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NEW PROOFS OF BING'S 1-ULC TAMING THEOREM AND BING'S SIDE APPROXIMATION THEOREM

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The main contents of this paper are new, shorter proofs of R.H. Bing's 1-ULC taming theorem [**5**] and Side Approximation theorem [**6**]. Most of this paper is drawn from the author's dissertation, which he completed under the direction of Professors William T. Eaton and Michael P. Starbird of The University of Texas at Austin. The author thanks them for their patience in teaching and guiding him and thanks Matthew V. Brahm and James W. Cannon for suggestions that have proven helpful in writing this paper. Ideas and theorems of James W. Cannon are used frequently in this paper, so the reader is urged to consult [**12**].

Theorem (Bing). *Suppose that* Σ *is a* 2*-sphere topologically embedded in* E^3 *and that* $\operatorname{Int}\Sigma$ *is* 1*-ULC. Then* $\Sigma \cup \operatorname{Int}\Sigma$ *is a* 3*-cell.*

Since the new proof makes no use of Bing's approximability-implies-tameness theorem [**4, 12**, pp. 361–362], the latter theorem follows as a corollary from his 1-ULC taming theorem.

Theorem (Bing). *Suppose that* Σ *is a* 2*-sphere topologically embedded in* E^3 *and that, for each* $\varepsilon > 0$*, there is an* ε*-homeomorphism from* Σ *into* $\operatorname{Int}\Sigma$*. Then* $\Sigma \cup \operatorname{Int}\Sigma$ *is a* 3*-cell.*

The proof of Bing's 1-ULC taming theorem follows from Lemma 1 and the tools of [**12**, pp. 373–376]. No proof of Lemma 1 will be given, as its proof is easier than and uses the same methods as the proof of Lemma 2.

Lemma 1. *Suppose that* Σ *is a* 2*-sphere topologically embedded in*

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E^3 and that $\operatorname{Int}\Sigma$ *is* 1-*ULC. Let* $D \subset \Sigma$ *be a disk, and let* W *be a neighborhood of* $\operatorname{Int} D$ *in* E^3*. Then there exists a homeomorphism* h *of* Σ *into* $\operatorname{Cl}(\operatorname{Int}\Sigma)$ *such that*:

1. *h is the identity on* $\Sigma \backslash \operatorname{Int} D$.
2. $h|(\operatorname{Int} D)$ *is* PL.
3. $h(\operatorname{Int} D) \subset W \cap \operatorname{Int}\Sigma$.

The proof of Bing's Side Approximation Theorem [**6**] follows from Lemma 2, [**11**, Theorem 2C.7(2)] and the tools of [**12**, pp. 373–376].

Lemma 2. *Suppose that* Σ *is a* 2-*sphere topologically embedded in* E^3 *and that* $F \subset \Sigma$ *is a* 0-*dimensional* F_σ-*set such that* $F \cup \operatorname{Int}\Sigma$ *is* 1-*ULC. Let* $D \subset \Sigma$ *be a disk, and let* W *be a neighborhood of* $\operatorname{Int} D$ *in* E^3*. Let* N *be a neighborhood of* $F \cap \operatorname{Int} D$ *in* $\operatorname{Int} D$*. Then there exist a homeomorphism* h *of* Σ *into* E^3 *and a locally finite collection* $\{D_i\}$ *of disjoint disks in* $\operatorname{Int} D$ *such that*:

1. *h is the identity on* $\Sigma \backslash \operatorname{Int} D$.
2. $h|(\operatorname{Int} D)$ *is* PL.
3. $h(\operatorname{Int} D) \subset W$.
4. $\cup\{D_i\} \subset N$.
5. $\operatorname{Int} D \backslash \cup \{\operatorname{Int} D_i\} \subset \operatorname{Ext} h(\Sigma)$.

