ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 3, Summer 1993

COVARIANT COMPLETELY POSITIVE MAPS AND LIFTINGS

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ABSTRACT. Covariant completely positive maps related to discrete unital C^*-dynamical systems are studied. We consider their application to the completely positive lifting problem for homomorphisms of certain reduced discrete group C^*-algebras into the Calkin algebra.

Let $\alpha : G \to \mathrm{Aut}\,(A)$ be an action of a discrete group G on a C^*-algebra A. Given a unitary representation $u : G \to \mathcal{U}(B)$ of G into the unitary group of a unital C^*-algebra B, a linear map $\varphi : A \to B$ is called *u-covariant* if $\varphi(\alpha_g(a)) = u_g\varphi(a)u_g^*$ for all $a \in A$ and $g \in G$. In this situation the triple (A, G, α) is called a discrete C^*-dynamical system, and φ will be referred to as a *(discrete) covariant map of* (A, G, α).

The purpose of this note is to explore the natural relation of discrete covariant completely positive maps to crossed products and to show their application to a certain lifting problem. We shall show that a discrete covariant completely positive map $\varphi : A \to B$ extends to a completely positive map on the crossed product $A \times_\alpha G$. This is done using the analog of the covariant version of Stinespring theorem [**8**] for bounded operators on Hilbert modules. Our main result states that, given a discrete unital C^*-dynamical system (A, G, α) such that $A \times_\alpha G$ nuclear and $A \times_\alpha G = A \times_{\alpha r} G$, a homomorphism $\sigma : C_r^*(G) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ has a completely positive lifting to $\mathcal{B}(\mathcal{H})$ precisely when there exists a covariant completely positive map $\varphi : A \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ with respect to the unitary representation of G in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, determined by σ. As an illustration, it follows that the invertible elements of the semigroup Ext of the Choi's algebra $C_r^*(\mathbf{Z}_2 * \mathbf{Z}_3)$ are determined by those monomorphisms $\sigma : C_r^*(\mathbf{Z}_2 * \mathbf{Z}_3) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ which can be extended to completely positive maps on the Cuntz algebra $\mathcal{O}_2$.

Recall that each covariant representation (π, u) of (A, G, α) on a Hilbert space canonically induces a representation $\pi \times u$ of the involutive

Received by the editors on July 23, 1991.

Banach convolution algebra $L^1(G,A)$ [**9**, 7.6]. The *crossed product*, $A\times_\alpha G$, of (A,G,α) is the universal enveloping C^*-algebra of $L^1(G,A)$. It is obtained as the completion of $L^1(G,A)$ with respect to the norm $||\cdot|| = \sup\{||(\pi\times u)(\cdot)|| : (\pi,u)\}$. The algebra $L^1(G,A)$ contains the dense subalgebra $K(G,A)$ consisting of all the functions with finite support. We shall use the same symbol $K(G,A)$ to denote its image in $A\times_\alpha G$. If A is unital, δ_g will denote the unitary element of $K(G,A)$, defined by $\delta_g(g) = I$ and $\delta_g(t) = 0$ for all $t \neq g$. Then every $y \in K(G,A)$ can be written in the form $y = \Sigma_g y(g)\delta_g$, and A can be identified with the C^*-subalgebra of $A\times_\alpha G$ via the embedding $a \to a\delta_e$.

The *reduced crossed product*, $A\times_{\alpha r} G$, of (A,G,α) is the C^*-algebra generated by $(\tilde{\pi}\times\tilde{\lambda})(L^1(G,A))$, where $(\tilde{\pi},\tilde{\lambda})$ is a regular covariant representation of (A,G,α) [**9**, 7.7]. Given a faithful representation π of A on a Hilbert space $\mathcal{H}$, the representation $(\tilde{\pi},\tilde{\lambda})$ is defined on a Hilbert space $l^2(G,\mathcal{H})$ by

$$(\tilde{\pi}(a)\xi)(g)=\pi(\alpha_{g-1}(a))\xi(g) \quad \text{and} \quad (\tilde{\lambda}_t\xi)(g)=\xi(t^{-1}g), \qquad \xi\in l^2(G,\mathcal{H}).$$

