,\infty}(X, r_n) \leq \sum_{m=n}^{\infty} 4kqc_m \leq 8kqc_n$. Thus property (4) holds. This completes the proof of Theorem 1. □

3. Proof of Theorem 2. Let us start with a definition.

Definition 3.1. For each integer $M \geq 2$, define the probability measures μ_M and ν_M on $\{-1,1\}^M$ as follows: For $w := (w_1, \dots, w_M) \in \{-1,1\}^M$,

$$\mu_M(w) := 1/2^M$$

and

$$\nu_M(w) := \begin{cases} 1/2^{M-1} & \text{if } w_1 \cdot \ldots \cdot w_M = 1, \\ 0 & \text{if } w_1 \cdot \ldots \cdot w_M = -1. \end{cases}$$

Lemma 3.2. *Suppose $0 \leq q \leq 1$, $M \geq 2$ is an integer, and $(W_1, \dots, W_M)$ is a random vector which takes its values in $\{-1,1\}^M$ and whose distribution is $(1-q)\mu_M + q\nu_M$. Then the following statements hold*:

(1) *For all $j = 1, \dots, M$, $P(W_j = -1) = P(W_j = 1) = 1/2$.*

(2) *Any $M-1$ of the random variables $W_1, \dots, W_M$ are independent.*

(3) *If S and T are nonempty disjoint sets whose union is $\{1, \dots, M\}$, then*

$$\alpha(\sigma(W_k, k \in S), \sigma(W_k, k \in T)) = q/4$$

and

$$\rho(\sigma(W_k, k \in S), \sigma(W_k, k \in T)) = q.$$

Proof. Properties (1) and (2) are easy to verify first for the "extreme" cases $q = 0$ and $q = 1$, and then (using these two "extreme" cases) for the case $0 < q < 1$. □

Proof of (3). First define the events $A := \{\prod_{k \in S} W_k = 1\}$ and $B := \{\prod_{k \in T} W_k = 1\}$. By an elementary calculation,

$$\alpha(\sigma(W_k, k \in S), \sigma(W_k, k \in T)) \geq P(A \cap B) - P(A)P(B) = q/4. \tag{3.1}$$

Now suppose that f and g are random variables with mean 0 and variance 1, with f being a function of $(W_k, k \in S)$ and g a function of

$(W_k, k \in T)$. By properties (1) and (2), the distribution (and hence the moments) of f and g do not depend on q. Hence (computing as though our probability space were $\{-1,1\}^M$ itself),

$$\begin{aligned} Efg &= (1-q)\int fg\,d\mu_M + q\int fg\,d\nu_M \\ &\le 0 + q\left[\int f^2\,d\nu_M \cdot \int g^2\,d\nu_M\right]^{1/2} \\ &= q\cdot 1. \end{aligned}$$

Hence $\rho(\sigma(W_k, k \in S), \sigma(W_k, k \in T)) \le q$. This and (3.1) and (1.1) force both equations in property (3) to hold. This completes the proof of Lemma 3.2. □

Definition 3.3. Suppose $0 < q \le 1$, and $M \ge 2$ is an integer. A random sequence (indexed by $\mathbf{Z}$) is said to have the $\mathcal{U}(q,M)$-*distribution* if it has the same distribution as the random sequence $Y := (Y_t, t \in \mathbf{Z})$ defined as follows: First let $V := (V_t, t \in \mathbf{Z})$ be i.i.d. random vectors taking their values in $\{-1,1\}^M$, with the distribution of V_0 being $(1-q)\mu_M + q\nu_M$. For each $t \in \mathbf{Z}$, represent V_t by $V_t := (V_{t,0}, \dots, V_{t,M-1})$. Now define the random sequence $Y := (Y_t, t \in \mathbf{Z})$ as follows:

$$\forall\, t \in \mathbf{Z}, \quad Y_t := V_{t,0} + 3V_{t-1,1} + 9V_{t-2,2} + \cdots + 3^{M-1}\cdot V_{t-(M-1),M-1}.$$

Lemma 3.4. *Suppose* $0 < q \le 1$, *and* $M \ge 2$ *is an integer. Suppose* $Y := (Y_t, t \in \mathbf{Z})$ *is a random sequence with the* $\mathcal{U}(q,M)$*-distribution. Then this random sequence* Y *has the following properties*:

(1) Y *is strictly stationary and* 1*-dependent.*

(2) *Any* $M-1$ *of the random variables* $Y_t, t \in \mathbf{Z}$ *are independent* (*if* $M \ge 3$).

