ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 2, Spring 1993

LINE BUNDLES AND NON-ALGEBRAICALLY CLOSED FIELDS

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ABSTRACT. Fix a non-algebraically closed field k, an algebraic closure K of k, an algebraic scheme X over k and a line bundle L on X defined over k. Here we relate properties of k and the order of the restriction of L to a neighborhood of $X(k)$ in $X(K)$.

Fix a field k and assume that k is *not* algebraically closed; let K be an algebraic closure of k. There are at least two very different theories trying to do algebraic geometry over k. One (the big one: schemes) includes as a very particular case Serre's fundamental paper [**4**], in which the base field is assumed to be algebraically closed; even if the base field for a scheme S is k, points with value in K appear as certain maximal ideals of the defining rings of the affine open subsets of S and their presence strongly influence the cohomological properties of S. Another theory, a more naive one, used essentially by some real algebraic geometers (e.g., see [**5, 3**]) consists in taking the definition in Serre's fundamental paper [**4**] substituting K with k in the definition of variety. We will use essentially this approach, but the scheme-oriented reader will have no trouble about that. Indeed, for what we are doing here, everything boils down to fixing a reduced algebraic scheme X over k and consider $X(k)$ as a subset of $X(K)$ (with the Zariski topology). The aim of this note is to give a good bound for a positive integer t (depending only on n and k) such that for every reduced scheme X of dimension n over k and every $L \in \mathrm{Pic}\,(X)$, L defined over k, there is a neighborhood U of $X(k)$ in $X(K)$ such that $L^{\otimes t}_{|U}$ is trivial (see Section 1 for a more precise statement).

Section 1. Fix a field k and assume that k is *not* algebraically closed; let K be the algebraic closure of k.

Definition 1.1. Fix an integer $n > 1$. Set $A_k(n) := \{t \in \mathbf{N} : t > 0$, and there is a homogeneous polynomial $p \in k[T_1, \dots , T_n]$ with no zero in $k^n\backslash\{0\}$ and $\deg(p) = t\}$. $A_k(n)$ is a *semigroup*; let $S_k(n)$ be the *subgroup* of $\mathbf{Z}$ *generated by* $A_k(n)$; let $d_k(n)$ be the *positive generator* of $S_k(n)$.

Note that $d_k(n) > 0$ since k is not algebraically closed. If k is the real number field we have $d_k(n) = 2$ for every n. For many interesting fields k, we have $d_k(n) = 1$ for every n. Here we check that $d_k(n) = 1$ for every n if $d_k(2) = 1$ (and in particular if k is a finite field). Take homogeneous polynomials p and q in two variables with no zero $k^2\backslash\{0\}$ and of coprime degree; set $a := \deg(p)$, $b := \deg(q)$; set $p_1(x_1, x_2, x_3) := p(x_1^a, p(x_2, x_3))$ and $q_1(x_1, x_2, x_3) := q(x_1^b, q(x_2, x_3))$; p_1 and q_1 show that $d_k(3) = 1$; then continue. As remarked by the referee, by Definition 1.1 if there exist irreducible polynomials u and v over k in one variable with $\deg(u) = 3$ and $\deg(v) = 2$, then homogenizing u and v one gets $d_k(2) = 1$; hence, $d_k(n) = 1$ for every n. In particular, this is the case if k is a finite extension either of the rational number field $\mathbf{Q}$ or of a p-adic field $\mathbf{Q}_p$.

Here is our result.

Theorem 1.2. *Fix a field k and positive integers n and d. The following conditions are equivalent*:

(i) $d \in S_k(n+1)$ (*i.e., d is a multiple of $d_k(n+1)$*);

(ii) *there are homogeneous polynomials p and q in $n+1$ variables with no zero in* $k^{n+1}\backslash\{0\}$ *and* $\deg(p) - \deg(q) = d$;

(iii) *for every quasi-projective scheme X over k with* $\dim(X) = n$ *and every* $L \in \mathrm{Pic}(X)(k)$ *there is a neighborhood U of $X(k)$ with* $L_{|U}^{\otimes d}$ *trivial* (*both the neighborhood and the trivialization being defined over* k);

(iv) *as in* (iii) *but with "projective" instead of "quasi-projective"*;

(v) *as in* (iii) *but just for one pair* $(X, L) : X = \mathbf{P}^n$ *and L the line bundle* $\mathbf{O}(1)$ *of degree* 1.