It can be shown that $A\times_{\alpha r} G$ is independent of the choice of π. If A is unital, $A\times_{\alpha r} G$ is generated by the operators $\tilde{\pi}(a), \tilde{\lambda}_g$ ($a \in A$, $g \in G$). In particular, in this case the reduced group C^*-algebra $C_r^*(G)$, generated by the image of the left regular representation λ of G on $l^2(G)$, can be identified with the C^*-subalgebra of $A\times_{\alpha r} G$ by means of the correspondence $\lambda_g \to \tilde{\lambda}_g$. From the universal property of $A\times_\alpha G$, there is a natural homomorphism Φ of $A\times_\alpha G$ onto $A\times_{\alpha r} G$. If A is unital, Φ can be described by $\Phi(\Sigma_g y(g)\delta_g) = \Sigma_g\tilde{\pi}(y(g))\tilde{\lambda}_g$. We say that $A\times_\alpha G = A\times_{\alpha r} G$ if Φ is an isomorphism.

We shall use some standard notations regarding Hilbert C^*-modules [**2, 6, 7**]. If E and F are Hilbert modules over a C^*-algebra B, $\mathcal{B}(E,F)$ will denote the set of all B-module homomorphisms from E into F, such that for each $T \in \mathcal{B}(E,F)$ there exists an adjoint (B-module homomorphism) $T^* : F \to E$ satisfying $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all $x\in E$, $y \in F$. Each $T\in\mathcal{B}(E,F)$ is a bounded operator with respect to the operator norm. In particular, the set $\mathcal{B}(E,E)(=\mathcal{B}(E))$ of all B-module endomorphisms of E that admit adjoints is a unital C^*-algebra. Note that $V^*TV \in \mathcal{B}(E)$ when $V\in\mathcal{B}(E,F)$ and $T\in\mathcal{B}(F)$. In the special case, when a unital C^*-algebra B is regarded as Hilbert module over itself with respect to the inner product $\langle x,y\rangle = x^*y$, $\mathcal{B}(B)$ is the C^*-algebra of all left multiplication operators on B.

The following is an "abstract" covariant version of Stinespring theorem. The proof essentially follows the lines of [**8**, Theorem 2.1] and [**6**, Theorem 4]; so we shall present only a sketch of it.

Proposition 1. *Let (A, G, α) be a unital C^*-dynamical system and $u : G \to \mathcal{U}(B)$ a unitary representation of G into a unital C^*-algebra B. If $\varphi : A \to B$ is a u-covariant completely positive map, then there exist*:

i) *a Hilbert B-module E and a covariant representation (π, w) of (A, G, α) into $\mathcal{B}(E)$;*

ii) *an element $V \in \mathcal{B}(B, E)$, such that*

1) *$\varphi(a)x = V^*\pi(a)Vx$ for all $a \in A$ and $x \in B$.*

2) *$w_g Vx = Vu_g x$ for all $g \in G$ and $x \in B$.*

Sketch of the proof. The algebraic tensor product $A \otimes B$ is a right B-module in the natural way. Define a B-valued positive semi-definite inner product on $A \otimes B$ by

$$\langle \Sigma_i a_i \otimes x_i, \Sigma_j c_j \otimes y_j \rangle = \Sigma_{i,j} x_i^* \varphi(a_i^* c_j) y_j, \qquad a_i, c_j \in A, x_i, y_j \in B.$$

The Hilbert B-module E is obtained as the completion of the quotient of $A \otimes B$ by the kernel $\mathcal{N}$ of $\langle , \rangle_{A\otimes B}$ with respect to the inner product given by $\langle \xi + \mathcal{N}, \zeta + \mathcal{N} \rangle = \langle \xi, \zeta \rangle_{A\otimes B}$, $\xi, \zeta \in A \otimes B$. The covariant representation (π, w) is determined by

$$\pi(a)(c \otimes y + \mathcal{N}) = ac \otimes y + \mathcal{N}; \qquad w_g(c \otimes y + \mathcal{N}) = \alpha_g(c) \otimes u_g y + \mathcal{N}.$$