(3) $\alpha_{1,\infty}(Y,1) = q/4$.

(4) $\rho_{\infty,\infty}(Y,1) = q$.

Proof. Without loss of generality, assume that $V_t := (V_{t,0}, \dots, V_{t,M-1})$, $t \in \mathbf{Z}$, are such that all conditions in Definition 3.3 are fulfilled.

For each $u \in \mathbf{Z}$, each $s \in \mathbf{Z} - \{0, 1, \dots, M-1\}$, define the constant random variable $V_{u,s}$ by $V_{u,s} := 0$. Note that

$$\forall\, t \in \mathbf{Z}, \quad \sigma(Y_t) = \sigma(V_{u,t-u}, u \in \mathbf{Z}). \tag{3.2}$$

We need the following claim:

Claim 1. *Suppose that S and T are nonempty disjoint subsets of $\mathbf{Z}$ such that* $\operatorname{dist}(S,T) \geq 2$. *Suppose that $u \in \mathbf{Z}$. Then the σ-fields $\sigma(V_{u,t}, t \in S)$ and $\sigma(V_{u,t}, t \in T)$ are independent.*

Proof of Claim 1. Define the sets $S^* := \{0, 1, \dots, M-1\} \cap S$ and $T^* := \{0, 1, \dots, M-1\} \cap T$. If either S^* or T^* is empty, then Claim 1 is trivial. Now suppose instead that neither S^* nor T^* is empty. Then $S^* \cup T^*$ cannot be $\{0, 1, \dots, M-1\}$ (for otherwise $\operatorname{dist}(S,T)$ would be 1). Hence, by Lemma 3.2 (2), the σ-fields $\sigma(V_{u,t}, t \in S^*)$ and $\sigma(V_{u,t}, t \in T^*)$ are independent, and Claim 1 follows trivially. □

Now property (1) in Lemma 3.4 is an elementary consequence of Definition 3.3, Claim 1, and (3.2).

Proof of (2). Suppose that $t_1, \dots, t_{M-1}$ are distinct integers. For each $u \in \mathbf{Z}$, the random variables $V_{u,t(1)-u}, \dots, V_{u,t(M-1)-u}$ are independent by Definition 3.3 and Lemma 3.2 (2). Hence by (3.2) and Definition 3.3, the random variables $Y_{t(1)}, \dots, Y_{t(M-1)}$ are independent. □

Proof of (3) *and* (4). First note that

$$\begin{aligned} \alpha_{1,\infty}(Y,1) &\geq \alpha(\sigma(Y_0), \sigma(Y_1, \dots, Y_{M-1})) \\ &\geq \alpha(\sigma(V_{0,0}), \sigma(V_{0,1}, \dots, V_{0,M-1})) \\ &= q/4 \end{aligned} \tag{3.3}$$

by (3.2) and Lemma 3.2 (3). On the other hand, for any two disjoint nonempty subsets S and $T \subset \mathbf{Z}$, one has

$$\begin{aligned} \rho(\sigma(Y_t, t \in S), \sigma(Y_t, t \in T)) &= \sup_{u \in \mathbf{Z}} \rho(\sigma(V_{u,t-u}, t \in S), \sigma(V_{u,t-u}, t \in T)) \\ &\leq q \end{aligned}$$

by (3.2), Lemma 2.1, and Lemma 3.2 (3). Hence $\rho_{\infty,\infty}(Y,1) \le q$. This and (3.3) and (1.1) together force both properties (3) and (4) to hold. This completes the proof of Lemma 3.4. □

For the next definition we use the terminology of Definition 2.4.

Definition 3.5. Suppose $0 < q \le 1$, $M \ge 2$ is an integer, d is a positive integer, and $L \ne 0$ is an element of $\mathbf{Z}^d$. A random field $Z := (Z_t, t \in \mathbf{Z})$ is said to have the $\mathcal{V}(q,M,d,L)$-*distribution* if it has the following two properties:

(1) For each $l \in \mathbf{Z}^d$, the random sequence $(Z_{l+jL}, j \in \mathbf{Z})$ has the $\mathcal{U}(q,M)$-distribution.

