ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 2, Spring 1993

LIMITS OF WEIGHTED SPLINES BASED ON PIECEWISE CONSTANT WEIGHT FUNCTIONS

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Weighted splines using piecewise constant weight functions as introduced by Salkauskas [**7**] have proven useful in the interpolation of rapidly varying data by C^1 piecewise cubics and have been successfully exploited by, for example, Foley [**2, 3, 4**], both in combination with Nielson's ν-splines [**6**] and in a bivariate analogue of tensor product interpolation.

In [**8**], the authors have shown that for any weight function which is piecewise constant on *some* partition (not necessarily that determined by the interpolation points), there exists an interpolant which minimizes the weighted semi-norm. As it turns out, it is also a piecewise cubic. We show here that for a broad class of weight functions there exist unique optimal interpolants which can be represented as uniform limits of piecewise cubic interpolants based on piecewise constant weight functions.

The existence proof is similar to the approach taken by Meinguet [**5**] in the construction of optimal multivariate interpolants in a semi-Hilbert space.

Theorem 1. *Suppose that we are given a data-set,* $(x_1, f_1), \dots, (x_N, f_N)$ *with* $x_1 < \cdots < x_N$. *Let* $w(x)$ *be a positive locally integrable weight function such that*

$$0 < m \le w(x) \le M < \infty \quad \text{on } [x_1, x_N],$$

$$w(x) = 1, \quad \forall\, x \notin [x_1, x_N],$$

and

$$1/w(x) \in L_1[x_1, x_N].$$

Received by the editors on August 18, 1989, and in revised form on March 14, 1991.
The first author supported by NSERC Canada grant A8389.
The second author supported by NSERC Canada grant A8521.

Further, let $X := \{v \in C^1(\mathbf{R}) : Dv$ *is absolutely continuous and* $D^2v \in L_2(\mathbf{R})\}$ *and*

$$(u,v)_w := \int_{\mathbf{R}} wD^2uD^2v \tag{1}$$

be a semi-inner product for X. *Then there is a unique* $\sigma \in X$ *which interpolates the data for which* $(\sigma,\sigma)_w$ *is a minimum. Moreover,* $D^2\sigma = 0$ *outside* $[x_1, x_N]$.

Proof. We first decompose X into a direct sum $X_0 \oplus \mathcal{P}_1$. Specifically, let $P : X \to \mathcal{P}_1$ be the projector defined by linear interpolation at x_1 and x_N, so that in Lagrange form

$$(Pv)(x) := l_1(x)v(x_1) + l_N(x)v(x_N).$$

Clearly, the kernel of $I - P$ is just $\mathcal{P}_1$. We set $X_0 = \mathrm{Im}\,(I-P)$. It is not difficult to see that X_0 is in fact a Hilbert space. From this, it follows that an optimal interpolant $p \in X$ has the form $u + Pf$, where $u \in X_0$ interpolates the reduced data $y_i := f_i - (Pf)(x_i)$, $i = 1, \dots, N$. Note that $y_1 = y_N = 0$. All $u \in X_0$ satisfy $u(x_1) = u(x_N) = 0$, so that u only need to interpolate at $x_2, \dots, x_{N-1}$. Suppose for the moment that there are representers for function evaluation in X_0; i.e., there are $K_i \in X_0$ such that $(K_i, u)_w = u(x_i)$ for all $u \in X_0$. Then it is well known (see Davis [**1**]) that u has least norm in X_0 if and only if $u \in \mathrm{span}\,\{K_i\}$. Equivalently, we wish to find $K_i \in X_0$ such that

$$(K_i, v)_w = (v - Pv)|_{x=x_i}, \qquad i = 2, \dots, N-1, \tag{2}$$

for all $v \in X$. It seems easiest to show the existence of such K_i by making use of distribution theory. Therefore, introduce the space of test functions $\mathcal{D} := \{\varphi \in C^\infty(\mathbf{R}) : \varphi$ has compact support$\}$. Now, for any $\varphi \in \mathcal{D}$, the condition (2) is equivalent to

$$\langle D^2(wD^2K_i), \varphi\rangle = \langle \delta_{x_i}, \varphi\rangle - l_1(x_i)\langle \delta_{x_1}, \varphi\rangle - l_N(x_i)\langle \delta_{x_N}, \varphi\rangle. \tag{3}$$