Theorem 2C.7(2) (J.W. Cannon). *Let* $\Sigma \subset E^3$ *be a* 2-*sphere. Then there is a* 0-*dimensional* F_σ-*set* F *in* Σ *such that* $F \cup \operatorname{Int}\Sigma$ *is* 0-*ULC and* 1-*ULC.*

The point of Lemma 2 is to create an embedded spanning disk with PL interior. In the proof of Lemma 2, a singular spanning disk with PL interior will be created, but [**9**, Theorem (2.2)] will not be applied to the interior of the singular spanning disk since the result will be a spanning plane whose union with $\operatorname{Bd} D$ may fail to be a disk.

In the statement of the following theorem, observe that if A is a complementary domain of a compact subset C of a 3-manifold M into which R^2 is properly mapped by f, then there exists a disk $D \subset R^2$

such that $f(R^2 \backslash D) \subset A$.

Theorem (2.2) (E.M. Brown and C.D. Feustel). *Let M be a 3-manifold and let $f : R^2 \to M$ be a proper map. Suppose, for some compact $C \subset M$, that if $D \subset R^2$ is the disk and A the complementary domain of C with $f(R^2 \backslash D) \subset A$, then $[f|\mathrm{Bd}\, D] \notin G$ for G a normal subgroup of $\pi_1(A)$. Then, for any neighborhood U of $f(R^2)$, there is a proper embedding $g : R^2 \to M$ and a disk $E \subset R^2$ so that $g(R^2) \subset U$, $g(R^2 \backslash E) \subset A$, and $[g|\mathrm{Bd}\, E] \notin G$.*

Instead, the singular spanning disk is desingularized in three steps. We only outline the three steps here. The proof of Lemma 2 follows later.

The singular spanning disk will be thought of as the image of a map $\Psi : D \to E^3$, where for every $x \in \mathrm{Bd}\, D$, $\Psi(x) = x$. Let $\mathcal{A}$ be a collection of arcs properly embedded in D as shown in Figure 1. The first step will be to alter Ψ so that, for every $a \in \mathcal{A}$, there will be a neighborhood of $\mathrm{Int}\, a$ in $\mathrm{Int}\, D$ such that, for every point x in the neighborhood, $\Psi^{-1}(\{\Psi(x)\}) = \{x\}$. For each $a \in \mathcal{A}$, we will construct a singular open annulus whose two ends map to the two ends of $E^3 \backslash \mathrm{Bd}\, a$. The singular open annulus will be constructed so that applying [**7**, Corollary IV] to it will produce an embedded open annulus whose closure will be a 2-sphere Σ_a such that:

i) $\mathrm{Bd}\, a \subset \Sigma_a$.

ii) $\Sigma_a \backslash \mathrm{Bd}\, a$ is PL.

iii) $(\Sigma_a \backslash \mathrm{Bd}\, a) \cap \Sigma \subset N$.

iv) $\Psi(\mathrm{Int}\, a) \subset \mathrm{Int}\, \Sigma_a$.

v) If $J \subset \Sigma_a$ is a simple closed curve that separates $\mathrm{Bd}\, a$, then $J \not\subset \Psi(D)$.

Corollary IV (Brin and Thickstun). *Let $f : (S^1 \times E^1) \to M$ be proper carrying the ends of $S^1 \times E^1$ to different ends of M, and let H be a normal subgroup of $\pi_1 M$. If $[f|S^1 \times \{0\}]$ lies in $\omega(\pi_1 M) \backslash H$, then there is a proper embedding $g : (S^1 \times E^1) \to M$ such that $[g|S^1 \times \{0\}] \not\subset H$.*

FIGURE 1.

After cutting the singular disk $\Psi(D)$ off near $\cup_{a\in\mathcal{A}}\{\Sigma_a\}$ in the manner of [**12**, p. 374], $\Psi(D)$ will remain singular, but it will have the property that, for every $a \in \mathcal{A}$, there will be a neighborhood of Int a in Int D such that, for every point x in the neighborhood, $\Psi^{-1} = (\{\Psi(x)\}) = \{x\}$. The first step will then be complete.

Let $\mathcal{T}$ be the set of closed triangular regions into which the arcs of $\mathcal{A}$ divide D. The second step will be to desingularize the restriction of Ψ to every closed triangular region by using Corollary 1, a consequence of Brin and Thickstun's Near Disk Theorem [**7**, Theorem 2].

