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POLYHEDRAL NORMS IN AN INFINITE DIMENSIONAL SPACE

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ABSTRACT. In finite dimensional linear spaces, polyhedral norms have been widely studied. Many extensions of such notions to infinite-dimensional spaces are possible: in fact, several different definitions have been given, leading to different classes of spaces; the comparison among these classes has not been studied in detail.

In the present paper we prove equivalences and inclusions among the classes considered in this context, and we indicate some counterexamples.

1. Introduction. In a finite dimensional linear space over the real field, a polytope is the convex hull of a finite set of points, or equivalently, the intersection of a finite number of closed half spaces. A finite dimensional normed space is said to be polyhedral if its unit ball is a polytope; note that such a space is polyhedral if and only if its dual space is also polyhedral.

In an infinite dimensional normed space X, we may consider several properties concerning the unit ball of X or of its dual space, which reduce to polyhedrality when the dimension of X is finite. In this paper we study eight properties of this kind, which have been introduced in the literature, and we establish relations among them. We study in depth two among the more important ones (polyhedrality according to Klee and quasi-polyhedrality according to Amir and Deutsch); several equivalent formulations of them are given. The particular case of Lindenstrauss spaces (spaces whose dual is linearly isometric to $L_1(\mu)$ for some measure μ) leads to a simpler situation.

2. Preliminaries. Throughout the paper, X is a normed space over $\mathbf{R}$, and unless otherwise stated, X is assumed to be infinite dimensional; B is its closed unit ball and S its unit sphere. The space X will always

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be endowed with the topology of the norm. For a subset $A \subset X$, $\overline{A}$ and $\operatorname{int}(A)$ will indicate, respectively, the closure and the interior of A; for $x \in X$, $\mathcal{V}(x)$ indicates the set of all neighborhoods of x. We denote by X^* the topological dual of X and by B^* and S^*, respectively, the closed unit ball and the unit sphere of X^*. The space X^* will always be endowed with the $\sigma(X^*, X)$-topology, also called the w^*-topology. With this topology, B^* is known to be compact; $\mathcal{E}(B^*)$ will indicate the set of all extreme points of B^* and $\mathcal{E}'(B^*)$ its derived set; $\mathcal{E}(B^*)$ is known to be infinite (X^* is infinite dimensional); thus, $\mathcal{E}'(B^*)$ is nonempty (due to the w^*-compactness of B^*).

For $x \in X$, we denote by $\partial\gamma(x)$ the subdifferential of the norm at x. If $x \neq 0$, $\partial\gamma(x)$ is the face of $B^* : \{x^* \in S^*/x^*(x) = ||x||\}$; this is also called the peak-set. We have, moreover, $\partial\gamma(0) = B^*$ and the correspondence $x \rightrightarrows \partial\gamma(x)$ is known to be upper semicontinuous. For $x \neq 0$, the set $\partial\gamma_e(x) = \partial\gamma(x) \cap \mathcal{E}(B^*)$ is the set of all extreme points of the convex compact set $\partial\gamma(x)$. The directional derivative of the norm at x in the direction $v \neq 0$ is

$$\gamma'(x; v) = \lim_{t \to 0^+} (||x + tv|| - ||x||)/t;$$

it is related to the subdifferential $\partial\gamma(x)$ by

$$\gamma'(x; v) = \max\{x^*(v)/x^* \in \partial\gamma(x)\}.$$

Let x be a point in S. We introduce now three cones with vertex at x:

$$\begin{aligned} K(x) &= \{y \in X/\exists\, \lambda > 0, ||x + \lambda(y - x)|| < 1\}, \\ C(x) &= \{y \in X/\exists\, \lambda > 0, ||x + \lambda(y - x)|| \leq 1\}, \\ \Gamma(x) &= \{y \in X/y \neq x, \gamma'(x; y - x) \leq 0\}. \end{aligned}$$

We define also on S the function l_x:

for $u \in S$, $\quad l_x(u) = \max\{k \geq 0/x + ku \in B\} = \inf\{k > 0/x + ku \notin B\}$.

Let v be a point in X, $v \neq 0$. We introduce three sets:

$$Q_v = \{x \in X/\forall\, \lambda > 0, ||x - \lambda v|| > ||x||\},$$

$$P_v \text{ is the complementary set of } Q_v,$$

$$U_v = \{x \in X / \gamma'(x; -v) > 0\}$$
$$= \{x \in X / \partial\gamma(x) \cap \{x^* \in X^* / x^*(v) < 0\} \neq \varnothing\}.$$

Concerning the three kinds of cones, these sets and the function l_x, we collect in Proposition 1 some properties useful in the sequel.