Of course, δ_x is the Dirac delta function, and for T, a distribution, $\langle T, \varphi\rangle := T(\varphi)$. Now such K_i's can be constructed from a solution of the distributional differential equation

$$D^2(wD^2E_i) = \delta_{x_i}$$

in the following manner. This equation is known to be satisfied by a function E_i such that

$$w(x)D^2E_i(x) = |x - x_i|/2,$$

and thus a solution is

$$E_i(x) := \frac{1}{2}\int_{x_i}^{x}\int_{x_i}^{t} \frac{|s - x_i|}{w(s)}\, ds\, dt. \tag{4}$$

It is not hard to see, in view of our assumptions about w, that $E_i \notin X$. However, the function

$$H_i(x) := E_i(x) - l_1(x_i)E_1(x) - l_N(x_i)E_N(x) \tag{5}$$

can be shown to be in X. To see this, consider $\int_{\mathbf{R}} w(D^2H_i)^2\,dx = \{\int_{-\infty}^{x_1} + \int_{x_1}^{x_N} + \int_{x_N}^{\infty}\}w(D^2H_i)^2\,dx$. Since $w(x) = 1$ for $x \notin [x_1, x_N]$, the first and last integrals vanish. To see this, suppose $x > x_N$. Then, from (4) and (5), it follows that

$$D^2H_i(x) = (1/2)(x-x_i)-(1/2)(x-x_1)l_1(x_i)-(1/2)(x-x_N)l_N(x_i) = 0,$$

since l_1 and l_N define the projector P onto $\mathcal{P}_1$. A similar argument shows that $D^2H_i(x) = 0$ for $x < x_1$. Consequently,

$$\int_{\mathbf{R}} w[D^2H_i]^2\,dx = \int_{x_1}^{x_N} w[D^2H_i]^2\,dx < \infty,$$

so $H_i \in X$. A simple calculation now reveals that H_i satisfies (3), from which it follows that

$$K_i := H_i - PH_i \in X_0$$

also satisfies (3) and D^2K_i vanishes outside $[x_1, x_N]$. Note too that $K(x_1) = K(x_N) = 0$.

We thus have shown that

$$(K_i, \varphi)_w = \varphi(x_i) - l_1(x_i)\varphi(x_1) - l_N(x_i)\varphi(x_N), \qquad i = 2, \ldots, N-1,$$

for all test functions $\varphi \in \mathcal{D}$, and this can be extended to all $v \in X$ by a density argument.

The problem of finding a $u \in X_0$ of least norm, satisfying the (hyperplane) equations

$$(K_j, u)_w = u(x_j) = y_j$$

is solved by a linear combination $\sum_{i=2}^{N-1} \alpha_i K_i(x)$ satisfying $\sum_{i=2}^{N-1} \alpha_i K_i(x_j) = y_j$. But $(K_i, K_j)_w = K_j(x_i) - (PK_j)(x_i)$ and $(PK_j)(x_i) = P(H_j - PH_j)|_{x=x_i} = (PH_j - P^2 H_j)|_{x=x_i} = 0$. It follows that Vandermondian $[K_i(x_j)]$ is in fact a Gram matrix of $N-2$ linearly independent functions and, hence, positive-definite. To see the independence, suppose that there are constants β_i, not all zero, such that $K := \sum_{i=2}^{N-1} \beta_i K_i = 0$. Then $(K, v) = 0$ for all $v \in X_0$. Choose v so that $v(x_i) = \beta_i$. Then $(K, v) = \sum \beta_i^2 \neq 0$. Hence, there is a unique interpolant of minimal semi-norm, of the form $\sigma = u + Pf$, and its second derivative vanishes outside $[x_1, x_N]$. As in the unweighted case, σ is orthogonal to every interpolant in X of zero data. □

Theorem 2. *Given a data set $(x_1, f_1), \dots, (x_N, f_N)$ with $x_1 < \dots < x_N$, suppose that $w(x)$ is a weight function satisfying the conditions of Theorem 1. Suppose further that $w_n(x)$, $n = 1, 2, \dots,$ is a sequence of weight functions, piecewise constant on finite partitions of $[x_1, x_N]$, such that $m \le w_n(x)$ on $[x_1, x_N]$ and that $\lim_{n\to\infty}\{\sup_{x\in[x_1,x_N]} |w(x) - w_n(x)|\} = 0$.*