(2) Letting $S_1, S_2, S_3, \ldots$ denote the members of the L-partition of $\mathbf{Z}^d$, one has that the σ-fields $\sigma(Z_t, t \in S_1), \sigma(Z_t, t \in S_2), \sigma(Z_t, t \in S_3), \ldots$ are independent.

Lemma 3.6. *Suppose* $0 < q \le 1$, $M \ge 2$ *is an integer, d is a positive integer, and* $L \ne 0$ *is an element of* $\mathbf{Z}^d$. *Define* $r := ||L||$. *Suppose* $Z := (Z_t, t \in \mathbf{Z}^d)$ *is a random field with the* $\mathcal{V}(q,M,d,L)$-*distribution. Then the random field Z has the following properties*:

(1) *Z is strictly stationary and r-dependent.*

(2) *Any* $M-1$ *of the random variables* Z_t, $t \in \mathbf{Z}^d$ *are independent* (*if* $M \ge 3$).

(3) $\alpha_{1,\infty}(Z,r) = q/4$.

(4) $\rho_{\infty,\infty}(Z,1) = q$.

Proof. By elementary arguments using Definition 3.5, Lemma 3.4, and Lemma 2.1, one can verify properties (1), (2) and (4), as well as $\alpha_{1,\infty}(Z,r) \ge q/4$. From this last fact and property (4) and (1.1), property (3) is also forced to hold. □

Proof of Theorem 2. Let $M_1, M_2, M_3, \ldots$ be a sequence of positive integers such that, for every $n \ge 1$,

$$M_n \ge 1/(4qc_n) \quad \text{and} \quad M_n \le M_{n+1}. \tag{3.4}$$

For each $n = 0, 1, 2, 3, \dots$, let $Z^{(n)} := (Z_t^{(n)}, t \in \mathbf{Z}^d)$ be a random field. Assume that these random fields have the following properties:

(3.5) The random field $Z^{(0)}$ has all of the properties in Theorem 1.

(3.6) For each $n \geq 1$, the random field $Z^{(n)}$ has the $\mathcal{V}(q, M_n, d, L_n)$-distribution.

(3.7) These random fields $Z^{(0)}, Z^{(1)}, Z^{(2)}, Z^{(3)}, \dots$ are independent of each other.

Let $h : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \cdots \to \mathbf{R}$ be a one-to-one bimeasurable Borel function. Define the random field $X := (X_t, t \in \mathbf{Z}^d)$ as follows:

$$\forall\, t \in \mathbf{Z}^d, \quad X_t := h(Z_t^{(0)}, Z_t^{(1)}, Z_t^{(2)}, \dots).$$

Note that

$$\forall\, t \in \mathbf{Z}^d, \quad \sigma(X_t) = \sigma(Z_t^{(0)}, Z_t^{(1)}, Z_t^{(2)}, \dots). \tag{3.8}$$

By an elementary argument using (3.5)–(3.7) and Lemma 3.6 (1), the random field X is strictly stationary. Property (1) in Theorem 2 follows from (3.5) (see property (1) in Theorem 1) and the nature of the function h. Property (6) follows easily from (3.5), (3.7), (3.8), Lemma 3.6 (4), and Lemma 2.1. Properties (2) and (3) follow from (3.5), (3.8) and property (6). All that remains is to prove properties (4) and (5).

Proof of (5). For each $n \geq 1$, $\alpha_{1,\infty}(X, r_n) \geq \alpha_{1,\infty}(Z^{(n)}, r_n) = q/4$ by (3.8), (3.6), and Lemma 3.6 (3). Under the hypothesis of Theorem 2, $r_n := ||L_n|| \to \infty$ as $n \to \infty$. Hence for all $r \geq 1$, $\alpha_{1,\infty}(X, r) \geq q/4$. Now property (5) follows from (6) and (1.1).

Proof of (4). Suppose that $n \geq 1$ and $k \geq 1$. Our task is to prove that

$$k^{-1}\beta_{k,k}(X, r_n) \leq 8qc_n. \tag{3.9}$$

If $k \geq M_n/2$, then (3.9) holds automatically because $k^{-1}\beta_{k,k}(X, r_n) \leq k^{-1} \leq 2/M_n \leq 8qc_n$ by (1.1) and (3.4).

