MATRIX TRANSFORMATIONS OF CLASSES OF GEOMETRIC SEQUENCES

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ABSTRACT. For any fixed $t$ satisfying $0 < t < 1$, let $G_t$ denote the set of all sequences which are dominated by a constant multiple of any sequence $\{r^n\}$ with $r < t$. In this paper we characterize three kinds of matrix transformations: (i) those from $G_t$ to the convergent sequences, (ii) those from $G_t$ to the null sequences, and (iii) those from $G_t$ to the bounded sequences. Also, the classes of three well-known summability methods are investigated as mappings on $G_t$.

1. Introduction. If $u$ is a complex number sequence and $A = [a_{n,k}]$ is an infinite matrix, then $Au$ is the sequence whose $n$th term is given by

$$(Au)_n = \sum_{k=0}^{\infty} a_{n,k}u_k.$$

The matrix $A$ is called an $X-Y$ matrix if $Au$ is in the set $Y$ whenever $u$ is in $X$. In [4] Selvaraj introduced the set $G_t$ for any fixed $t$ satisfying $0 < t < 1$ as

$$G_t = \{ u : u_n = O(r^n) \text{ for some } r \in (0, t) \}$$

and gave the characterization as follows:

**Theorem 1.1.** The sequence $u$ is in $G_t$ if and only if

$$(1) \quad \limsup_{k} |u_k|^{1/k} < t.$$

In Section 2 we investigate $G_t - c$, $G_t - c_0$, and $G_t - l^\infty$ matrices. The characterizations of such matrices are established in terms of their
rows and columns. Section 3 examines $G_t - c, G_t - c_0$ and $G_t - l^\infty$ mapping properties of the classical summability methods of Euler-Knopp, Nörlund, and Borel matrices.

2. Matrix transformations of $G_t$ into $c, c_0$ and $l^\infty$. First we will prove the necessary and sufficient conditions for a matrix to be a $G_t - c$ matrix. In order to characterize such a matrix, we need the following preliminary result.

**Lemma 2.1.** Let $x$ be a complex sequence such that, for any $u \in G_t$, $\sum_{n=0}^{\infty} u_n x_n$ converges. Then for each $\varepsilon > 0$ there exists a constant $B > 0$ such that, for all $k$, $|x_k| \leq B(1/t + \varepsilon)^k$.

**Proof.** Suppose the conclusion of the lemma is false. This implies that there is an $\varepsilon > 0$ so that for every $B > 0$ there exists $k = k(B)$ satisfying

$$
|x_k| > B\left(\frac{1}{t} + \varepsilon\right)^k.
$$

We now choose an increasing sequence $\{k(i)\}_{i=0}^\infty$ as follows. Choose $k(0)$ satisfying $|x_{k(0)}| > 0$. After selecting $k(p)$ for all $p < i$, we choose $k(i)$ as follows. For $N = k(i) - 1$, there exists a constant $B = \max_{0 \leq j \leq N} |x_j(t/(1 + \varepsilon t))^j|$ such that

$$
|x_j| \leq B\left(\frac{1}{t} + \varepsilon\right)^j, \quad \text{for } j \leq N.
$$

Let $B' = B + 1$. Now we can find $k(i)$ such that

$$
|x_{k(i)}| > B'\left(\frac{1}{t} + \varepsilon\right)^{k(i)},
$$

using (2). Thus,

$$
|x_{k(i)}| > \left(\frac{1}{t} + \varepsilon\right)^{k(i)}.
$$
This $k(i) > k(i-1)$ because, if not, $k(i) \leq N$ and hence, by (3), $|x| < B(1/t + \varepsilon)^{k(i)}$ which would contradict (4).

