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K-THEORY AND EXT-THEORY FOR RECTANGULAR UNITARY C^*-ALGEBRAS

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1. Introduction. Much study has been done on the C^*-algebras O_n generated by n isometries $S_1, S_2, \dots, S_n$ such that $S_1S_1^* + \cdots + S_nS_n^* = 1$. These algebras were introduced by Cuntz in [**9**] (see also [**6, 7, 8, 11, 15, 16**]). The K-theory of these algebras has been computed by Cuntz in [**7**]. The Ext-groups have been computed by Pimsner and Popa in [**16**] (see also [**15**]). In [**3**], Brown introduced the C^*-algebra U_n^{nc} generated by elements u_{ij}, $1 \le i, j \le n$, satisfying the relations which make the matrix $[u_{ij}]$ a unitary matrix. The K-groups of U_n^{nc} were computed in [**14**], where it was also shown that U_n^{nc} has no nontrivial projections. In [**18**], Voiculescu defined the $m \times n$ version of U_n^{nc} which we will denote $U_{(m,n)}^{\mathrm{nc}}$. The algebras O_n and U_n^{nc} correspond to $U_{(1,n)}^{\mathrm{nc}}$ and $U_{(n,n)}^{\mathrm{nc}}$, respectively. We will show that $U_{(m,n)}^{\mathrm{nc}}$ is isomorphic to the commutant of the $m+n$ by $m+n$ matrices in a certain amalgamated free product C^*-algebra. We will also prove some partial results about the K-theory of $U_{(m,n)}^{\mathrm{nc}}$ and also compute their Ext-groups.

2. The C^*-algebra $U_{(m,n)}^{\mathrm{nc}}$. We define $U_{(m,n)}^{\mathrm{nc}}$ as follows. $U_{(m,n)}^{\mathrm{nc}}$ is generated by elements u_{ij}, $1 \le i \le m$, $1 \le j \le n$, subject to the following relations on $u = [u_{ij}]$: $u^*u = I_n$ and $uu^* = I_m$, where I_k denotes the k by k identity matrix. $U_{(m,n)}^{\mathrm{nc}}$ has the universal property that if B is any unital C^*-algebra with elements v_{ij} for which $v = [v_{ij}]$ satisfies the same relations as u, then there is a unique unital $*$-homomorphism $\phi : U_{(m,n)}^{\mathrm{nc}} \to B$ such that $\phi(u_{ij}) = v_{ij}$. Clearly, any two C^*-algebras which satisfy the above property are canonically isomorphic. If u_{ij} and v_{kl} denote the generators of $U_{(m,n)}^{\mathrm{nc}}$ and $U_{(n,m)}^{\mathrm{nc}}$, respectively, then the map $u_{ij} \mapsto v_{ji}^*$ induces an isomorphism from $U_{(m,n)}^{\mathrm{nc}}$ onto $U_{(n,m)}^{\mathrm{nc}}$. As a result of this observation, we will restrict our attention to the $m \le n$ cases.

There are two special cases of interest. If $m = n$, then $U_{(n,n)}^{\mathrm{nc}}$ is the C^*-algebra U_n^{nc} defined by Brown in [**4**]. If $m = 1$, then let $S_j = u_{1j}$.

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The relations defining $U^{\mathrm{nc}}_{(1,n)}$ are equivalent to the relations $S_i^* S_j = \delta_{ij} 1$ and $\sum_{i=1}^n S_i S_i^* = 1$. Thus, $U^{\mathrm{nc}}_{(1,n)}$ is isomorphic to the Cuntz algebra O_n [**9**, 1.12]. It was observed by Paschke that O_n is isomorphic to the relative commutant of M_n in the amalgamated free product C^*-algebra (see [**3**] for a definition) $M_{1+n*\mathbf{C}^2} M_2$, where $(\alpha, \beta) \in \mathbf{C}^2$ is identified with $\alpha \oplus \beta \oplus \cdots \oplus \beta \in M_{1+n}$ and $\alpha \oplus \beta \in M_2$. The following theorem extends this result to $U^{\mathrm{nc}}_{(m,n)}$ for all cases of $1 \leq m \leq n$.

Theorem 2.1. *Let $M_{m+n} *_{\mathbf{C}^2} M_2$ be the amalgamated product C^*-algebra where $(\alpha, \beta) \in \mathbf{C}^2$ is identified with $\alpha \oplus \cdots \oplus \alpha \oplus \beta \oplus \cdots \oplus \beta$ in M_{m+n} (m copies of α and n copies of β) and $\alpha \oplus \beta$ in M_2. Then $U^{\mathrm{nc}}_{(m,n)}$ is isomorphic to the relative commutant M^c_{m+n} of M_{m+n} in $M_{m+n} *_{\mathbf{C}^2} M_2$.*

Proof. The amalgamated free product of the unital C^*-algebras A and B over a common subalgebra D with $1_A = 1_B = 1_D$ (denoted $A *_D B$) can be described by the following universal property. $A *_D B$ contains isomorphic copies of A and B, $1_A = 1_B =$ the unit of $A *_D B$ and for any unital C^*-algebra E containing an isomorphic copy of D (also denoted D) with $1_D = 1_E$ and any pair of homomorphisms $\alpha : A \to E$ and $\beta : B \to E$ satisfying $\alpha|_D = \beta|_D = \mathrm{Id}_D$, there is a unique unital $*$-homomorphism $\alpha * \beta : A *_D B \to E$ such that $\alpha * B|_A = \alpha$ and $\alpha * \beta|_B = \beta$. This property will be used in order to produce the elements $u_{ij} \in M^c_{m+n}$ which satisfy the universal property defining $U^{\mathrm{nc}}_{(m,n)}$.