Corollary 1. *Let* $X \subset \operatorname{Bd} D^2$ *be a closed* 0*-dimensional set. Let* N *be a neighborhood of* $\operatorname{Bd} D^2 \backslash X$ *in* $D^2 \backslash X$*. Let* $f : D^2 \to E^3$ *be a map such that* $f|(D^2 \backslash X)$ *is* PL *and such that* $f^{-1}(\{f(x)\}) = \{x\}$ *for every* $x \in \operatorname{Bd} D^2 \cup N$*. Let* V *be an open neighborhood of* $f(\operatorname{Int} D^2)$ *in* $E^3 \backslash f(\operatorname{Bd} D^2)$*. Then there exist an open neighborhood* N' *of* $\operatorname{Bd} D^2 \backslash X$ *in* $D^2 \backslash X$ *and an embedding* $g : D^2 \to \operatorname{Cl}(V)$ *such that* $g|(D^2 \backslash X)$ *is* PL *and such that, for every* $x \in \operatorname{Bd} D^2 \cup N'$*,* $g(x) = f(x)$*.*

Theorem 2 (Brin and Thickstun). *Let* M *be a noncompact* 3*-manifold, let* $Z \subset M$ *be compact, let* V *be a component of* $M \backslash Z$*, and let* H *be a normal subgroup of* $\pi_1 V$*. If there is a proper allowable map* $f : G \to M$ *of a near disk that is* (Z, H)*-essential, and if* U *is an open set in* M *containing* $f(G)$*, then there is a proper embedding* $g : E \to M$ *that is an allowable replacement for* f*, that is* (Z, H)*-essential, and that has* $g(E) \subset U$*.*

A near disk is a disk from whose boundary a closed, nonempty subset has been removed. From the near disk G it is possible that boundary components are eliminated to obtain the near disk E; however, if f embeds a boundary component of G that is not dropped in passing from f to g, then g embeds that boundary component, too. The proof of Corollary 1 from [**7**, Theorem 2] is left for the reader.

If we define Ψ carefully enough, then the images under Ψ of the interiors of two nonadjacent closed triangular regions will be disjoint. Therefore, in the third step, we will cut the image of each "odd" triangular region off near the images of its neighboring "even" triangular regions to finish the desingularization of $\Psi(D)$.

Proof of Lemma 2. In this paragraph, identify D with a flat round disk so that sense can be made of the following constructions. Let $\mathcal{A}$ be a collection of arcs properly embedded in D as shown in Figure 1, and let $\mathcal{T}$ be the set of closed triangular regions into which the elements of $\mathcal{A}$ divide D. For every $a \in \mathcal{A}$, let a_+ and a_- be two arcs near a properly embedded in D as shown in Figure 1. Let Δ_a be the disk in D bounded by $a_+ \cup a_-$. By [**11**, Theorem 2C.7(2).1], we may assume that no arc in $\mathcal{A}$ intersects F.

Now we describe a collection of open sets that will give us control of the map $\Psi : D \to E^3$ to be defined. Let $\{N_r\}_{r\in A\cup\mathcal{T}}$ be a collection of open sets in E^3 such that:

i) For every $r \in \mathcal{A}\cup\mathcal{T}$, $N_r \subset W$.

ii) For every $a \in \mathcal{A}$, $\Delta_a\backslash\operatorname{Bd} D \subset N_a \subset \operatorname{Int} D \cup (E^3\backslash\Sigma)$.

iii) For every $T \in \mathcal{T}$, $T\backslash\operatorname{Bd} D \subset N_T \subset \operatorname{Int} D \cup (E^3\backslash\Sigma)$.

iv) For every $r, s \in \mathcal{A}\cup\mathcal{T}$, if $r \cap s \subset \operatorname{Bd} D$, then $N_r \cap N_s = \varnothing$.

v) For every $\delta > 0$, there exists a finite subset $\mathcal{Q}$ of $\mathcal{A}\cup\mathcal{T}$ such that if $r \in \mathcal{A}\cup\mathcal{T}$ and $\operatorname{diam}(N_r) > \delta$, then $r \in \mathcal{Q}$.