Proposition 1. A. *Let $x \in S$. Then*

i) *$K(x)$ is open, $\Gamma(x)$ is closed, and we have*

$$\mathrm{int}\,(B) \subset K(x),\ B \subset C(x), \quad and \quad K(x) \subset C(x) \subset \overline{C(x)} = \overline{K(x)} \subset \Gamma(x).$$

ii) *Let $y \neq x$. Then*

$$y \in C(x) \text{ if and only if } x \in P_{x-y}$$
$$y \in \Gamma(x) \text{ if and only if } x \notin U_{x-y}.$$

iii) *The function l_x is upper semicontinuous on S, and, for $u \in S$, $l_x(u - x) > 0$ if and only if $u \in C(x)$.*

B. *Let $v \neq 0$. Then we have*

$$U_v \subset Q_v \subset \overline{U_v} = \overline{Q_v}.$$

Proof. The proofs are straightforward. For B, see [**8**]. □

3. Eight kinds of polyhedrality. We recall four definitions of polyhedrality for an infinite dimensional normed space X, which have been proposed.

Polyhedrality according to Klee [**15**]: every intersection of B with any finite dimensional subspace of X is a polytope. It is the most widely studied definition in the area. Some results concerning this notion will be given in Section 4.

Polyhedrality according to Bastiani [**3**–**4**]: if τ denotes the finest topology on X for which X is a locally convex Hausdorff vector space, B is τ-closed and, for every $x \in S$, $C(x)$ is τ-closed. In fact, in [**3**–**4**], Bastiani defines "pyramids" and "polyhedra" in an abstract setting.

Polyhedrality according to Maserick [**21**]: B is a "convex polytope" if, for every $x \in X$, the set $\{x^* \in \mathcal{E}(B^*)/x^*(x) > 1\}$ is finite. The following assertions are clearly equivalent:

(i) B is not a "convex polytope" according to Maserick,

(ii) there exist $x \in X$ and $x^* \in \mathcal{E}'(B^*)$ such that $x^*(x) > 1$,

(iii) there exists $x^* \in \mathcal{E}'(B^*)$ with $x^* \neq 0$.

Thus, polyhedrality according to Maserick means, in fact:

$$\mathcal{E}'(B^*) = \{0\}, \quad \text{or equivalently,} \quad \mathcal{E}'(B^*) \subset \{0\}.$$

Quasi-polyhedrality: for every $x \in S$, there exists $V \in \mathcal{V}(x)$ such that $K(x) \cap V = \operatorname{int}(B) \cap V$. This is the definition initially given by Amir and Deutsch [**1**]. This concept will be intensively studied in Section 5.

The following theorem establishes implications and equivalences among these properties and four other ones which appear in the literature.

Theorem 1. *With the following notations*:

(1) $\mathcal{E}'(B^*) \subset \{0\}$.

(2) *For all* $x^* \in \mathcal{E}'(B^*)$, *for all* $x \in S$, $x^*(x) < 1$,

(3) *For all* $x \in S$, *there exists* $I_x \subset \mathcal{E}(B^*)$, I_x *finite, such that* $\sup\{x^*(x)/x^* \in \mathcal{E}(B^*)\backslash I_x\} < 1$,

(4) *For all* $x \in S$, $\sup\{x^*(x)/x^* \in \mathcal{E}(B^*)\backslash \partial\gamma_e(x)\} < 1$,

(5) *X is quasi-polyhedral,*

(6) *For all* $\nu \neq 0$, U_ν *is closed,*

(7) *X is polyhedral according to Klee,*

(8) *X is polyhedral according to Bastiani,*

we have the implications:

$$1 \Rightarrow \begin{pmatrix} (2) \\ \Updownarrow \\ (3) \end{pmatrix} \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow \begin{pmatrix} (7) \\ \Updownarrow \\ (8) \end{pmatrix}.$$

The proof of the theorem is postponed to Section 6.

If X is finite dimensional, then each of the eight properties of Theorem 2 means that X is polyhedral and all these properties are equivalent.

If $X = c_0(\mathbf{N})$, or more generally, $X = c_0(\Gamma)$ where Γ is an abstract infinite set, then X satisfies (1) and therefore all other properties; but the duals of these spaces satisfy none of these properties.