The B-module homomorphism $V : B \to E$ is determined by $V(x) = I \otimes x + \mathcal{N}$. It is straightforward to check that $V^* : E \to B$ is given by $V^*(a \otimes x + \mathcal{N}) = \varphi(a)x$. In particular, $V \in \mathcal{B}(B, E)$, and $V^*\pi(a)V \in \mathcal{B}(B)$ is a left multiplication operator on B for each $a \in A$. The rest is routine. $\square$

Proposition 2. *Let (A, G, α) be a discrete unital C^*-dynamical system. If $u : G \to \mathcal{U}(B)$ is a unitary representation of G into a unital C^*-algebra B and $\varphi : A \to B$ is a u-covariant completely positive map of (A, G, α), then there is a completely positive map $\Psi : A \times_\alpha G \to B$ uniquely defined by $\Psi(y) = \Sigma_g \varphi(y(g)) u_g$ for all $y \in K(G, A)$.*

Proof. By Proposition 1 there is a covariant representation (π, w) of (A, G, α) in the C^*-algebra $\mathcal{B}(E)$ and an operator $V \in \mathcal{B}(B, E)$ satisfying 1) and 2). (π, w) gives rise to a homomorphism $\pi \times w : L^1(G, A) \to \mathcal{B}(E)$ uniquely determined by

$$(\pi \times w)(y) = \Sigma_g \pi(y(g)) w_g, \qquad y \in K(G, A).$$

From the universal property of $A \times_\alpha G$, $\pi \times w$ extends to a representation $\pi \times w : A \times_\alpha G \to \mathcal{B}(E)$. Consider a completely positive map $\phi : A \times_\alpha G \to \mathcal{B}(B)$ given by $\phi(x) = V^*(\pi \times w)(x)V$. By 1) and 2) of Proposition 1 we have

$$\phi(y)b = \Sigma_g V^* \pi(y(g)) w_g V b = \Sigma_g V^* \pi(y(g)) V u_g b = \Sigma_g \varphi(y(g)) u_g b$$

for all $y \in K(G, A)$ and $b \in B$. Composing ϕ with the natural isomorphism between $\mathcal{B}(B)$ and B, we obtain the required completely positive map $\Psi : A \times_\alpha G \to B$. $\square$

Corollary 3. *Let (A, G, α) be a discrete unital C^*-dynamical system such that $A \times_\alpha G = A \times_{\alpha r} G$, and B a unital C^*-algebra. Suppose $\tau : C_r^*(G) \to B$ is a unital $*$ homomorphism, and let $u : G \to \mathcal{U}(B)$ be the unitary representation defined by $u_g = \tau(\lambda_g)$. If $\varphi : A \to B$ is a unital, u-covariant completely positive map of (A, G, α), then there is a completely positive map $\Psi : A \times_{\alpha r} G \to B$ extending both τ and φ.*

Proof. By Proposition 2 there is a completely positive map $\Psi : A \times_{\alpha r} G \to B$ defined by $\Psi(\Sigma_g a_g \delta_g) = \Sigma_g \varphi(a_g) \tau(\lambda_g)$. Since φ is unital, the map Ψ extends both φ and τ. $\square$

Proposition 4. *Suppose α and β are actions of a discrete group G on unital C^*-algebras A and B, respectively. If $\varphi : A \to B$ is a completely positive map satisfying $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $a \in A$ and $g \in G$, then there is a completely positive map $\Psi : A \times_\alpha G \to B \times_\beta G$ given by*

$$\Psi(y) = \Sigma_g \varphi(y(g)) \delta_g, \qquad y \in K(G, A).$$

Proof. Identifying B with the C^*-subalgebra of $K(G, B)$ $(\subset B \times_\alpha G)$ via the embedding $b \to b\delta_e$, we have $\beta_g(b) = \delta_g b \delta_g^*$ for each $b \in B$ and

$g \in G$. Therefore, $\varphi(\alpha_g(a)) = \beta_g(\varphi(a)) = \delta_g \varphi(a) \delta_g^*$ for each $a \in A$, $g \in G$; and so φ is a δ-covariant completely positive map from A into $B \times_\beta G$. The required conclusion follows directly from Proposition 2. $\square$

Our next goal is to characterize the existence of completely positive liftings for extensions of the algebra of compact operators by certain reduced discrete group C^*-algebras. First we recall several key facts concerning extensions.