We later define the map $\Psi : D \to E^3$ and build the collection $\{\Sigma_a\}_{a\in\mathcal{A}}$ of 2-spheres so that $\Sigma_a\backslash\operatorname{Bd} D \subset N_a$ for every $a \in \mathcal{A}$ and so that initially $\Psi(T\backslash\operatorname{Bd} D) \subset N_T$ for every $T \in \mathcal{T}$.

For every $a \in \mathcal{A}$, let $a' \subset \Sigma$ be an arc such that $a'\cap D = \operatorname{Bd} a' = \operatorname{Bd} a$.

Step 1A. *Building* S_a *and* S'_a, *two "singular surfaces bounded by* $a' \cup a$". For every $a \in \mathcal{A}$, we build in $\operatorname{Int}\Sigma$ a singular surface which we use, after we have built the 2-sphere Σ_a and defined the map $\Psi : D \to E^3$, to ensure (1) that $\Psi(\operatorname{Int} a) \subset \operatorname{Int}\Sigma_a$, (2) that no simple closed curve in Σ_a separating $\operatorname{Bd} a$ is a subset of $\Psi(D)$, and (3) that the images under Ψ of two as-yet-to-be specified components of $\operatorname{Int} D\cap\Psi^{-1}(\Sigma_a)$ are disjoint. If we triangulate a disk D^2 modulo $\operatorname{Bd} D^2$, then by [**11**, Theorem 2A], there is an embedding of the union of $\operatorname{Bd} D^2$ and the set underlying the 1-skeleton of the triangulation such that the restriction of the embedding to the set underlying the 1-skeleton is PL into $\operatorname{Int}\Sigma$ and such that the image of $\operatorname{Bd} D^2$ is $a' \cup a$. Then, since $\operatorname{Int}\Sigma$ is 1-ulc [**11**, Theorem 2A] and has trivial first homology group [**11**, Theorem 2B.2], the image of the boundary of every 2-simplex in the triangulation bounds a singular surface in $\operatorname{Int}\Sigma$ that is small if the image of the boundary of the 2-simplex is near Σ. Since $a'\cup a$ may fail to be homologically trivial in $a'\cup a\cup\operatorname{Int}\Sigma$, $a'\cup a$ does not really bound the union of the singular surfaces in the homological sense, but we will abuse language by calling the union of $a'\cup a$ and the singular surfaces "the singular surface bounded by $a'\cup a$," which we will denote by S_a.

Also, for every $a \in \mathcal{A}$, we build in $a' \cup a \cup \operatorname{Ext}\Sigma$ a singular surface S'_a bounded by $a' \cup a$. After we build the map $\Psi : D \to E^3$ and the singular open annuli, neither $\Psi(D)$ nor any singular open annulus intersects $\operatorname{Ext}\Sigma$; however, when we PL approximate Ψ and the singular open annuli, we use S'_a to maintain control of Ψ and the singular open annuli by insisting that $\Psi(\operatorname{Int} D)$ and the singular open annuli remain disjoint from S'_a.

Step 1B. *Building the "top" of the singular open annulus near a.* Now, for every $a \in \mathcal{A}$, we begin building the 2-sphere Σ_a in N_a. As mentioned previously, we do this by building a singular open annulus running between the endpoints of a, desingularizing the singular open annulus, and then taking the union of the resulting embedded open annulus and the endpoints of a to get the 2-sphere Σ_a.

We begin building the singular open annulus near a.

First, we build over Δ_a a singular cap in which the "top" of the singular open annulus will lie. We let Δ_a be the domain of the singular cap. We may assume that the map from Δ_a to the singular cap is the identity map on $\operatorname{Bd}\Delta_a$, that the map PL embeds into $\operatorname{Int}\Sigma$ the set underlying the 1-skeleton of some triangulation of Δ_a modulo $\operatorname{Bd}\Delta_a$, and that the singular cap lies in $(\operatorname{Int}\Sigma \cup \operatorname{Bd}\Delta_a \cup F) \cap N_a$.