Suppose instead that $k < M_n/2$. By (3.7), (3.8), and Lemma 2.1,

$$\beta_{k,k}(X, r_n) \leq \sum_{m=0}^{\infty} \beta_{k,k}(Z^{(m)}, r_n). \tag{3.10}$$

For each m such that $1 \le m < n$ (if $n \ge 2$), one has $\beta_{k,k}(Z^{(m)}, r_n) = 0$ by (3.6), Lemma 3.6 (1), and the fact that $r_m < r_n$ (from the hypothesis of Theorem 2). For each $m \ge n$, one has that $2k < M_n \le M_m$ by (3.4), and hence $\beta_{k,k}(Z^{(m)}, r_n) = 0$ by (3.6) and Lemma 3.6 (2). Also, $\beta_{k,k}(Z^{(0)}, r_n) \le 8qc_n k$ by (3.5). Thus, by (3.10), one has (3.9). This completes the proof of (4) and of Theorem 2. □

4. Proof of Theorem 3. We shall first prove the following lemma.

Lemma 4.1. *Suppose that d and g are as in the hypothesis of Theorem 3. Then there exists a strictly stationary random field $Y := (Y_t, t \in \mathbf{Z}^d)$ with properties (1), (2), and (3) in Theorem 3.*

Proof. Referring to Definition 2.5, for each nonzero element, $L \in \mathbf{Z}^d$, let $Z^{(L)} := (Z_t^{(L)}, t \in \mathbf{Z}^d)$ be a random field with the $\mathcal{T}(1/2, g(||L||), d, L)$-distribution. Assume that these random fields $Z^{(L)}$, $L \in \mathbf{Z}^d - \{0\}$ are independent of each other. Referring to Lemma 2.6, we assume without loss of generality that for all $L \in \mathbf{Z}^d - \{0\}$, for all $t \in \mathbf{Z}^d$, the event $\{Z_t^{(L)} \notin \{0,1,2,3\}\}$ is empty. Let

$$h : \{0,1,2,3\}^{\mathbf{Z}^d - \{0\}} \to \mathbf{R}$$

be a one-to-one bimeasurable Borel function. Define the random field $Y := (Y_t, t \in \mathbf{Z}^d)$ as follows:

$$\forall\, t \in \mathbf{Z}^d, \quad Y_t := h(Z_t^{(L)}, L \in \mathbf{Z}^d - \{0\}).$$

(The notation $h(\cdots)$ here is somewhat informal, but its meaning should be clear.) Note that, under our assumptions,

$$\forall\, t \in \mathbf{Z}^d, \quad \sigma(Y_t) = \sigma(Z_t^{(L)}, L \in \mathbf{Z}^d - \{0\}). \tag{4.1}$$

From Lemma 2.6, each of the random fields $Z^{(L)}$ is strictly stationary, and hence the random field Y is strictly stationary by an elementary argument.

From Lemma 2.6, each random variable $Z_t^{(L)}$ is uniformly distributed on the set $\{0,1,2,3\}$. Hence, by an elementary argument, the common distribution of the random variables Y_t does not have any atoms. That is, the random field Y satisfies property (1) in Theorem 3.

Next let us prove property (3) in Theorem 3 for the random field Y. Suppose $r \in [1, \infty)$. For each $L \in \mathbf{Z}^d$ such that $||L|| < r$, one has that $\rho_{\infty,\infty}(Z^{(L)}, r) = 0$ by Lemma 2.6 (1). For each $L \in \mathbf{Z}^d$ such that $||L|| \geq r$, one has by Lemma 2.6 (6),

$$\rho_{\infty,\infty}(Z^{(L)}, r) \leq \rho_{\infty,\infty}(Z^{(L)}, 1) = g(||L||) \leq g(r)$$

(since g is nonincreasing). Using Lemma 2.1, we now have that

$$\rho_{\infty,\infty}(Y, r) = \sup \rho_{\infty,\infty}(Z^{(L)}, r) \leq g(r)$$

(where the sup is taken over all $L \in \mathbf{Z}^d - \{0\}$). Taking note of (1.1), we thus have property (3) for Y.