If $\sigma(x)$ is that element of X which interpolates the given data $\{(x_i, f_i) : 1 \le i \le N\}$ for which $|\sigma|_w$ is a minimum and S_n the corresponding minimizer with weight $w_n(x)$, then

$$S_n \to \sigma \quad \textit{uniformly on } [x_1, x_N].$$

Proof. First, given $\varepsilon > 0$, choose M so that $n > M \Rightarrow |w(x) - w_n(x)| \le \varepsilon m$ on $[x_1, x_N]$. Then, for $n > M$,

$$\begin{aligned} &|(S_n, S_n)_w - (S_n, S_n)_{w_n}| \\ &\quad = \left| \int_{x_1}^{x_N} (D^2 S_n(x))^2 (w(x) - w_n(x))\, dx \right| \\ &\quad \le \int_{x_1}^{x_N} (D^2 S_n(x))^2 |w(x) - w_n(x)|\, dx \end{aligned}$$

$$\le \int_{x_1}^{x_N} (D^2 S_n(x))^2 \varepsilon m \, dx$$

$$\le \varepsilon \int_{x_1}^{x_N} (D^2 S_n(x))^2 w_n(x) \, dx \qquad (\text{as } m \le w_n(x))$$

$$\le \varepsilon \int_{x_1}^{x_N} (D^2 \sigma(x))^2 w_n(x) \, dx \qquad (\text{by the minimality of } S_n).$$

But

$$\int_{x_1}^{x_N} (D^2 \sigma(x))^2 w_n(x) \, dx \to \int_{x_1}^{x_N} (D^2 \sigma(x))^2 w(x) \, dx,$$

and so we see that

$$|(S_n, S_n)_w - (S_n, S_n)_{w_n}| \to 0.$$

Now, by the minimality of S_n, $(S_n, S_n)_{w_n} \le (\sigma, \sigma)_{w_n}$, and by the minimality of σ, $(\sigma, \sigma)_w \le (S_n, S_n)_w$. But, also, $(\sigma, \sigma)_{w_n} \to (\sigma, \sigma)_w$ and, thus, we must have $(S_n, S_n)_w \to (\sigma, \sigma)_w$. Further, the optimality of σ implies that $(\sigma, \sigma - S_n)_w = 0$ and so $(S_n, S_n)_w = (\sigma - (\sigma - S_n), \sigma - (\sigma - S_n))_w = (\sigma, \sigma)_w + (\sigma - S_n, \sigma - S_n)_w$, and we see that $(\sigma - S_n, \sigma - S_n)_w = (S_n, S_n)_w - (\sigma, \sigma)_w \to 0$. Therefore, as $w(x)/m \ge 1$ on $[x_1, x_N]$, $(\sigma - S_n, \sigma - S_n)_1 \le (1/m)(\sigma - S_n, \sigma - S_n)_w \to 0$. (Here, $(\cdot, \cdot)_1$ denotes the semi-inner product with weight function 1.) This shows convergence in an L_2 sense. To show pointwise convergence, consider

$$E_t(x) := (1/12)\{|x - t|^3 - l_1(t)|x - x_1|^3 - l_N(t)|x - x_N|^3\}$$

where l_1 and l_N are, as before, the linear Lagrange interpolating polynomials for the points x_1 and x_N, respectively. It is easily seen that, in fact, $E_t \in C^2(\mathbf{R}) \subset X$ for any $t \in \mathbf{R}$. Moreover, an integration by parts reveals that

$$(h, E_t)_1 = \int_{x_1}^{x_N} D^2 h(x) D^2 E_t(x) \, dx = h(t)$$

for $t \in [x_1, x_N]$, and $h \in X_0$. (E_t is essentially a representer of function evaluation at t in X_0.) Thus, as σ and S_n are interpolants of the same data,

$$|(\sigma - S_n)(t)|^2 = |(\sigma - S_n, E_t)_1|^2 \le (\sigma - S_n, \sigma - S_n)_1 (E_t, E_t)_1 \to 0.$$

The convergence is uniform on $[x_1, x_N]$ for $(E_t, E_t)_1$ is evidently uniformly bounded on $[x_1, x_N]$. □

A natural question arising at this point concerns the choice of weight function. We will now show that if the data originates from a function f which is spline-like in a sense defined below, and if its second derivative is known, then there is a weight function w such that the optimal interpolant $\sigma = f$. This w can be approximated by piecewise constant weight functions w_n converging uniformly to w. It is known [**8**] that the corresponding optimal interpolating splines are piecewise cubics which, by Theorem 2, converge to f.