Let E be a unital C^*-algebra, and suppose $v_{ij} \in E$ are such that the matrix $v = [v_{ij}]$ satisfies $v^* v = I_n$ and $v v^* = I_m$. The C^*-algebra $E \otimes M_{m+n} \cong M_{m+n}(E)$ contains an isomorphic copy of $\mathbf{C}^2$ through the following correspondence.

$$(\alpha, \beta) \leftrightarrow 1 \otimes (\alpha \oplus \cdots \oplus \alpha \oplus \beta \oplus \cdots \oplus \beta).$$

Let e_{ij} and f_{kl} be matrix units for M_2 and M_{m+n}, respectively. Consider the following homomorphisms

$$\phi_1 : M_{m+n} \to E \otimes M_{m+n}$$

$$\phi_2 : M_2 \to E \otimes M_{m+n}$$

defined as follows;

$$\phi_1(x) = 1 \otimes x$$

$$\phi_2(e_{12}) = \begin{bmatrix} 0_{m\times m} & [v_{ij}] \\ 0_{n\times m} & 0_{n\times n} \end{bmatrix}.$$

Note that ϕ_2 does in fact extend to a unital homomorphism of M_2 since $P = \phi_2(e_{12})$ is a partial isometry satisfying $PP^* + P^*P = 1$. Since $\phi_1|_{\mathbf{C}^2} = \phi_2|_{\mathbf{C}^2} = \mathrm{Id}_{\mathbf{C}^2}$, it follows that there is a unique $*$-homomorphism $\phi_1 * \phi_2 : M_{m+n} *_{\mathbf{C}^2} M_2 \to E \otimes M_{m+n}$ extending both ϕ_1 and ϕ_2. Since $(\phi_1 * \phi_2)(M^c_{m+n}) \subset (1 \otimes M_{m+n})^c = E \otimes I_{m+n}$, $\phi = \phi_1 * \phi_2|_{M^c_{m+n}}$ is a unital $*$-homomorphism from M^c_{m+n} into $E \otimes I_{m+n} \cong E$. Since

$$v_{ij} \otimes I_{m+n} = \sum_{k=1}^{m+n} (1 \otimes f_{ki}) P (1 \otimes f_{m+j,k}),$$

it follows that $\phi(u_{ij}) = v_{ij} \otimes I_{m+n} \leftrightarrow v_{ij}$, where

$$u_{ij} = \sum_{k=1}^{m+n} f_{ki} e_{12} f_{m+j,k}.$$

It is routine to check that $u_{ij} \in M^c_{m+n}$ and $u = [u_{ij}]$ satisfies $u^*u = I_n$ and $uu^* = I_m$.

All that remains is to show that if $\psi : M^c_{m+n} \to E$ satisfies $\psi(u_{ij}) = v_{ij}$, then $\psi = \phi$. Suppose ψ is such a unital $*$-homomorphism. Consider ψ as a map from $M^c_{m+n} \otimes I_{m+n}$ into $E \otimes M_{m+n}$. Extend ψ to

$$\bar{\psi} : M^c_{m+n} \otimes M_{m+n} \to E \otimes M_{m+n}$$

by letting $\bar{\psi}(1 \otimes x) = 1 \otimes x$ for $x \in M_{m+n}$. Now it is routine to check that the map $a \otimes b \mapsto ab$ defines an isomorphism of $M^c_{m+n} \otimes M_{m+n}$ and $M_{m+n} *_{\mathbf{C}^2} M_2$. Thus, we have a homomorphism

$$\tilde{\psi} : M_{m+n} *_{\mathbf{C}^2} M_2 \to E \otimes M_{m+n}$$

induced from $\bar{\psi}$ and the above isomorphism. It is enough to show that $\tilde{\psi} = \phi_1 * \phi_2$ since $\phi_1 * \phi_2|_{M^c_{m+n}} = \phi$ and $\tilde{\psi}|_{M^c_{m+n}} = \bar{\psi}|_{M^c_{m+n} \otimes I_{m+n}} = \psi$. To see this, we need only check that $\tilde{\psi}|_{M_{m+n}} = \phi_1$ and $\tilde{\psi}|_{M_2} = \phi_2$. Now if $x \in M_{m+n}$, then

$$\tilde{\psi}(x) = \bar{\psi}(1 \otimes x) = 1 \otimes x = \phi_1(x).$$

To check that the two maps agree on M_2, it suffices to check that they agree on e_{12}. Now e_{12} corresponds to $\sum_{i=1}^m \sum_{j=1}^n u_{ij} \otimes f_{i,m+j}$ under the isomorphism of $M_{m+n}^c \otimes M_{m+n}$ and $M_{m+n} *_{\mathbf{C}^2} M_2$. To see this, first notice that $u_{ij} \in M_{m+n}^c$ and $e_{11} = \sum_{i=1}^m f_{ii}$, $e_{22} = \sum_{j=1}^m f_{m+j,m+j}$ in $M_{m+n} *_{\mathbf{C}^2} M_2$. The correspondence just mentioned follows from the following computation.

$$\begin{aligned} e_{12} &= e_{11}e_{12}e_{22} \\ &= \bigg(\sum_{i=1}^m f_{ii}\bigg) e_{12} \bigg(\sum_{j=1}^n f_{m+j,m+j}\bigg) \\ &= \sum_{i=1}^m \sum_{j=1}^n f_{ii} e_{12} f_{m+j,m+j} \\ &= \sum_{i=1}^m \sum_{j=1}^n \bigg(\sum_{k=1}^{m+n} f_{ki} e_{12} f_{m+j,k}\bigg) f_{i,m+j} \\ &= \sum_{i=1}^m \sum_{j=1}^n u_{ij} f_{i,m+j}. \end{aligned}$$

The computation below completes the proof of the theorem.

$$\begin{aligned} \tilde{\psi}(e_{12}) &= \bar{\psi}\bigg(\sum_{i=1}^m \sum_{j=1}^n u_{ij} \otimes f_{i,m+j}\bigg) \\ &= \sum_{i=1}^m \sum_{j=1}^n \psi(u_{ij}) \otimes f_{i,m+j} \\ &= \sum_{i=1}^m \sum_{j=1}^n v_{ij} \otimes f_{i,m+j} = \phi_2(e_{12}). \qquad \square \end{aligned}$$

Remark. By [**12**, Proposition 3.10], it follows that the algebras $U_{(m,n)}^{\mathrm{nc}}$ are semi-projective as is defined in [**12**]. Thus, if A is isomorphic to the C^*-algebraic direct limit $\varinjlim A_n$ and A has elements v_{ij} satisfying the same relations as the u_{ij} in $U_{(m,n)}^{\mathrm{nc}}$, then, for some n, A_n also has elements w_{ij} satisfying these relations. In fact, there is a norm continuous path $v_{ij}(t)$ in A of elements satisfying these relations such that $v_{ij}(0) = v_{ij}$ and $v_{ij}(1) = w_{ij}$.