Now consider the sequence $u$ given by

$$ u_j = \begin{cases} t \cdot \frac{x_j}{x_{j-1}}, & \text{if } j = k(i) \text{ for } i = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases} $$

It is obvious that $u \in G_t$. But, for each positive integer $m$,

$$ \sum_{j=0}^{k(m)} u_j x_j > m, $$

using (5). Thus, we have a contradiction to the hypothesis.

\begin{proof}

We first assume that $A$ satisfies both the conditions of the theorem and let $u$ be a sequence in $G_t$, say $|u| \leq M s^k$ for some $s \in (0, t)$. Choose $\varepsilon > 0$ such that $\varepsilon < 1/s - 1/t$. Then we have $|a_{nk}| \leq B(1/t + \varepsilon)^k$ for all $n$ and $k$. Since, for each $k$, $\lim_{n} a_{nk} = L_k$, we have $|L_k| \leq B(1/t + \varepsilon)^k$ for all $k$. Also, we can find a positive integer $l$ satisfying

$$ 2B \sum_{k=l}^{\infty} |u| \left( \frac{1}{k} + \varepsilon \right)^k \leq \frac{\varepsilon}{2}. $$

This is possible because the geometric series $\sum_{k=0}^{\infty} s^k (1/t + \varepsilon)^k$ converges. Also, by condition (i), we can find an $N$ such that for $k = 0, 1, \ldots, l - 1$,

$$ |a_{nk} - L_k| |u_k| \leq \frac{\varepsilon}{2l} \quad \text{for } n > N. $$

\end{proof}
Now, for \( n > N \),

\[
\left| (Au)_n - \sum_{k=0}^{\infty} u_k L_k \right| \leq \sum_{k=0}^{t-1} |a_{nk} - L_k| |u_k| + \sum_{k=t}^{\infty} |a_{nk} - L_k| |u_k| < \varepsilon
\]

using (6) and (7). Thus, we have proved that the sequence \( \{(Au)_n\} \) converges to \( \sum_{k=0}^{\infty} u_k L_k \); this series converges. Hence, \( Au \in c \).

Conversely, if \( A \) is a \( G_t - c \) matrix, then the basis sequences \( \{\delta_n^{(k)}\}_{n=0}^{\infty} \) are mapped into \( c \). Thus, condition (i) holds.

Suppose that condition (ii) does not hold. Then there is an \( \varepsilon > 0 \) so that for every \( B > 0 \) there exist \( n = n(B) \) and \( k = k(B) \) such that

\[
|a_{nk}| > B \left( \frac{1}{t} + \varepsilon \right)^k.
\]

As \( G_t \) is in the domain of the matrix \( A \), by Lemma 2.1, for each \( j \) there exists \( B(j) > 0 \) such that \( |a_{jk}| \leq B(j)(1/t + \varepsilon)^k \) for all \( k \). So, for \( j = 0, 1, \ldots, N \), we can find \( B'(N) > 0 \) satisfying \( |a_{jk}| \leq B'(N)(1/t + \varepsilon)^k \) for all \( k \). Since each column of the matrix \( A \) is bounded, for \( k = 0, 1, \ldots, N \), there exists a constant \( M'(N) \) such that \( |a_{jk}| \leq M'(N)(1/t + \varepsilon)^k \) for all \( j \). Thus, given any \( N \) there exists \( M = M(N) > 1 \) such that

\[
|a_{jk}| \leq M(1/t + \varepsilon)^k, \quad \text{for } j \leq N \text{ or } k \leq N.
\]

Now we choose increasing sequences \( \{u(n)\}_{n=0}^{\infty} \) and \( \{v(n)\}_{n=0}^{\infty} \) as follows. Choose \( u(0) \) and \( v(0) \) such that \( |a_{u(0),v(0)}| > 0 \). After selecting \( u(p) \) and \( v(p) \) for all \( p < i \), we choose \( u(i) \) and \( v(i) \) as follows. For \( N = u(i-1) + v(i-1) \), there exists \( M > 1 \) such that for \( j \leq N \) or \( k \leq N \),

\[
|a_{jk}| \leq M(1/t + \varepsilon)^k
\]

using (9). Let \( H = M + i \). Now we can find \( u(i) \) and \( v(i) \) such that

\[
|a_{u(i),v(i)}| > H(1/t + \varepsilon)^v(i)
\]
using (8). Thus,

\[(12) \quad |a_{u(i),v(i)}| > i(1/t + \varepsilon)^{v(i)}.\]

This \(u(i)\) and \(v(i)\) each exceed \(u(i - 1) + v(i - 1)\) because, if not, either \(u(i) \leq N\) or \(v(i) \leq N\) and hence, by (10), \(|a_{u(i),v(i)}| < M(1/t + \varepsilon)^{v(i)}\) which would contradict (11).