The following implications between some pairs of properties are proved in the literature. (1) $\Rightarrow$ (8) is due to Maserick [**21**]. (2) is used by Gleit and McGuigan [**13**] and (3) by Lindenstrauss [**18**, Lemma 7.12] as sufficient conditions for a space to be polyhedral according to Klee. (5) $\Rightarrow$ (7), (1) $\Rightarrow$ (5), but (5) $\not\Rightarrow$ (1) are due to Amir and Deutsch [**1**]. (4) is introduced by Brosowski and Deutsch [**5**], and (6) appears in [**8**] where the implication (5) $\Rightarrow$ (6) is proved.

4. Polyhedrality according to Klee.

Theorem 2. *The following properties are equivalent:*

(K_1) *X is polyhedral according to Klee,*

(K_2) *For all $x \in S$, $C(x)$ is closed,*

(K_3) *For all $x \in S$, $\{u \in S/l_x(u) > 0\}$ is closed,*

(K_4) *For all $x \in S$, $C(x) = \Gamma(x)$,*

(K_5) *For all $\nu \neq 0$, $Q_\nu = U_\nu$.*

Proof. In a finite dimensional normed space, $(K_1) \Leftrightarrow (K_2)$ is proved in [**14**, Proposition 5.8]. As an immediate consequence, in our setting, $(K_2) \Rightarrow (K_1)$; indeed, if Y is a finite dimensional subspace of X, for $x \in S \cap Y$, $C(x) \cap Y$ is closed and then Y is polyhedral. We prove $(K_1) \Rightarrow (K_5)$ by the contrapositive; assume $Q_v \neq U_v$ for some $v \neq 0$; then there exists some $y \in Q_v \backslash U_v$, $y \neq 0$ and the two-dimensional subspace generated by v and y is not polyhedral, so X cannot be polyhedral. To prove $(K_5) \Rightarrow (K_4)$, let $x \in S$ and $y \in \Gamma(x)$, thus $y \neq x$; then (cf. Proposition 1), $x \notin U_{x-y}$ and (due to (K_5)) $x \notin Q_{x-y}$; i.e., $y \in C(x)$. Thus, (K_5) implies $\Gamma(x) \subset C(x)$ for every $x \in S$; it is sufficient since $C(x) \subset \Gamma(x)$ always. Now $(K_4) \Rightarrow (K_2)$, since $\Gamma(x)$ is closed, and finally (K_3) is obviously equivalent to (K_2). □

The above theorem allows us to answer a question raised in [**2**, p. 306]. Does the condition that B is polyhedral according to Klee imply that each $C(x)$ is closed, when x is an extreme point of B? The answer is clearly positive.

Since its introduction by Klee, polyhedrality in his sense has been widely studied. Instead of quoting all the articles on this subject, we indicate only some recent papers of Fonf [**9, 10, 11, 12**], who considers several different aspects of polyhedral spaces.

5. Quasi-polyhedral spaces.

Theorem 3. *The following properties are equivalent.*

(a_1) *X is quasi-polyhedral according to Amir and Deutsch, i.e., for all $x \in S$, there exists $V \in \mathcal{V}(x)$, $K(x) \cap V = int(B) \cap V$,*

(a_2) *for all $x \in S$, there exists $V \in \mathcal{V}(x)$, $\overline{K(x)} \cap V = B \cap V$,*

(a_3) *for all $x \in S$, there exists $V \in \mathcal{V}(x)$, $C(x) \cap V = B \cap V$,*

(a_4) *for all $x \in S$, there exists $\delta > 0$, for all $u \in S$, $l_x(u) > 0 \Rightarrow l_x(u) \geq \delta$,*

(b_1) *for all $x \in S$, there exists $V \in \mathcal{V}(x)$, for all $y \in V$, $\partial\gamma(y) \subset \partial\gamma(x)$,*

(b_2) *for all $x \in S$, there exists $V \in \mathcal{V}(x)$, for all $y \in V$, $\partial\gamma(x) \cap \partial\gamma(y) \neq \varnothing$,*

(b_3) *for all $x \in S$, there exists $V \in \mathcal{V}(x)$, for all $y \in V$, $\partial\gamma(x+y) \subset \partial\gamma(x) \cap \partial\gamma(y)$,*

(b'_1) *for all $x \in S$, there exists $V \in \mathcal{V}(x)$, for all $y \in V$, $\partial\gamma_e(y) \subset \partial\gamma_e(x)$,*

(b'_2) *for all $x \in S$, there exists $V \in \mathcal{V}(x)$, for all $y \in V$, $\partial\gamma_e(x) \cap \partial\gamma_e(y) \neq \varnothing$,*