3. K-Theoretic results for $U^{\rm nc}_{(m,n)}$. It was remarked earlier that if $m = 1$, then $U^{\rm nc}_{(m,n)} \cong O_n$. In [**7**] it was shown that $K_0(O_n) \cong \mathbf{Z}_{n-1}$ and $K_1(O_n) \cong \delta_{1n}\mathbf{Z}$. In [**14**] it was shown that $K_0(U^{\rm nc}_n) \cong \mathbf{Z} \cong K_1(U^{\rm nc}_n)$. In light of these known facts, we make the following conjecture.

Conjecture.

$$K_0(U^{\rm nc}_{(m,n)}) \cong \mathbf{Z}_{n-m}$$

$$K_1(U^{\rm nc}_{(m,n)}) \cong \delta_{mn}\mathbf{Z}.$$

We will prove some results which support the above conjecture in the unknown $1 < m < n$ cases. Let

$$\psi_{mn} : U^{\rm nc}_{(m,n)} \to O_{n-m+1}$$

be defined as follows

$$[\psi_{mn}(u_{ij})] = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & S_1 \cdots S_{n-m+1} \end{pmatrix}$$

where the u_{ij} are the generators of $U^{\rm nc}_{(m,n)}$ and the S_k are the generators of O_{n-m+1}. Because $[\psi_{mn}(u_{ij})]^*[\psi_{mn}(u_{ij})] = I_n$ and $[\psi_{mn}(u_{ij})][\psi_{mn}(u_{ij})]^* = I_m$, the above rule induces a unital $*$-homomorphism. Note that this implies that $U^{\rm nc}_{(m,n)}$ is not simple for $m \neq 1$ (contrary to the $m = 1$ case where Cuntz showed that O_n is simple [**9**]). The following proposition shows that $K_0(U^{\rm nc}_n)$ contains $\mathbf{Z}_{n-m}$ as a direct summand.

Proposition 3.1. *There is an isomorphism $K_0(U^{\rm nc}_{(m,n)}) \cong \mathbf{Z}_{n-m} \oplus K_0(\ker \psi_{mn})$ such that $[1]_0$ in $K_0(U^{\rm nc}_{(m,n)})$ corresponds to $1 \oplus 0$.*

Proof. Let P denote the n by n matrix whose top m by n block is equal to $[u_{ij}]$ and whose bottom $n-m$ rows are identically zero. Then $P \in M_n(U^{\rm nc}_{(m,n)})$ and $PP^* = I_m \oplus 0_{n-m}$, $P^*P = I_n$. Thus, $m[1]_0 = [I_m]_0 = [I_n]_0 = n[1]_0$. So $(n-m)[1]_0 = 0$. Now, since $K_0(O_{n-m+1}) \cong \mathbf{Z}_{n-m}$ with $[1]_0$ as a generator [**7**], it follows that the group homomorphism

$$\gamma : K_0(O_{n-m+1}) \to K_0(U^{\rm nc}_{(m,n)})$$

defined by $\gamma([1]_0) = [1]_0$ satisfies $\psi_{mn,*,0} \circ \gamma = \mathrm{Id}$, where $\psi_{mn,*,0}$ is the map on K_0-groups induced by ψ_{mn}. Consider the following short exact sequence,

$$0 \to \ker \psi_{mn} \to U^{\mathrm{nc}}_{(m,n)} \overset{\psi_{mn}}{\to} O_{n-m+1} \to 0$$

and the induce six term exact sequence on K-groups,

$$\begin{array}{ccccc} K_0(\ker \psi_{mn}) & \longrightarrow & K_0(U^{\mathrm{nc}}_{(m,n)}) & \xrightarrow{\psi_{mn,*,0}} & \mathbf{Z}_{n-m} \\ \uparrow & & & & \downarrow 0 \\ 0 & \longleftarrow & K_1(U^{\mathrm{nc}}_{(m,n)}) & \longleftarrow & K_1(\ker \psi_{mn}) \end{array}$$

The vertical map on the right side is zero because $\psi_{mn,*,0}$ is surjective. The top row is thus a short exact sequence which splits because of the map γ. The conclusion now follows easily. $\square$

In the next theorem the map $\phi_{mn} : U^{\mathrm{nc}}_{(m,n)} \to M_m(U^{\mathrm{nc}}_{(m,n)})$ will be needed.

$$\phi_{mn}(x) = [u_{ij}] \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} [u_{ij}]^*.$$

Notice that, in the case $m = 1$, if we write $S_j = u_{1j}$, then $\phi_{1n}(x) = \sum_{j=1}^n S_j x S_j^*$. This coincides with the map $\phi_n : O_n \to O_n$ used by Cuntz in [**7**] in order to compute $K_j(O_n)$.

We claim that $\phi_{mn,*,j} : K_j(U^{\mathrm{nc}}_{(m,n)}) \to K_j(U^{\mathrm{nc}}_{(m,n)})$ is multiplication by n for $j = 0, 1$. To see this, let $x \otimes I_n$ denote $x \oplus \cdots \oplus x$ (n times) and $u = [u_{ij}]$. Thus, $\phi_{mn}(x) = u(x \otimes I_n)u^*$. Let $V = [v_{ij}]$ be a unitary in $M_r(U^{\mathrm{nc}}_{(m,n)})$. Then

$$\begin{aligned} (\mathrm{Id}_{M_r} \otimes \phi_{mn})(V) &= [\phi_{mn}(v_{ij})] \\ &= [u(v_{ij} \otimes I_n)u^*] \\ &= u_0[v_{ij} \otimes I_n]u_0^* \end{aligned}$$

where $u_0 = u \oplus \cdots \oplus u$ (r times). Let $U = u_0 \oplus u_0*$, a unitary in $M_{r(n+m)}(U^{\mathrm{nc}}_{(m,n)})$. Then, since $u_0^* u_0 = I_{rn}$, $u_0 u_0^* = I_{rm}$, it follows that

$$\begin{aligned} &(\mathrm{Id}_{M_r} \otimes \phi_{mn})(V) \oplus I_{r(2m+n)} \\ &\qquad = (U \oplus U^*)([v_{ij} \otimes I_n] \oplus I_{r(2m+n)})(U^* \oplus U). \end{aligned}$$