Step 1C. *Building the singular spanning disk $\Psi(D)$.* Since the singular cap over Δ_a does not intersect Int a, there is an arc $\alpha_a \subset S_a$, with $\operatorname{Int}\alpha_a$ lying in the image of the set underlying the 1-skeleton of the triangulation of D^2 modulo $\operatorname{Bd} D^2$ and with $\operatorname{Bd}\alpha_a = \operatorname{Bd} a$, lying so close to a that the piece of S_a bounded by $a \cup \alpha_a$ intersects the singular cap over Δ_a only in $\operatorname{Bd} a$.

For every $a \in \mathcal{A}$, there is an arc $\beta_a \subset S_a$, with $\operatorname{Int}\beta_a$ lying in the image of the set underlying the 1-skeleton of the triangulation of D^2 modulo $\operatorname{Bd} D^2$ and with $\operatorname{Bd}\beta_a = \operatorname{Bd} a$ lying so close to a that β_a intersects the piece of S_a bounded by $a' \cup \alpha_a$ only in $\operatorname{Bd} a$. If β_a is chosen close enough to a for every $a \in \mathcal{A}$, we may triangulate D modulo $\operatorname{Bd} D$ in such a way that $\cup\mathcal{A}$ is a subset of the set underlying the 1-skeleton of the triangulation and then build the map $\Psi : D \to E^3$ so that:

i) For every $x \in \operatorname{Bd} D$, $\Psi(x) = x$.

ii) The restriction of Ψ to the set underlying the 1-skeleton of the triangulation of D modulo $\operatorname{Bd} D$ is a PL embedding into $\operatorname{Int}\Sigma$.

iii) For every $a \in \mathcal{A}$, $\Psi(a) = \beta_a$.

iv) $\Psi(\mathrm{Int}\, D) \subset (\mathrm{Int}\, \Sigma) \cup (F \cap \mathrm{Int}\, D)$.

v) For every $a \in \mathcal{A}$, the piece of S_a bounded by $a' \cup \alpha_a$ intersects $\Psi(D)$ only in $\mathrm{Bd}\, a$.

vi) For every $T \in \mathcal{T}$, $\Psi(T \backslash \mathrm{Bd}\, D) \subset N_T$.

Step 1D. *Building the "bottom" of the singular open annulus near a.* Now we finish building the singular open annulus that will be used to give us the 2-sphere Σ_a near a. We have already build the singular cap over Δ_a in which the "top" of the singular open annulus will lie, so it remains for us to build the "bottom" of the singular open annulus.

For every $a \in \mathcal{A}$, there is an arc γ_a in S_a, with Int γ_a lying in the image of the set underlying the 1-skeleton of the triangulation of D^2 modulo $\mathrm{Bd}\, D^2$ and with $\mathrm{Bd}\, \gamma_a = \mathrm{Bd}\, a$, lying so close to a that the piece of S_a bounded by $a \cup \gamma_a$ intersects $\Psi(D)$ only in $\mathrm{Bd}\, a$.

In the union of $\mathrm{Bd}\, a$ and the set underlying the 1-skeleton of the triangulation of Δ_a modulo $\mathrm{Bd}\, \Delta_a$ that was PL embedded into the singular cap over Δ_a, there are an arc $a_{-0.5}$ so close to a_- and an arc $a_{+0.5}$ so close to a_+, with $\mathrm{Bd}\, a_{-0.5} = \mathrm{Bd}\, a_{+0.5} = \mathrm{Bd}\, a$, that the images in the singular cap over Δ_a of the disks bounded by $a_- \cup a_{-0.5}$ and $a_+ \cup a_{+0.5}$ intersect S_a only in $\mathrm{Bd}\, a$. If $a_{-0.5}$ is close enough to a_- and $a_{+0.5}$ is close enough to a_+, then we may find a new map from Δ_a into E^3 such that:

i) The restriction of the new map to the piece of Δ_a not in the interior of the disk bounded by $a_{-0.5} \cup a_{+0.5}$ agrees with the restriction of the original map from Δ_a into the singular cap over Δ_a to that same piece of Δ_a.

ii) The image of Δ_a under the new map intersects the piece of S_a bounded by $a' \cup \gamma_a$ only in $\mathrm{Bd}\, a$.

iii) The image of Δ_a under the new map lies in $(\mathrm{Int}\, \Sigma \cup \mathrm{Bd}\, \Delta_a \cup F) \cap N_a$.