(b'_3) *for all $x \in S$, there exists $V \in \mathcal{V}(x)$, for all $y \in V$, $\partial\gamma_e(x+y) \subset \partial\gamma_e(x) \cap \partial\gamma_e(y)$,*

(c_1) *for all* $x \in S$*, there exists* $V \in \mathcal{V}(x)$*, for all* $y \in V$*,* $||x+y|| = ||x|| + ||y||$*,*

(c_2) *for all* $x \in S$*, there exists* $V \in \mathcal{V}(x)$*, for all* $y \in S \cap V$*, the segment* $[x, y]$ *is included in* S*.*

Each of these properties may have its own interest. $(a_1), (a_2), (a_3)$ and (a_4) have a clear geometric meaning; (a_1) and (a_2) were introduced in [**1**] and [**7**], (a_3) in [**22**]. Formulation (b_1) which is probably the most useful is due to Wegmann [**23**]. The other formulations are more or less new; (c_1) links up quasi-polyhedrality with the wedges in which the norm is an additive function (cf. Day [**6**, p. 116]).

The following results: $(a_1) \Leftrightarrow (a_2) \Leftrightarrow (a_3) \Leftrightarrow (b_1)$ are known [**1, 7, 23**]. For the sake of completeness, we give here independent proofs.

Proof of Theorem 3. $(a_1) \Leftrightarrow (a_2) \Leftrightarrow (a_3)$: it is sufficient to observe that, if V is an open neighborhood of $x \in S$, then, according to Proposition 1,

$$\begin{aligned} \operatorname{int}(B) \cap V \supset K(X) \cap V &\Rightarrow B \cap \overline{V} \supset \overline{K(x)} \cap \overline{V} \\ &\Rightarrow B \cap \overline{V} \supset C(x) \cap \overline{V} \\ &\Rightarrow \operatorname{int}(B) \cap V \supset \operatorname{int} C(x) \cap V \Rightarrow \operatorname{int}(B) \cap V \supset K(x) \cap V. \end{aligned}$$

$(a_3) \Leftrightarrow (a_4)$: (a_4) is another formulation of (a_3). By considering the norm as the gauge of B, we obtain $(c_1) \Leftrightarrow (c_2)$. We prove now $(c_1) \Rightarrow (a_3)$; let $x \in S$ and $V \in \mathcal{V}(x)$ such that, for every $y \in V$, $||x+y|| = ||x|| + ||y||$. Let $y \in C(x) \cap V$, then for some $\lambda \in (0,1)$, $z = x + \lambda(x - y)$ is a member of B. As z belongs to the segment $[x, y]$, we have $||z|| = (1-\lambda)||x|| + \lambda||y||$, which implies $||y|| \leq 1$. Thus, $C(x) \cap V \subset B$. To establish (a_1) and $(a_2) \Rightarrow (c_2)$, we take in (a_1) and (a_2) the same neighborhood V of $x \in S$ and we assume that V is convex; let $y \in S \cap V$, so $y \in \overline{K(x)} \backslash K(x)$; then each z of the segment $[x, y]$ satisfies $z \in \overline{K(x)}$ and $z \notin K(x)$, whence $z \in B$ and $z \notin \operatorname{int}(B)$. Thus, $[x, y]$ is included in S.

We still have to prove the chain of implications:

$$\begin{array}{ccccc} (b_3) & \Longrightarrow & (b_1) & \Longrightarrow & (b_2) \\ \Uparrow & & & & \Downarrow \\ (c_1) & & & & (c_1) \\ \Downarrow & & & & \Uparrow \\ (b'_3) & \Longrightarrow & (b'_1) & \Longrightarrow & (b'_2) \end{array}$$

We will use the following lemma:

Lemma. *Let $x \neq 0$ and $y \neq 0$. The following are equivalent*:

(i) $||x+y|| = ||x|| + ||y||$

(ii) $\partial\gamma(x+y) \subset \partial\gamma(x) \cap \partial\gamma(y)$ (ii′) $\partial\gamma_e(x+y) \subset \partial\gamma_e(x) \cap \partial\gamma_e(y)$

(iii) $\partial\gamma(x+y) = \partial\gamma(x) \cap \partial\gamma(y)$ (iii′) $\partial\gamma_e(x+y) = \partial\gamma_e(x) \cap \partial\gamma_e(y)$

(iv) $\partial\gamma(x) \cap \partial\gamma(y) \neq \varnothing$ (iv′) $\partial\gamma_e(x) \cap \partial\gamma_e(y) \neq \varnothing$

Proof of the lemma. (i) $\Rightarrow$ (ii). Let $x^* \in \partial\gamma(x+y)$; then, from (i), $||x||+||y|| = ||x+y|| = x^*(x+y) = x^*(x)+x^*(y) \leq ||x||+||y||$. So we have $x^*(x) = ||x||$ and $x^*(y) = ||y||$; i.e., $x^* \in \partial\gamma(x) \cap \partial\gamma(y)$.