Since $U \oplus U^*$ is path connected to the identity matrix $I_{2r(m+n)}$ by a path of unitaries, it follows that

$$\phi_{mn,*,1}[V]_1 = [(\mathrm{Id}_{M_r} \otimes \phi_{mn})(V)]_1 = [[v_{ij} \otimes I_n]]_1 = n[V]_1.$$

If $P = [p_{ij}]$ is a projection in $M_r(U^{\mathrm{nc}}_{(m,n)})$, then let $X = u_0[p_{ij} \otimes I_n]$. Then

$$XX^* = u_0[p_{ij} \otimes I_n]u_0^* = (\mathrm{Id}_{M_r} \otimes \phi_{mn})(P)$$
$$X^*X = [p_{ij} \otimes I_n].$$

Hence,

$$\phi_{m,n,*,0}[P]_0 = [(\mathrm{Id}_{M_r} \otimes \phi_{mn})(P)]_0 = [[p_{ij} \otimes I_n]]_0 = n[P]_0.$$

The following theorem shows that the K-groups of $U^{\mathrm{nc}}_{(m,n)}$ are torsion groups.

Theorem 3.2. *$(n-m)K_j(U^{\mathrm{nc}}_{(m,n)}) = 0$ for $j = 0, 1$.*

Proof. We follow closely the proof given in [**7**] of the fact that $(n-1)K_j(O_n) = 0$. Let $\phi_{mn} : U^{\mathrm{nc}}_{(m,n)} \to M_m(U^{\mathrm{nc}}_{(m,n)})$ be defined as in the preceding remarks. Let u_{ij} be the generators of $U^{\mathrm{nc}}_{(m,n)}$. Let $x \otimes I_m$ denote the matrix $x \oplus \cdots \oplus x$ where there are m copies of x. If we let W denote the matrix $[\phi_{mn}(u_{ij})][u_{ij} \otimes I_m]^*$, then we have the following equation

$$[\phi_{mn}(u_{ij})] = [\phi_{mn}(u_{ij})][u_{ij} \otimes I_m]^*[u_{ij} \otimes I_m] = W[u_{ij} \otimes I_m].$$

Notice that W is a unitary in $M_{m^2}(U^{\mathrm{nc}}_{(m,n)}) \cong U^{\mathrm{nc}}_{(m,n)} \otimes M_m \otimes M_m$. Now we will prove the following claim.

Claim. *There is a path w_t of unitaries in $U^{\mathrm{nc}}_{(m,n)} \otimes M_m \otimes M_m$ from $w_0 = 1 \otimes I_m \otimes I_m$ to $w_1 = W$.*

To do this, we must first compute the entries of W. W has the form

$$W = \sum_{i,j=1}^{m} w_{ij} \otimes e_{ij}$$

where the e_{ij} are matrix units for M_m. By the definition of W, one has

$$w_{ij} = \sum_{k=1}^{n} \phi_{mn}(u_{ik})(u_{jk}^* \otimes I_m).$$

Elementary computations show that

$$w_{ij} = \sum_{r,s=1}^{m} w_{ij}^{rs} \otimes e_{rs},$$

where

$$w_{ij}^{rs} = \sum_{k,l=1}^{n} u_{rl}u_{ik}u_{sl}^* u_{jk}^*.$$

So we have that W is given by the following formula,

$$W = \sum_{k,l=1}^{n} \sum_{i,j,r,s=1}^{m} u_{rl}u_{ik}u_{sl}^* u_{jk}^* \otimes e_{rs} \otimes e_{ij}.$$

Define the following matrix $Z \in 1 \otimes M_m \otimes M_m$

$$Z = \sum_{i,j=1}^{m} 1 \otimes e_{ij} \otimes e_{ji}.$$

Then it is easily seen that Z is a self-adjoint unitary. Computations show that WZ is given by the following expression

$$WZ = \sum_{k,l=1}^{n} \sum_{i,j,r,s=1}^{m} u_{rl}u_{ik}u_{sl}^* u_{jk}^* \otimes e_{rj} \otimes e_{is}.$$

An elementary computation shows that WZ is a self adjoint unitary. Hence, there exist paths v_t, z_t of unitaries in $U_{(m,n)}^{\mathrm{nc}} \otimes M_m \otimes M_m$ such that $v_0 = z_0 = 1 \otimes I_m \otimes I_m$, $v_1 = WZ$, and $z_1 = Z$. Now let $w_t = v_t z_t$. Then w_t is a path of unitaries in $U_{(m,n)}^{\mathrm{nc}} \otimes M_m \otimes M_m$ from $w_0 = v_0 z_0 = 1 \otimes I_m \otimes I_m$ to $w_1 = v_1 z_1 = WZZ = W$. This establishes the claim.

The unitaries w_t in the above claim induce unital $*$-homomorphisms $\psi_t : U_{(m,n)}^{\mathrm{nc}} \to M_m(U_{(m,n)}^{\mathrm{nc}})$, where $\psi_0(X) = X \otimes I_m$ and $\psi_1 = \phi_{mn}$. It is easy to check that $\psi_t(X)$ is a norm continuous path for each X in the $*$-algebra generated by the u_{ij}'s and, hence, for each $X \in U^{\text{nc}}_{(m,n)}$. Thus, $\psi_{0,*,j} = \psi_{1,*,j}$ for $j = 0, 1$. So for $X \in K_j(U^{\text{nc}}_{(m,n)})$,

$$mX = \psi_{0,*,j}(X) = \psi_{1,*,j}(X) = nX,$$

and the proof is complete. □

Corollary 3.3. $K_j(U_{(n-1,n)}) = 0$ *for* $j = 0, 1$.