Now consider the sequence \(x\) given by

\[x_k = \begin{cases} 
\left(\frac{t}{1+\varepsilon t}\right)^{v(i)}, & \text{if } k = v(i) \text{ for } i = 1, 2, \ldots, \\
0, & \text{otherwise.}
\end{cases}\]

It is obvious that \(x \in G_t\). Define a matrix \(A'\) by \(a'_n k = a_{n,k} x_k\). For any \(u \in c\), we have \(x u \in G_t\). Since \(A\) is a \(G_t - c\) matrix, it follows that \(A'\) is a \(c - c\) matrix. But for each positive integer \(m\), we have \(|a'_{u(m),v(m)}| > m\) by (12). This contradicts that \(A'\) is a \(c - c\) matrix. \(\square\)

We state below two theorems on the characterization of \(G_t - c_0\) and \(G_t - l^\infty\) matrices. The proof of Theorem 2.1 can be easily applied to these two theorems.

**Theorem 2.2.** The matrix \(A\) is a \(G_t - c_0\) matrix if and only if

(i) each column sequence is in \(c_0\) and

(ii) for each \(\varepsilon > 0\) there exists a constant \(B > 0\) such that \(|a_{n,k}| \leq B(1/t + \varepsilon)^k\) for all \(n\) and \(k\).

**Theorem 2.3.** The matrix \(A\) is a \(G_t - l^\infty\) matrix if and only if for each \(\varepsilon > 0\) there exists a constant \(B > 0\) such that \(|a_{n,k}| \leq B(1/t + \varepsilon)^k\) for all \(n\) and \(k\).

In [2] Jacob derived similar characterizations of the above matrix transformations using the topological properties of the spaces \(G_t\).

3. **Well-known summability mappings on \(G_t\).** In this final section we shall apply the results of Section 2 to find the necessary and
sufficient conditions for some well-known matrix methods to be $G_t - c$, $G_t - c_0$, and $G_t - l^\infty$ matrices.

The Euler-Knopp means [3, p. 54] are given by

$$E_r[n, k] = \begin{cases} 
\binom{n}{k} (1 - r)^{n-k} r^k, & \text{if } k \leq n, \\
0, & \text{if } k > n,
\end{cases}$$

where $r$ is any complex number. In the following theorem, we shall consider the Euler matrices $E_r$ with only real values of the parameter $r$.

**Theorem 3.1.** The following statements are equivalent:

(i) $r \in [0, 2/(1 + t)]$;

(ii) $E_r$ is a $G_t - c$ matrix;

(iii) $E_r$ is a $G_t - l^\infty$ matrix.

**Proof.** When $r = 0$, the matrix $E_r$ has all ones in the first column and zeros elsewhere. So, by Theorem 2.1, $E_r$ is a $G_t - c$ matrix. If $0 < r \leq 2/(1 + t)$, then a simple calculation shows that $|1 - r| \leq 1 - rt$. Thus, for any $x \in G_t$, say $|x_k| \leq Mu^k$ where $u \in (0, t)$, we have

$$|(E_r x)_n| \leq M[(1 - r) + ru]^n.$$ 

Now $|1 - r| + ru \leq 1$ implies that $E_r x \in c_0$ and, hence, $E_r$ is a $G_t - c$ matrix. We have shown that (i) implies (ii).

The fact that (ii) implies (iii) is obvious from the set inclusion $c \subset l^\infty$. Next, to see that $r \in [0, 2/(1 + t)]$ whenever $E_r$ is a $G_t - l^\infty$ matrix, suppose $r < 0$. Then the first column sequence $\{(1 - r)^{n}\}_{n=0}^{\infty}$ is not bounded. Consequently, by Theorem 2.3, $E_r$ is not a $G_t - l^\infty$ matrix. Now, suppose that $r > 2/(1 + t)$. If we choose $u$ satisfying $2t/r(1 + t) < u < t$, then $\{x_k\} = \{(-u)^k\} \in G_t$ and

$$|(E_r x)_n| = \left|(-1)^n \sum_{k=0}^{n} \binom{n}{k} (r - 1)^{n-k} (ru)^k \right|
= |r - 1 + ru|^n.$$
Since $r - 1 + ru > 1$, it follows that $E_r$ is not a $G_t - l^\infty$ matrix. \(\square\)

It is easy to see that the following result is also true.