Let A be a separable unital C^*-algebra. Let $\mathcal{C}$ denote the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and $\eta : \mathcal{B}(\mathcal{H}) \to \mathcal{C}$ be the quotient homomorphism. A linear map $\Psi : A \to \mathcal{C}$ is said to have a lifting, if there exists a map $\phi : A \to \mathcal{B}(\mathcal{H})$ such that $\Psi = \eta \circ \phi$. Then ϕ is called a *lifting* of Ψ. Recall that an *extension* of the algebra of compact operators by A is a unital $*$ monomorphisms of A into $\mathcal{C}$. Two extensions are called (weakly) equivalent if they are conjugate by a unitary in $\mathcal{C}$. The set of all equivalence classes of extensions modulo this equivalence relation is denoted by $\mathrm{Ext}\,(A)$. $\mathrm{Ext}\,(A)$ is a commutative semigroup, where the sum $[\sigma] + [\tau]$ of two equivalence classes is defined to be the equivalence class of the extension obtained by composing the map $\sigma \oplus \tau : A \to \mathcal{C} \oplus \mathcal{C} \subset M_2(\mathcal{C})$ with the natural isomorphism of $M_2(\mathcal{C})$ onto $\mathcal{C}$. As a consequence of [**10**], $\mathrm{Ext}\,(A)$ has the unique identity element, which turns out to be the equivalence class of all the extensions liftable to homomorphisms from A to $\mathcal{B}(\mathcal{H})$ (such extensions are called trivial). In [**1**] it is shown that an element $[\sigma]$ in $\mathrm{Ext}\,(A)$ is invertible precisely when σ has a unital completely positive lifting. In the case when A is nuclear, every completely positive contraction $\varphi : A \to \mathcal{C}$ has a completely positive lifting [**4**]. In particular, in this case $\mathrm{Ext}\,(A)$ is a group.

Theorem 5. *Let (A, G, α) be a discrete unital C^*-dynamical system such that $A \times_\alpha G = A \times_{\alpha r} G$, and $A \times_{\alpha r} G$ is nuclear. Suppose $\sigma : C_r^*(G) \to \mathcal{C}$ is a unital $*$ homomorphism, and let $u : G \to \mathcal{U}(\mathcal{C})$ be the unitary representation defined by $u_g = \sigma(\lambda_g)$. Then σ has a completely positive lifting if and only if there exists a unital u-covariant completely positive map $\varphi : A \to \mathcal{C}$.*

Proof. Suppose $\varphi : A \to \mathcal{C}$ is a unital, u-covariant completely positive map. From Corollary 3, there is a completely positive map $\Psi : A \times_{\alpha r} G \to \mathcal{C}$ extending σ. Since $A \times_{\alpha r} G$ is nuclear, Ψ has a completely positive lifting $\phi : A \times_{\alpha r} G \to \mathcal{B}(\mathcal{H})$ [**4**, Corollary 3.11]. Then the restriction $\phi|C_r^*(G)$, of ϕ to $C_r^*(G)$, is a completely positive lifting of σ.