A few useful observations can be made about the proof of Proposition 3.1. First of all, the map ϕ_{mn} is equal to $\alpha_{mn} \circ (x \mapsto x \otimes I_n)$ where α_{mn} is the map from $M_n(U^{\text{nc}}_{(m,n)})$ into $M_m(U^{\text{nc}}_{(m,n)})$ defined as follows

$$\alpha_{mn}([x_{kl}]) = [u_{ij}][x_{kl}][u_{ij}]^*.$$

This map is an isomorphism. In fact, the inverse is given by the following formula

$$\alpha_{mn}^{-1}([x_{kl}]) = [u_{ij}]^*[x_{kl}][u_{ij}].$$

This proves the following result, which in the case $m = 1$ reduces to the known result $O_n \cong M_n \otimes O_n$ [**15**].

Proposition 3.4. $M_m \otimes U^{\text{nc}}_{(m,n)} \cong M_n \otimes U^{\text{nc}}_{(m,n)}$.

Another useful idea in the proof of Proposition 3.1 is that of associating a unitary matrix with a homomorphism. Given any homomorphism ϕ from $U^{\text{nc}}_{(m,n)}$ into $M_k \otimes U^{\text{nc}}_{(m,n)}$, we can associate a unitary W_ϕ in $M_{mk} \otimes U^{\text{nc}}_{(m,n)}$ as follows

$$W_\phi = [\phi(u_{ij})][u_{ij} \otimes I_k]^*.$$

Thus,

$$[\phi(u_{ij})] = W_\phi[u_{ij} \otimes I_k].$$

Conversely, given a unitary W in $M_{mk} \otimes U^{\text{nc}}_{(m,n)}$, we can associate a homomorphism ϕ_W from $U^{\text{nc}}_{(m,n)}$ into $M_k \otimes U^{\text{nc}}_{(m,n)}$

$$[\phi_W(u_{ij})] = W[u_{ij} \otimes I_k].$$

The correspondences $W \leftrightarrow \phi_W$ and $\phi \leftrightarrow W_\phi$ are clearly inverses. Let $\mathrm{Hom}\,(U^{\mathrm{nc}}_{(m,n)}, M_k \otimes U^{\mathrm{nc}}_{(m,n)})$ have the point-norm topology, i.e., the weakest topology making the maps $x \mapsto ||\phi(x)||$ continuous. Then it is easy to see that the correspondence just mentioned is a homeomorphism. We have just proven the following proposition.

Proposition 3.5. *The map* $W \mapsto \phi_W$ *is a homeomorphism of* $\mathcal{U}(M_{mk} \otimes U^{\mathrm{nc}}_{(m,n)})$ *onto* $\mathrm{Hom}\,(U^{\mathrm{nc}}_{(m,n)}, M_k \otimes U^{\mathrm{nc}}_{(m,n)})$.

Considering the special case $k = 1$, we have the following corollary.

Corollary 3.6. *The map* $W \mapsto \phi_W$ *is a homeomorphism of* $\mathcal{U}(M_m \otimes U^{\mathrm{nc}}_{(m,n)})$ *onto* $\mathrm{End}\,(U^{\mathrm{nc}}_{(m,n)})$.

This corollary generalizes the known result that $\mathcal{U}(O_n)$ is homeomorphic to $\mathrm{End}\,(O_n)$ [**7**, Proposition 2.1] by letting $m = 1$.

4. Ext groups of $U^{\mathrm{nc}}_{(m,n)}$. We will now compute the strong and weak Ext groups of $U^{\mathrm{nc}}_{(m,n)}$ which will be denoted $\mathrm{Ext}^{\,s}(U^{\mathrm{nc}}_{(m,n)})$ and $\mathrm{Ext}^{\,w}(U^{\mathrm{nc}}_{(m,n)})$, respectively. First, we will give a brief discussion of the Ext semigroups $\mathrm{Ext}^{\,s}(A)$ and $\mathrm{Ext}^{\,w}(A)$ for a separable unital C^*-algebra A following the exposition presented in [**15**]. Let H denote a fixed separable infinite dimensional Hilbert space, and, once and for all, we will make a fixed identification of H with $H \otimes \mathbf{C}^n$ for each $n \in \mathbf{N}$. This induces identifications of $B(H \otimes \mathbf{C}^n) \cong B(H) \otimes M_n$ with $B(H)$ as well as $Q(H \otimes \mathbf{C}^n) \cong Q(H) \otimes M_n$ with $Q(H)$, where $Q(H)$ is the quotient of $B(H)$ by the ideal of compact operators on H. We let $E(A)$ be the set of all unital $*$-monomorphisms or *extensions* of A into $Q(H)$. Let $\pi : B(H) \to Q(H)$ denote the quotient map. We say that extensions τ_1 and τ_2 are strongly (respectively weakly) equivalent if there is a unitary $U \in B(H)$ (respectively $u \in Q(H)$) such that $\tau_1(a) = \pi(U)\tau_2(a)\pi(U^*)$ (respectively $\tau_1(a) = u\tau_2(a)u^*$) for all $a \in A$. We write $[\tau]_s$ (respectively $[\tau]_w$) for the strong (respectively weak) equivalence class of τ. Let $\mathrm{Ext}^{\,s}(A)$ (respectively $\mathrm{Ext}^{\,w}(A)$) denote the set of strong (respectively weak) equivalence classes on $E(A)$. If $\tau_1, \tau_2 \in E(A)$, define $\tau_1 \oplus \tau_2 \in E(A)$ by $\tau_1 \oplus \tau_2(a) = \tau_1(a) \oplus \tau_2(a)$.