The original map from the subdisk bounded by $a_{-0.5} \cup a_{+0.5}$ into the singular cap over Δ_a and the new map from the subdisk into E^3 form a singular open annulus whose ends map to the endpoints of a.

Step 1E. *Finding the locally finite collection* $\{D_i\}$ *of disjoint disks.* Now that we have the map $\Psi : D \to E^3$ and, for every $a \in \mathcal{A}$, the singular open annulus near a, we take for the collection $\{D_i\}$ of disjoint disks in N any locally finite (in $\operatorname{Int} D$) collection of disjoint disks in N such that the union of the disks' interiors contains the 0-dimensional, closed (in $\operatorname{Int} D$) intersection of Σ and the union of the singular open annuli and $\Psi(\operatorname{Int} D)$. We may assume that, for every $a \in \mathcal{A}$, $a \cap \cup\{D_i\} = \varnothing$. in the rest of the proof, we take care never to introduce intersections between $\Psi(\operatorname{Int} D)$ and Σ outside the set $\cup\{\operatorname{Int} D_i\}$. In other words, the intersection of $\Psi(\operatorname{Int} D)$ and Σ will no longer be a source of worry. Since $\Psi(\operatorname{Int} D)$ and the singular open annuli will all miss S'_a, we will know at the end of the proof that $\operatorname{Int} D \backslash \cup \{\operatorname{Int} D_i\} \subset \operatorname{Ext}\ h(\Sigma)$.

Step 1F. *Desingularizing the singular open annulus near* a *to get* Σ_a. Next, for each $a \in \mathcal{A}$, we want to use [**7**, Corollary IV] to desingularize the singular open annulus that we have just finished building near a, for the 2-sphere Σ_a is to be the closure in E^3 of the resulting embedded open annulus. For the normal subgroup H found in the hypotheses of [**7**, Corollary IV], we use the preimage of $H_1(E^3\backslash(a' \cup \beta_a))$ under the homomorphism between fundamental groups induced by an inclusion of a neighborhood of the singular open annulus into $E^3\backslash(a' \cup \beta_a)$.

To apply [**7**, Corollary IV] to the singular open annulus near a, we must first PL approximate it. If the approximation is close enough, then:

i) The singular PL open annulus is homologically nontrivial in $E^3\backslash(a' \cup \beta_a)$.

ii) The singular PL open annulus misses S'_a and the piece of S_a bounded by $\alpha_a \cup \gamma_a$.

iii) The singular PL open annulus lies in N_a.

iv) The singular PL open annulus intersects Σ in a subset of $\cup\{\operatorname{Int} D_i\}$.

Then we apply [**7**, Corollary IV] to the singular PL open annulus near a to get an embedded PL open annulus near a such that:

i) The embedded PL open annulus is homologically nontrivial in $E^3\backslash(a' \cup \beta_a)$.

ii) The embedded PL open annulus misses S'_a and the piece of S_a bounded by $\alpha_a \cup \gamma_a$.

iii) The embedded PL open annulus lies in N_a.

iv) The embedded PL open annulus intersects Σ in a subset of $\cup\{\operatorname{Int} D_i\}$.

For the 2-sphere Σ_a we take the closure of the embedded PL open annulus. Then Σ_a has the following properties:

a) $\Sigma_a \backslash \operatorname{Bd} a$ is PL.

b) $\Psi(\operatorname{Int} a) = \operatorname{Int} \beta_a \subset \operatorname{Int} \Sigma_a$, and $\Psi(\operatorname{Int} D) \cap S_a \subset \operatorname{Int} \Sigma_a$.

c) If a simple closed curve in Σ_a separates $\operatorname{Bd} a$, then the simple closed curve is not a subset of $\Psi(D)$.

d) $\Sigma_a \backslash \operatorname{Bd} a \subset N_a$.

e) $(\Sigma_a \backslash \operatorname{Bd} a) \cap \Sigma \subset \cup\{\operatorname{Int} D_i\}$.