The following definition is motivated by noting that the second derivative of a C^2 cubic spline is a continuous linear spline with the same knots. We will say that a function is spline-like if its second derivative has the same sign pattern as the second derivative of such a cubic spline. More precisely, we make the

Definition. Suppose $x_1 < \cdots < x_N$, and $f \in C^2[x_1, x_N]$. Then f is *spline-like on this partition* if and only if $D^2 f(x_1) = D^2 f(x_N) = 0$ and there exists a continuous linear spline λ with knots $x_1, \ldots, x_N$ such that $\lambda / D^2 f \geq m > 0$, and is continuous on $[x_1, x_N]$.

It follows that $\lambda / D^2 f$ can be extended to a weight function w satisfying the conditions of Theorem 1 by defining

$$w(x) := \begin{cases} \lambda(x)/D^2 f(x), & x \in [x_1, x_N], \\ 1, & x \notin [x_1, x_N]. \end{cases}$$

Of course, at any point where $D^2 f(x) = 0$, we must have $\lambda(x) = 0$ and, in general w must have removable singularities.

Theorem 3. *The optimal interpolant of the data $(x_i, f(x_i))$, $i = 1, \ldots, N$, corresponding to the weight function w above is f.*

Proof. We extend f by its tangent lines at x_1 and x_N (and continue to refer to it as f) to a function $f \in X$, and recall that f is optimal if and only if it is orthogonal in the weighted semi-inner product to every interpolant of zero data at the given points. Let $z \in X$ be such an interpolant. Then

$$\begin{aligned}\int_{\mathbf{R}} w(x)D^2f(x)D^2z(x)\,dx &= \sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} w(x)D^2f(x)D^2z(x)\,dx\\ &= \sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} \lambda(x)D^2z(x)\,dx\\ &= \sum_{i=1}^{N-1}\{\lambda(x)Dz(x)\Big|_{x=x_i}^{x_{i+1}}\\ &\qquad -\int_{x_i}^{x_{i+1}} D\lambda(x)Dz(x)\,dx\}\\ &= \{\lambda(x_N)Dz(x_N)-\lambda(x_1)Dz(x_1)\}\\ &\qquad - D\lambda(x)\Big|_{(x_i,x_{i+1})}\int_{x_i}^{x_{i+1}} Dz(x)\,dx\\ &= 0,\end{aligned}$$

since $D^2f(x_1) = D^2f(x_N) = 0$ implies that $\lambda(x_1) = \lambda(x_N) = 0$, and $\int_{x_i}^{x_{i+1}} Dz(x)\,dx = z(x_{i+1}) - z(x_i) = 0$ by hypothesis. Hence, f satisfies the orthogonality condition and is therefore optimal. □

We note that similar results hold for clamped splines.

Example. For the purpose of illustrating the above results, we have chosen the *arbitrary* function, constructed from a hand-drawn sketch, and constrained by the precise data points shown in Figure 1. This function is spline-like, but its natural spline interpolant shown in Figure 2 displays undesirable oscillations. Figure 3 shows an approximation to the second derivative, obtained from second divided differences, and a suitable $\lambda(x)$. Figure 4 is a plot of the corresponding weight function. Near 1.2 it is small but nonzero. A sequence of C^1 piecewise cubic interpolating splines with piecewise constant weight functions which approximate $w(x)$, approaching the given function, is shown in

Figure 5. The initial piecewise constant approximation was formed by taking 4 equally spaced knots from 0 to 1, 10 additional equally spaced knots from 1 to 3.5 and then the two knots 4 and 5 for a total of 16. On each subinterval so formed, $w(x)$ was approximated by the average of its endpoint values. An improved approximation was obtained by placing two additional knots at 1.12 and 1.37 and a third approximation by placing a further two knots at 1.18 and 1.31. The solid curve is the original function and the broken curves are the splines.

FIGURE 1. Original function and data points.

FIGURE 2. Natural spline interpolant.

FIGURE 3. Second derivative of original function and associated piecewise linear.

FIGURE 4. The weight function $w = \lambda / D^2 f$.

FIGURE 5. The original function and a sequence of interpolating splines with piecewise constant weight functions.

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