Note that we have used the identification of $Q(H)\otimes M_2$ with $Q(H)$ in the definition. This addition is associative, commutative, and respects both the strong and weak equivalence classes. Thus, $\mathrm{Ext}^{\,s}(A)$ and $\mathrm{Ext}^{\,w}(A)$ become commutative semigroups when equipped with this addition. The zero elements of these semigroups can be defined as follows. We say that an extension τ is trivial if there is a unital $*$-representation $\theta : A \to B(H)$ such that $\tau = \pi \circ \theta$. Voiculescu showed in [**17**] (see also [**2**]) that all trivial extensions of A are strongly (and hence weakly) equivalent and that, if τ_0 is a trivial extension, then $[\tau_0]_s$ (respectively $[\tau_0]_w$) is the zero element of $\mathrm{Ext}^{\,s}(A)$ (respectively $\mathrm{Ext}^{\,w}(A)$). Trivial extensions always exist (see [**2**, Section 4]). However, additive inverses do not always exist [**1**]. In the case that additive inverses do exist in $\mathrm{Ext}^{\,s}(A)$ (and hence in $\mathrm{Ext}^{\,w}(A)$), $\mathrm{Ext}^{\,w}(A)$ is isomorphic to the quotient of the group $\mathrm{Ext}^{\,s}(A)$ by the subgroup consisting of all elements of the form $[\tau]_s$ where τ is weakly equivalent to a trivial extension. For further references on extension theory of C^*-algebras, see [**2, 4, 5**].

Now we are ready to determine $\mathrm{Ext}^{\,s}(U^{\mathrm{nc}}_{(m,n)})$ and $\mathrm{Ext}^{\,w}(U^{\mathrm{nc}}_{(m,n)})$. It has been shown by Pimsner and Popa [**16**] that $\mathrm{Ext}^{\,s}(O_n) \cong \mathbf{Z}$ and $\mathrm{Ext}^{\,w}(O_n) \cong \mathbf{Z}_{n-1}$ for $n \geq 2$. We will follow closely the approach used by Paschke and Salinas in [**15**] in their computation of the Ext groups of O_n. Much of what is presented here is repeated from [**15**] for the sake of completeness. The most significant differences occur in the proofs of Lemmas 4.2 and 4.3 where adjustments had to be made in order to circumvent the problem of $U^{\mathrm{nc}}_{(m,n)}$ not being simple for $m \neq 1$.

In order to compute the Ext semigroups of $U^{\mathrm{nc}}_{(m,n)}$ we need the following lemma, the proof of which can be found in [**15**, Lemma 1.1].

Lemma 4.1. *Let P and Q be projections in $B(H)$ and v a partial isometry in $Q(H)$ such that $vv^* = \pi(P)$ and $v^*v = \pi(Q)$. There is a partial isometry V in $B(H)$ such that*

(a) $\pi(V) = v$; *and*

(b) $VV^* \leq P$ *and* $V^*V \leq Q$.

Moreover, the integer $\dim(Q - V^*V) - \dim(P - VV^*)$ *is uniquely determined by these conditions.*

Now let $\tau \in E(U^{\mathrm{nc}}_{(m,n)})$. Let $v_\tau \in M_n(Q(H)) \cong Q(H^n)$ be defined as follows

$$v_\tau = \begin{bmatrix} \tau(u_{11}) & \cdots & \tau(u_{1n}) \\ \vdots & & \vdots \\ \tau(u_{m1}) & \cdots & \tau(u_{mn}) \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

Let $P = I_m \oplus 0_{n-m}$. Then $v_\tau v_\tau^* = \pi(P)$ and $v_\tau^* v_\tau = \pi(I_n)$. By Lemma 4.1, there exists a partial isometry $V_\tau \in B(H^n) \cong M_n(B(H))$ such that $\pi(V_\tau) = v_\tau$ and $V_\tau V_\tau^* \leq P$. Also the integer

$$m(\tau) = \dim\,(I_n - V_\tau^* V_\tau) - \dim\,(P - V_\tau V_\tau^*)$$

is well defined. We now show that m is constant on strong equivalence classes. Suppose σ is strongly equivalent to τ. Let U be a unitary on H such that

$$\sigma(\cdot) = \pi(U)\tau(\cdot)\pi(U^*).$$

Let $\overline{U}$ be the direct sum of n copies of U. Then $\overline{U}P = P\overline{U}$. Let v_σ be defined as v_τ is with σ in place of τ. Then $v_\sigma = \pi(\overline{U})v_\tau\pi(\overline{U}^*)$. Let $V_\sigma^0 = \overline{U}V_\tau\overline{U}^*$. Then $\pi(V_\sigma^0) = v_\sigma$ and $V_\sigma^0 V_\sigma^{0*} \leq P$ follows from $V_\tau V_\tau^* \leq P$ and $\overline{U}P = P\overline{U}$. So in the definition of $m(\sigma)$ we can take $V_\sigma = V_\sigma^0$. The fact that $m(\sigma) = m(\tau)$ now also follows from the fact that $\overline{U}$ and P commute. We also have that $m(\tau \oplus \sigma) = m(\tau) + m(\sigma)$. This can be seen by observing that there exists a unitary matrix u in M_{2n} such that $v_{\tau\oplus\sigma} = u(v_\tau \oplus v_\sigma)u^*$ and taking $V_{\tau\oplus\sigma}$ to be $u(V_\tau \oplus V_\sigma)u^*$. Thus m induces a semigroup homomorphism

$$\bar{m} : \mathrm{Ext}^{\,s}(U^{\mathrm{nc}}_{(m,n)}) \to \mathbf{Z}.$$

In the following two lemmas, we compute the kernel and range of $\bar{m}$.

Lemma 4.2. *$m(\tau) = 0 \Leftrightarrow \tau$ is trivial.*

Proof. ($\Rightarrow$). Suppose $\tau \in E(U^{\mathrm{nc}}_{(m,n)})$ and $m(\tau) = 0$. Let V_τ be as in Lemma 4.1. Then

$$\dim\,(I - V_\tau^* V_\tau) = \dim\,(P - V_\tau V_\tau^*).$$

So if we replace V_τ with $V_\tau + X_\tau$, where X_τ is a finite rank partial isometry with initial space $(I - V_\tau^* V_\tau)H$ and final space $(P - V_\tau V_\tau^*)H$, then we can assume $V_\tau^* V_\tau = I$ and $V_\tau V_\tau^* = P$. Now, since $V_\tau = V_\tau V_\tau^* V_\tau = PV_\tau$, it follows that the last $n-m$ rows of V_τ are identically zero. If we let T be the m by n matrix $[T_{ij}]$ consisting of the first m rows of V_τ, then the relations on V_τ imply that $T^*T = I_n$ and $TT^* = I_m$. Then, by the universal property of $U^{\mathrm{nc}}_{(m,n)}$, there is a unique unital $*$-homomorphism

$$\tau_0 : U^{\mathrm{nc}}_{(m,n)} \to B(H)$$

such that $\tau_0(u_{ij}) = T_{ij}$. Now $\pi(V_\tau) = v_\tau$ is equivalent to saying that $\pi(T_{ij}) = \tau(u_{ij})$ for all i,j. Thus, $(\pi \circ \tau_0)(u_{ij}) = \pi(T_{ij}) = \tau(u_{ij})$ for all i,j. Since $U^{\mathrm{nc}}_{(m,n)}$ is generated by the u_{ij}, it follows that $\tau = \pi \circ \tau_0$ and τ is trivial.