Now we PL approximate $\Psi|(\operatorname{Int} D)$ without changing $\Psi|\operatorname{Bd} D$. If the PL approximation is close enough, the following are true:

i) For every $a \in \mathcal{A}$, $\Psi(\operatorname{Int} a) \subset \operatorname{Int} \Sigma_a$, and $\Psi(\operatorname{Int} D) \cap S_a \subset \operatorname{Int}\ \Sigma_a$, and $\Psi(\operatorname{Int} D) \cap S'_a = \varnothing$.

ii) If a simple closed curve in Σ_a separates $\operatorname{Bd} a$, then the simple closed curve is not a subset of $\Psi(D)$.

iii) For every $T \in \mathcal{T}$, $\Psi(T \backslash \operatorname{Bd} D) \subset N_T$.

iv) $\Psi(\operatorname{Int} D) \cap \Sigma \subset \cup\{\operatorname{Int}\ D_i\}$.

Assume that $\Psi(\operatorname{Int} D)$ and $\Sigma_a \backslash \operatorname{Bd} a$ are in general position.

Step 1G. *Cutting* $\Psi(D)$ *off near* $\cup_{a \in \mathcal{A}}\{\Sigma_a\}$. Methods like those of [**12**, p. 374] are used both in Step 1G and in Step 3.

For every $a \in \mathcal{A}$, there are closed triangular regions T_1 and T_2 in $\mathcal{T}$ whose intersection is a. Then $\operatorname{Int} D \cap \Psi^{-1}(\Sigma_a) \subset \operatorname{Int} T_1 \cup \operatorname{Int} T_2$. There are three types of components of $\operatorname{Int} D \cap \Psi^{-1}(\Sigma_a)$. The first type is a PL simple closed curve. There are infinitely many components of the first type. The second type is a simple closed curve that contains one of the endpoint of a and is PL modulo that endpoint of a, but from which that endpoint of a has been removed. There are infinitely many components of the second type. The third type is the interior of an arc joining the endpoints of a. There are an odd number of components of the third type in each of $\operatorname{Int} T_1$ and $\operatorname{Int} T_2$.

Here is the plan that we will execute next. We will cut $\Psi(D)$ off near $\cup_{a\in\mathcal{A}}\{\Sigma_a\}$ so that, for every $a \in \mathcal{A}$, there will remain in $\operatorname{Int} D \cap \Psi^{-1}(\Sigma_a)$ no components of the first or second types and only the two components of the third type that were originally farthest from a, one in each of Int T_1 and $\operatorname{Int} T_2$. The images under Ψ of the two components of the third type that will remain after the cutting off will be disjoint; otherwise, since $\Psi(\operatorname{Int} D) \cap S_a \subset \operatorname{Int} \Sigma_a$ and $\Psi(\operatorname{Int} D) \cap S'_a = \varnothing$ before any cutting off has occurred, we will be able to join points in different components of $\Sigma\backslash(a' \cup a)$ by an arc in E^3 missing $S_a \cup S'_a$. We will cut the singularities of the restriction of Ψ to neighborhoods of these two remaining components off near Σ_a so that, for every x in each of the neighborhoods, $\Psi^{-1}(\{\Psi(x)\}) = \{x\}$. Then, we will identify one of these two remaining components with a.