**Corollary 3.1.** $E_r$ is a $G_t - c_0$ matrix if and only if $r \in (0, 2/(1+t)]$.

The Nörlund mean $Np$ is represented by a lower triangular matrix in which

$$Np[n, k] = p_{n-k}/P_n \quad \text{if } k \leq n,$$

where $P_n = \sum_{k=0}^{n} p_k$ and $p$ is a nonnegative sequence such that $p_0 > 0$.

**Theorem 3.2.** Let $Np$ be a Nörlund matrix.

(i) $Np$ is a $G_t - c_0$ matrix if and only if $\lim_{n} p_n/P_n = 0$;

(ii) $Np$ is a $G_t - c$ matrix if and only if each column sequence converges;

(iii) $Np$ is a $G_t - l^\infty$ matrix for all $p$.

**Proof** (i) If $\lim_{n} p_n/P_n = 0$, then $Np$ is a regular matrix and thereby maps $G_t$ into $c_0$. Conversely, if $Np$ is a $G_t - c_0$ matrix, then by Theorem 2.2, the first column is a null sequence.

(ii) Since the absolute row sums of the matrix $Np$ are equal to 1 and $1/t > 1$, the second condition of Theorem 2.1 is always true. Hence, the result follows.

(iii) It is obvious that the condition of Theorem 2.3 is satisfied by $Np$ matrices. \(\square\)

Fricke and Fridy [1] introduced the extended form of Borel matrix by the following definition. For any real number $\delta$,

$$B_\delta[n, k] = e^{-n^\delta} (n^\delta)^k/k!$$

for $k = 0, 1, \ldots$, and $n = 0, 1, \ldots$. When $\delta = 0$, the matrix is defined by

$$B_0[n, k] = e^{-1}/k!, \quad \text{for all } n \text{ and } k.$$
**Theorem 3.3.** The matrix \( B_\delta \) is a \( G_t - c_0 \) matrix if and only if \( \delta > 0 \); also, \( B_\delta \) is a \( G_t - c \) matrix for all \( \delta \).

*Proof.* It is known [4, Table 3.2, Theorem 3] that if \( \delta > 0 \) then \( B_\delta \) is a \( G_t - l^1 \) matrix, whence \( B_\delta \) is a \( G_t - c_0 \) matrix. Conversely, suppose that \( \delta \leq 0 \). In the case of \( \delta < 0 \), we have \( B_\delta[n,0] = e^{-n^\delta} \) converging to 1 as \( n \to \infty \). Thus, the first column of \( B_\delta \) is not in \( c_0 \). Therefore, \( B_\delta \) cannot be a \( G_t - c_0 \) matrix. Similarly, if \( \delta = 0 \) then the first column converges to \( 1/e \) so that \( B_\delta \) is not a \( G_t - c_0 \) matrix.

Now, in order to show that \( B_\delta \) is a \( G_t - c \) matrix for all \( \delta \), it is enough to consider the cases \( \delta \leq 0 \). When \( \delta \leq 0 \), the preceding argument shows that the first column of \( B_\delta \) is in \( c \). When \( \delta = 0 \), for each \( k \geq 1 \), \( B_\delta[n,k] \) converges to \( 1/(k!e) \) as \( n \to \infty \) and, when \( \delta < 0 \), for each \( k \geq 1 \), \( B_\delta[n,k] \) converges to zero as \( n \to \infty \). So, in both cases, \( B_\delta[n,k] < (1/t + \varepsilon)^k \) for any \( \varepsilon > 0 \). Thus, both conditions of Theorem 2.1 are true. Hence, \( B_\delta \) is a \( G_t - c \) matrix. \( \Box \)

**REFERENCES**


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