($\Leftarrow$). Now suppose that $\tau \in E(U^{\mathrm{nc}}_{(m,n)})$ is trivial. Let $\tau_0 : U^{\mathrm{nc}}_{(m,n)} \to B(H)$ be a unital $*$-homomorphism such that $\tau = \pi \circ \tau_0$. Let

$$V_\tau^0 = \begin{bmatrix} \tau_0(u_{11}) & \cdots & \tau_0(u_{1n}) \\ \vdots & & \vdots \\ \tau_0(u_{m1}) & \cdots & \tau_0(u_{mn}) \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

Then $V_\tau^0 V_\tau^{0*} = P$ and $V_\tau^{0*} V_\tau^0 = I_n$ since $\tau_0(1) = 1$. Also, $\pi(V_\tau^0) = v_\tau$ follows from $\pi \circ \tau_0 = \tau$. Hence, we can take $V_\tau = V_\tau^0$ in the definition of $m(\tau_0)$. Since

$$\dim(I_n - V_\tau^* V_\tau) = \dim(P - V_\tau V_\tau^*) = 0,$$

it follows that $m(\tau) = 0$. $\square$

Lemma 4.3. $\bar{m}(\mathrm{Ext}^s(U^{\mathrm{nc}})_{(m,n)})) = \mathbf{Z}$.

Proof. Since $\bar{m}$ is a semigroup homomorphism, it suffices to show that $\pm 1 \in \bar{m}(\mathrm{Ext}^s(U^{\mathrm{nc}}_{(m,n)}))$. To see this, let Q be a one-dimensional projection in $B(H)$. Decompose H into the following two internal direct sums

$$H = H_1 \oplus \cdots \oplus H_m$$
$$H = H^1 \oplus \cdots \oplus H^n \oplus QH,$$

where H_i and H^j are infinite dimensional for all $1 \le i \le m$ and $1 \le j \le n$. For $2 \le i \le m$, let R_{in} be an isometry of H_i onto $H^n \oplus QH$ which vanishes on H_k for $k \ne i$. For all other (i, j), let R_{ij} be an isometry of H_i onto H^j which vanishes on H_k for $k \ne i$. If R is the m by n matrix $[R_{ij}]$ and $\bar{Q} = Q \oplus 0 \oplus \cdots \oplus 0 \in M_m(B(H))$, then the following relations hold.

$$RR^* = I_m - \overline{Q}, \qquad R^*R = I_n.$$

Let $\tilde{Q} = Q \oplus 0 \oplus \cdots \oplus 0 \in M_n(B(H))$. A similar construction yields an m by n matrix $T = [T_{ij}]$ such that the following relations hold

$$TT^* = I_m, \qquad T^*T = I_n - \tilde{Q}.$$

The above relations on R and T imply the following relations on $\pi(R) = [\pi(R_{ij})]$ and $\pi(T) = [\pi(T_{ij})]$

$$\pi(R)\pi(R)^* = \pi(T)\pi(T)^* = I_m$$
$$\pi(R)^*\pi(R) = \pi(T)^*\pi(T) = I_n.$$

So there are unique unital $*$-homomorphisms

$$\tau, \sigma : U^{\mathrm{nc}}_{(m,n)} \to Q(H)$$

such that $\tau(u_{ij}) = \pi(R_{ij})$ and $\sigma(u_{ij}) = \pi(T_{ij})$. By adding trivial extensions to each, we can assume that τ and σ are $*$-monomorphisms. Let V_τ^0 be the n by n matrix whose first m rows are the rows of R and whose remaining rows are zero. Similarly, define V_σ^0 with respect to the matrix T. It not follows easily that $\pi(V_\tau^0) = v_\tau$ and $\pi(V_\sigma^0) = v_\sigma$. We also have the following inequalities

$$V_\tau^0 V_\tau^{0*} = RR^* \oplus 0_{n-m} = P - \tilde{Q} \le P$$
$$V_\tau^{0*} V_\tau^0 = R^*R = I_n.$$

Similarly, $V_\sigma^{0*} V_\sigma^0 = I_n - \tilde{Q}$ and $V_\sigma^0 V_\sigma^{0*} = P$. Thus, we can take V_τ and V_σ as in Lemma 4.1 to be V_τ^0 and V_σ^0, respectively. Hence

$$m(\tau) = \dim(I_n - V_\tau^* V_\tau) - \dim(P - V_\tau V_\tau^*) = 0 - \dim(\tilde{Q}) = -1$$

and

$$m(\sigma) = \dim(I_n - V_\sigma^* V_\sigma) - \dim(P - V_\sigma V_\sigma^*) = \dim(\tilde{Q}) - 0 = +1. \qquad \square$$

Theorem 4.4. $\operatorname{Ext}^s(U^{\mathrm{nc}}_{(m,n)})$ *is a group isomorphic to* $\mathbf{Z}$.

Proof. To see that $\operatorname{Ext}^s(U^{\mathrm{nc}}_{(m,n)})$ is a group, let $\tau \in E(U^{\mathrm{nc}}_{(m,n)})$. Then, by Lemma 4.3, there is a $\tau' \in E(U^{\mathrm{nc}}_{(m,n)})$ such that $m(\tau') = -m(\tau)$. Hence, $m(\tau \oplus \tau') = m(\tau) + m(\tau') = 0$. By Lemma 4.2, $\tau \oplus \tau'$ is trivial and hence $[\tau]_s + [\tau']_s = 0$. Now $\bar{m}$ is a group homomorphism from $\operatorname{Ext}^s(U^{\mathrm{nc}}_{(m,n)})$ to $\mathbf{Z}$ which is injective by Lemma 4.2 and surjective by Lemma 4.3. $\square$

Theorem 4.5. $\operatorname{Ext}^w(U^{\mathrm{nc}}_{(m,n)}) \cong \mathbf{Z}_{n-m}$.