Now we complete the first step of the proof by carrying out the plan. For each $T \in \mathcal{T}$, there are arcs a_1, a_2, and a_3 in $\mathcal{A}$ whose union is $\operatorname{Bd} T$. Then $\operatorname{Int} T \cap \Psi^{-1}(\cup_{a\in\mathcal{A}}\{\Sigma_a\}) = \operatorname{Int} T \cap \Psi^{-1}(\Sigma_{a_1} \cup \Sigma_{a_2} \cup \Sigma_{a_3})$. For $i = 1, 2, 3$, there are in $\operatorname{Int} T \cap \Psi^{-1}(\Sigma_{a_i})$ an odd number of components of the third type. The component closest to a_i and the component next-to-farthest from a_i bound a disk from which the endpoints of a_i have been removed. We cut off near Σ_{a_i} to remove from $\operatorname{Int} T \cap \Psi^{-1}(\Sigma_{a_i})$ every component of the third type except the component that began farthest from a_i and to desingularize the restriction of Ψ to a neighborhood of the component that began farthest from a_i. (This component has a one-half chance of being the component that we will have identified with a_i when we have finished.) Using outermost remaining components of $\operatorname{Int} T \cap \Psi^{-1}(\Sigma_{a_1} \cup \Sigma_{a_2} \cup \Sigma_{a_3})$, which are of types one and two, we cut off near $\Sigma_{a_1} \cup \Sigma_{a_2} \cup \Sigma_{a_3}$. Then, after carrying out the foregoing plan for each $T \in \mathcal{T}$, for each $a \in \mathcal{A}$, we identify a with one of the two third-type components whose closures are arcs joining the endpoints of a.

The first step of the proof is complete. For every $a \in \mathcal{A}$, there is a neighborhood of $\operatorname{Int} a$ such that, for every x in that neighborhood, $\Psi^{-1}(\{\Psi(x)\}) = \{x\}$. Also, if $T_1, T_2 \in \mathcal{T}$ and $T_1 \cap T_2 \subset \operatorname{Bd} D$, then $\Psi(T_1\backslash\operatorname{Bd} D) \cap \Psi(T_2\backslash\operatorname{Bd} D) = \varnothing$.

Step 2. *Desingularizing the restriction of* Ψ *to each closed triangular region* T. Now, for each $T \in \mathcal{T}$, apply Corollary 1 to $\Psi|T$. Then the second step of the proof is complete, and

a) for every $T \in \mathcal{T}$, $\Psi|T$ is an embedding,

b) if $T_1, T_2 \in \mathcal{T}$ and $T_1 \cap T_2 \subset \operatorname{Bd} D$, then $\Psi(T_1 \backslash \operatorname{Bd} D) \cap \Psi(T_2 \backslash \operatorname{Bd} D) = \varnothing$, and

c) for every $a \in \mathcal{A}$, the singular set of Ψ (a union of disjoint double curves) does not intersect $\operatorname{Int} a$.

Step 3. *Desingularizing* Ψ. The third step of the proof resembles the end of the proof of [**9**, Theorem (2.2)], and again we employ the method of [**12**, p. 374].

The closed triangular regions of $\mathcal{T}$ we label "even" and "odd," where each "odd" region is surrounded by three "even" regions, and vice versa. The goal of the third step is to cut off the image of each "odd" triangular region near the images of its neighboring "even" triangular regions.

Now we begin cutting off. By removing "rolls," we may assume that, for every $a \in \mathcal{A}$, there are no double curves joining the endpoints of a. It remains for us to eliminate the other double curves.

We assign to each remaining double curve in each "even" triangular region a nonnegative integer. One is assigned to each outermost double curve. Two is assigned to each double curve that is outermost when we ignore outermost double curves. The number n is assigned to each double curve that is outermost when we ignore double curves to which integers less than n have been assigned.

For each double curve in each "odd" triangular region, we change Ψ in a neighborhood of the disk or disk-with-one-point-removed bounded by the double curve by cutting off near the image of the "even" triangular region T_e that contains the double curve's partner double curve. The nearness to the "even" triangular region of the replacement disk or disk-with-one-point-removed is determined by the number that has been assigned to the partner double curve. Identify $T_e \backslash \operatorname{Bd} D$ with $(T_e \backslash \operatorname{Bd} D) \times \{0\}$ in a product neighborhood $(T_e \backslash \operatorname{Bd} D) \times [-1, 1]$, and if n has been assigned to the partner double curve, put the replacement disk or disk-with-one-point-removed in $(T_e \backslash \operatorname{Bd} D) \times \{\pm(1 - 2^{-n})\}$.

Let h be the identity on $\Sigma \backslash \operatorname{Int} D$, and let $h = \Psi$ on $\operatorname{Int} D$. The proof is complete. □

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