Finally, let us prove property (2) for Y. For each nonzero $L \in \mathbf{Z}^d$,

$$\alpha(\sigma(Y_0), \sigma(Y_L)) \geq \alpha(\sigma(Z_0^{(L)}), \sigma(Z_L^{(L)})) \geq (1/4)g(||L||)$$

by (4.1) and Lemma 2.6 (3). By property (3) (proved above) we have, for each nonzero $L \in \mathbf{Z}^d$, $\rho(\sigma(Y_0), \sigma(Y_L)) \leq g(||L||)$. Now (1.1) forces both equations in property (2) to hold for Y. This completes the proof of Lemma 4.1. □

The next lemma is very elementary and very well known in various guises. (It plays a role in, e.g., the examples given by Dobrushin [**6**, p. 205] and Zhurbenko [**16**, p. 8, Example 2.1].)

Lemma 4.2. *Suppose that* (V_k, W_k), $k = 1, 2, 3, \ldots$ *are independent, identically distributed random vectors, such that the random variables* V_1 *and* W_1 *fail to be independent. Then*

$$\beta(\sigma(V_k, k \geq 1), \sigma(W_k, k \geq 1)) = 1.$$

Proof. Changing our probability space, if necessary, we assume without loss of generality that there exists a probability measure Q on that space such that the two σ-fields $\sigma(V_k, k \geq 1)$ and $\sigma(W_k, k \geq 1)$

are independent under Q, and on each of these two σ-fields the measure Q coincides with P.

Let A and B be Borel subsets of $\mathbf{R}$ such that $P(V_1 \in A, W_1 \in B) \neq P(V_1 \in A) \cdot P(W_1 \in B)$. By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I(V_k \in A, W_k \in B) = P(V_1 \in A, W_1 \in B) \quad \text{a.s.} \quad -P \quad \text{and}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I(V_k \in A, W_k \in B) = P(V_1 \in A) \cdot P(W_1 \in B) \quad \text{a.s.} \quad -Q.$$

Hence the measures P and Q are mutually singular on $\sigma((V_k, W_k), k = 1, 2, 3, \ldots)$. Lemma 4.2 follows by an elementary argument. □

Proof of Theorem 3. Using Lemma 4.1, for each $n \geq 1$, let $Y^{(n)} := (Y_t^{(n)}, t \in \mathbf{Z}^d)$ be a strictly stationary random field with properties (1), (2) and (3) in Theorem 3. Further, assume that these random fields $Y^{(1)}, Y^{(2)}, Y^{(3)}, \ldots$ are independent of each other and have the same distribution.

Let $f : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \cdots \to \mathbf{R}$ be a one-to-one bimeasurable Borel function. Define the random field $X := (X_t, t \in \mathbf{Z}^d)$ as follows:

$$\forall\, t \in \mathbf{Z}^d, \quad X_t := f(Y_t^{(1)}, Y_t^{(2)}, Y_t^{(3)}, \ldots).$$

Note that

$$\forall\, t \in \mathbf{Z}^d, \quad \sigma(X_t) = \sigma(Y_t^{(n)}, n \geq 1). \tag{4.2}$$

By elementary arguments, the random field X is strictly stationary and satisfies property (1) in Theorem 3. By (4.2) and Lemma 2.1, for each $r \geq 1$,

$$\rho_{\infty,\infty}(X, r) = \rho_{\infty,\infty}(Y^{(1)}, r) \leq g(r).$$

By (4.2), for each nonzero $L \in \mathbf{Z}^d$,

$$\alpha(\sigma(X_0), \sigma(X_L)) \geq \alpha(\sigma(Y_0^{(1)}), \sigma(Y_L^{(1)})) = (1/4)g(||L||).$$

Now (1.1) forces the random field X to satisfy both properties (2) and (3) in Theorem 3. Now for each $L \in \mathbf{Z}^d$ (including 0) we already have that the random variables $Y_0^{(1)}$ and $Y_L^{(1)}$ fail to be independent. Hence the random field X satisfies property (4) in Theorem 3 by (4.2) and Lemma 4.2. This completes the proof of Theorem 3. □

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Note added in proof. Dobrushin [**6**] identified some strictly stationary random fields (in particular, some Gibbs fields) that satisfy (1.3) but not (1.2). An extensive treatment of mixing conditions and limit theory for random fields is given by A.V. Bulinskii, *Limit theorems under weak dependence conditions*, Moscow University Press, 1989 (in Russian). An analog of the Proposition in *Remark* 6, with the assumption $\lim q_n < 1/128$ weakened to $\lim q_n < 1$, was established by W. Bryc and W. Smolenski, *On the convergence of averages of mixing sequences*, preprint 1991. The author thanks I.G. Zhurbenko and W. Bryc for pointing out these things.

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