Proof. Let G denote the subgroup of $\operatorname{Ext}^s(U^{\mathrm{nc}}_{(m,n)})$ consisting of the elements $[\tau]_s$ where τ is weakly equivalent to some trivial extension. It was remarked earlier that $\operatorname{Ext}^w(U^{\mathrm{nc}}_{(m,n)}) \cong \operatorname{Ext}^s(U^{\mathrm{nc}}_{(m,n)})/G$. So we need to compute the subgroup G. To this end, let $\tau_0 = \pi \circ \theta$ be a trivial extension where $\theta : U^{\mathrm{nc}}_{(m,n)} \to B(H)$. Let $u \in Q(H)$ be a unitary such that

$$\tau(\cdot) = u\tau_0(\cdot)u^*.$$

The unitary u lifts to either an isometry or a coisometry in $B(H)$. Let V be such a lift of u. By the von Neumann-Wold decomposition [**13**, Problem 118], V is unitarily equivalent to $U_+^n \oplus W$ where W is a unitary and U_+ is the unilateral shift operator. So we can replace τ by a strongly equivalent extension and write

$$\tau(\cdot) = \pi(U_+^n)\tau_0(\cdot)\pi(U_+^{*n}).$$

So G is the subgroup of $\operatorname{Ext}^s(U^{\mathrm{nc}}_{(m,n)})$ generated by $[\tau_1]_s$, where

$$\tau_1(\cdot) = \pi(U_+)\tau_0(\cdot)\pi(U_+^*).$$

Let $V^0_{\tau_1} \in M_n(B(H))$ be defined by

$$V^0_{\tau_1} = \begin{bmatrix} U_+\theta(u_{11})U_+^* & \cdots & U_+\theta(u_{1n})U_+^* \\ \vdots & & \vdots \\ U_+\theta(u_{m1})U_+^* & \cdots & U_+\theta(u_{mn})U_+^* \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

Then $\pi(V^0_{\tau_1}) = v_{\tau_1}$. Also $V^0_{\tau_1}V^{0*}_{\tau_1}$ is the direct sum of m copies of $U_+U_+^*$ (and is thus majorized by P) and $V^{0*}_{\tau_1}V^0_{\tau_1}$ is the direct sum of n copies of $U_+U_+^*$. So we can take $V_{\tau_1} = V^0_{\tau_1}$ in the definition of $m(\tau_1)$. Letting p denote the one-dimensional projection $1 - U_+U_+^*$ we see that

$$\begin{aligned} m(\tau_1) &= \dim\,(I_n - V^*_{\tau_1}V_{\tau_1}) - \dim\,(P - V_{\tau_1}V^*_{\tau_1}) \\ &= \dim \begin{bmatrix} p & & \\ & \ddots & \\ & & p \end{bmatrix} - \dim \begin{bmatrix} p & & & \\ & \ddots & & \\ & & p & \\ & & & 0_{n-m} \end{bmatrix} \\ &= n - m. \end{aligned}$$

So $G = (n-m)\mathbf{Z}$ and hence $\operatorname{Ext}^w(U^{\mathrm{nc}}_{(m,n)}) \cong \mathbf{Z}_{n-m}$. □

5. Concluding remarks. In the computation of $K_j(O_n)$ in [**8**], Cuntz made much use of the algebra $\mathcal{E}_n$ defined as follows. If O_{n+1} is generated by the isometries $S_1, \dots, S_{n+1}$, then $\mathcal{E}_n$ is defined to be $C^*(S_1, \dots, S_n)$. $\mathcal{E}_n$ was shown to be isomorphic to $C^*(T_1, \dots, T_n)$ where the T_j are any isometries satisfying $T_1T_1^* + \cdots + T_nT_n^* < 1$. If we let $T = [T_1 \cdots T_n]$, then we can describe the above relations as $TT^* < 1$ and $T^*T = I_n$. So $\mathcal{E}_n$ has a natural generalization $\mathcal{E}_{mn}$ with respect to the algebra $U^{\mathrm{nc}}_{(m,n)}$. Let $\mathcal{E}_{mn}$ be $C^*(u_{ij} \mid 1 \le i \le m, 1 \le j \le n)$ where u_{ij} are the generators of $U^{\mathrm{nc}}_{(m,n+1)}$. Then the generators u_{ij} of $\mathcal{E}_{mn}$ satisfy the relations $UU^* < I_m$ and $U^*U = I_n$, where U is the $m \times n$ matrix $[u_{ij}]$. Cuntz used the map from $\mathcal{E}_n$ to O_n which sends $S_i \in O_{n+1}$ to $S_i \in O_n$ in order to compute the K-groups of O_n. He showed that the kernel of the above map is the ideal generated by the projection $P = 1 - S_1S_1^* - \cdots - S_nS_n^*$. The elements of the form

$$S_{i_1} \cdots S_{i_r} P S^*_{j_1} \cdots S^*_{j_r} \tag{5.1}$$

form a system of matrix units $e_{(i_1,\dots,i_r),(j_r,\dots,j_1)}$. This fact follows from the relation $S_i^*S_j = \delta_{ij}1$ which is not the case for the u_{ij}'s in $U^{\rm nc}_{(m,n)}$ when $m \neq 1$. Thus, the span of the elements in (5.1) is isomorphic to M_{n^r}. This was then used to show that the kernel of the map from $\mathcal{E}_n$ to O_n is isomorphic to the algebra of compact operators on a separable Hilbert space. This fact was used often in the K-theory computations. In the case of the algebra $U^{\rm nc}_{(m,n)}$ there does not appear to be a simple method of determining the kernel of the natural map from $\mathcal{E}_{mn}$ to $U^{\rm nc}_{(m,n)}$ or its K-groups.

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