Proof. By definition, (i) and (ii) are equivalent.

(ii) implies (iii) (hence (iv) and (v)). Fix X and $L \in \mathrm{Pic}(X)(k)$ as in (iii). The set $X(k)$ has an affine neighborhood U in X (defined over k) (see, e.g., [**3**, Lemma 2.3]); fix any such U. Since U is affine,

$L_{|U}$ is spanned by its global sections. Since U is quasi-compact, one easily checks the existence of a finite dimensional k-vector subspace $V \subseteq H^0(U, L_{|U})$ such that V spans $L_{|U}$.

First assume k is infinite. We claim that $L_{|U}$ is spanned by $n+1$ sections $s_1, \ldots, s_{n+1}$ defined over k. To prove the claim, we may assume $\dim(V) > n+1$. Let E be the vector bundle on U which is the kernel of the natural surjection $V \otimes \mathbf{O}_U \to L_{|U}$. Note that V^* spans E^*. Since $\mathrm{rank}\,(E^*) = \dim(V^*) - 1 > \dim(U)$, we may apply to E^* and V^* a lemma of Serre (see [**1**, Theorem 2], whose proof works if and only if the base field is assumed to be infinite), and prove the claim. By definition, there are integers $u, v \in A_k(n)$ with $u - v = d$. Choose homogeneous polynomials p and q, respectively, of degree u and degree v, p and q never vanishing in $k^{n+1}\backslash\{0\}$. As in the proof of [**3**, 2.4], if $p = \Sigma c_a T^a$, a multi-index of weight u, we may form a section s' of $L_{|U}^{\otimes u}$ substituting formally T_i with s_i in the expansion of p. In the same way, q gives a section, s'', of $L_{|U}^{\otimes v}$. By the choice of p (respectively, q), s' (respectively, s'') generates $L_{|U}^{\otimes u}$ (respectively $L_{|U}^{\otimes v}$) at each point of $X(k)$, hence in a neighborhood of $X(k)$. The section s'/s'' of $L_{|U}^{\otimes d}$ induces a trivialization of $L^{\otimes d}$ in a neighborhood of $X(k)$.

Now assume k finite (hence $d_k(m) = 1$ for every m). Note that $L_{|U}$ is spanned by a finite number of sections, say by $s_1, \ldots, s_m$, defined over k. Since $d_k(m) = 1$, the proof given for an infinite field works without any change.

(v) implies (ii). Fix a neighborhood U of $\mathbf{P}^n(k)$ with $\mathbf{O}(d)_{|U}$ trivial. Thus, there is a nowhere vanishing section s of $\mathbf{O}(d)_{|U}$. By [**3**, Lemma 2.3], we may assume that $Y := \mathbf{P}^n\backslash U$ is a hypersurface. Let q be a defining equation of Y; set $a := \deg(q)$. Hence, q is a homogeneous polynomial of degree a with no zero in $k^{n+1}\backslash\{0\}$. We may see s as a rational section of $\mathbf{O}(d)$ with zeros and poles only on Y. thus, for large w, $q^w s$ extends to a section of $\mathbf{O}(wa+d)$ (exactly of this line bundle by the definition of the line bundles on a Proj-scheme (see, e.g., [**2**, p. 116]) over $\mathbf{P}^n(K)$ which does not vanish at any point of $\mathbf{P}^n(k)$). Hence, $d \in S(n+1)$. $\square$

One could call *test pair* (for the integer n and the field k) a pair (X, L) which can be used instead of $(\mathbf{P}^n, O(1))$ in the statement of 1.2 (v). We say that X is a test scheme (or test variety) if there is $L \in \mathrm{Pic}(X)(k)$ such that (X, L) is a test pair. One can easily construct test varieties (for instance as cyclic covers of $\mathbf{P}^n$), but not every variety is a test variety (for example, $\mathbf{P}^k \times \mathbf{P}^{n-k}$ with $0 < k < n$).

Acknowledgment. I want to thank the referee whose stimulating report and suggestions helped in an essential way to improve this paper, both in content and form.

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