Conversely, let $\theta : C_r^*(G) \to \mathcal{B}(\mathcal{H})$ be a completely positive lifting of σ. We can assume that θ is unital (see [**1**]). Since $\mathcal{B}(\mathcal{H})$ is injective, θ extends to a completely positive map defined on $A \times_{\alpha r} G$. Composing this map with the quotient homomorphism η, we get a unital completely positive map $\zeta : A \times_{\alpha r} G \to \mathcal{C}$ extending σ. Let φ be the restriction of ζ to A. Since δ_g $(g \in G)$ are in the multiplicative domain of ζ,

$$\varphi(\alpha_g(a)) = \zeta(\delta_g a \delta_g^*) = \zeta(\delta_g)\zeta(a)\zeta(\delta_g)^* = \sigma(\lambda_g)\varphi(a)\sigma(\lambda_g)^*$$

for all $a \in A$ and $g \in G$ (see [**3**, Proposition 3.2]). Consequently, φ is a u-covariant completely positive map. □

Remark. If (A, G, α) of the preceding theorem is such that A admits an α-invariant state ω, then there is a trivial u-covariant map $\varphi : A \to \mathcal{C}$ given by $\varphi(a) = \omega(a)I$. However, with the assumption $A \times_\alpha G = A \times_{\alpha r} G$ of the theorem, this implies the amenability of G (see [**11**, 5.2]).

Example. Let $G = \mathbf{Z}_2 * \mathbf{Z}_3$, and let $\mathcal{O}_2$ denote the C^*-algebra generated by two isometries with complementary ranges [**5**]. Then there is a Cantor set S and an action α of G on $\mathcal{C}(S)$, the continuous functions on S, such that $\mathcal{O}_2 = \mathcal{C}(S) \times_\alpha G = \mathcal{C}(S) \times_{\alpha r} G$.

Let a and b denote the generators of G satisfying $a^2 = b^3 = e$. Define S to be the set of all "infinite amalgamated words" in a and b; that is,

$$S = \{(ab^{i_1}ab^{i_2}\dots ab^{i_n}\dots) \text{ or } (b^{i_1}ab^{i_2}a\dots b^{i_n}a\dots) : i_n \in \{1, 2\} \text{ for each } n\}.$$

There is a natural bijection between S and the countably-infinite product of copies of the set $\{1, 2\}$. By means of this bijection S is provided with the topology coming from the product topology; so that S becomes a totally disconnected compact metrizable space. The left multiplication by elements of G defines an action of G by homeomorphisms on S. Let E denote the (closed and open) subset of S consisting of all the elements of S originating with a. It is easy to see that the sets E and $a(E)$ are disjoint and $E \cup a(E) = S$. Similarly, the sets $E, b(E)$ and $b^2(E)$ are pairwise disjoint and $E \cup b(E) \cup b^2(E) = S$. Let α be the induced action of G on $\mathcal{C}(S)$, given by $\alpha_g(f) = f \circ g^{-1}$, and let p denote the projection in $\mathcal{C}(S)$ corresponding to the characteristic function of E. If (π, u) is any covariant representation of $(\mathcal{C}(S), G, \alpha)$, then, from the preceding identities, $\pi(p) + u_a \pi(p) u_a^* = I$ and $\pi(p) + u_b \pi(p) u_b^* + u_b^* \pi(p) u_b = I$. By [**3**, Theorem 2.6], this shows that $\mathcal{C}(S) \times_\alpha G = \mathcal{O}_2$. Since $\mathcal{O}_2$ is simple, $\mathcal{C}(S) \times_\alpha G = \mathcal{C}(S) \times_{\alpha r} G = \mathcal{O}_2$.

The following corollary follows directly from Theorem 5 and the preceding example.

Corollary 6. *Let $\sigma : C_r^*(\mathbf{Z}_2 * \mathbf{Z}_3) \to \mathcal{C}$ be a unital $*$-monomorphism, and let $u : \mathbf{Z}_2 * \mathbf{Z}_3 \to \mathcal{U}(\mathcal{C})$ be the unitary representation given by $u(g) = \sigma(\lambda_g)$. The following are equivalent*:

(i) *The element $[\sigma]$ is invertible in* $\operatorname{Ext}(C_r^*(\mathbf{Z}_2 * \mathbf{Z}_3))$.

(ii) *There exists a unital u-covariant completely positive map $\varphi : \mathcal{C}(S) \to \mathcal{C}$.*

(iii) *σ extends to a completely positive map $\Psi : \mathcal{O}_2 \to \mathcal{C}$.*

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