(ii) $\Rightarrow$ (iii). Indeed, $\partial\gamma(x) \cap \partial\gamma(y) \subset \partial\gamma(x+y)$ is always true: $x^*(x) = ||x||$ and $x^*(y) = ||y||$ imply $x^*(x+y) = ||x||+||y|| \geq ||x+y||$, and thus equality $x^*(x+y) = ||x+y||$ is the only one possible.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i). Let $x^* \in \partial\gamma(x) \cap \partial\gamma(y)$; then $||x+y|| \geq x^*(x+y) = x^*(x)+x^*(y) = ||x||+||y|| \geq ||x+y||$, so we have (i).

The same reasoning is available with $\partial\gamma_e$ instead of $\partial\gamma$.

We now finish the proof of Theorem 3. The lemma implies $(c_1) \Rightarrow (b_3) \Rightarrow (b_2) \Rightarrow (c_1)$ and $(c_1) \Rightarrow (b'_3) \Rightarrow (b'_2) \Rightarrow (c_1)$. Obviously, $(b_1) \Rightarrow (b_2)$ and $(b'_1) \Rightarrow (b'_2)$. It remains to establish $(b_3) \Rightarrow (b_1)$, (and analogously, $(b'_3) \Rightarrow (b'_1)$). Let $x \in S$ and $V_0 \in \mathcal{V}(0)$ such that, for $y \in x+V_0$, $\partial\gamma(x+y) \subset \partial\gamma(x) \cap \partial\gamma(y)$. Since, for $v \in X$ and $k > 0$, $\partial\gamma(kv) = \partial\gamma(v)$, then $\partial\gamma((x+y)/2) \subset \partial\gamma(x) \cap \partial\gamma(y)$. But $y \in x+V_0$

means $(x+y)/2 \in x+V_0/2$, then we have $\partial\gamma(z) \subset \partial\gamma(x)$ for every $z \in x+V_0/2$. □

6. Proof of Theorem 1. Some counterexamples.

Proof of Theorem 1. (1) $\Rightarrow$ (2) is obvious.

We prove (3) $\Rightarrow$ (2), and then (2) $\Rightarrow$ (3) by the contrapositive. Let X not satisfy (2); thus there exists $x^* \in \mathcal{E}'(B^*)$ such that $x^*(\bar{x}) = 1$ for some $\bar{x} \in S$. We want to show that $\sup\{y^*(\bar{x})/y^* \in \mathcal{E}(B^*)\backslash I\} \geq 1$ for every finite subset I of $\mathcal{E}(B^*)$. In fact, given a finite set $I \subset \mathcal{E}(B^*)$, there exists a net $\{x^*_\alpha\}$ in $\mathcal{E}(B^*)\backslash I$ which w^*-converges to x^*; thus, $x^*_\alpha(\bar{x})$ converges to $x^*(\bar{x}) = 1$. This implies $1 \leq \sup_\alpha x^*_\alpha(\bar{x})$ and, therefore, $\sup\{y^*(\bar{x})/y^* \in \mathcal{E}(B^*)\backslash I\} \geq 1$; so (3) does not hold.

Let now X not satisfy (3). Thus, there exists $\bar{x} \in S$ such that, for every finite set $I \subset \mathcal{E}(B^*)$ and every $\varepsilon > 0$, some x^*_ε exists in $\mathcal{E}(B^*)\backslash I$ with $x^*_\varepsilon(\bar{x}) > 1-\varepsilon$. By choosing $\varepsilon_n = 1/n$, we may construct a sequence (x^*_n) such that, for all $n \geq 1$, $x^*_n \in \mathcal{E}(B^*)$, $x^*_n(\bar{x}) > 1-1/n$ and $x^*_{n+1} \notin \{x^*_1, \dots, x^*_n\}$. Let x^* be an accumulation point of the infinite family $\{x^*_n\} \subset \mathcal{E}(B^*)$; then $x^* \in \mathcal{E}'(B^*)$ and we have $x^*(\bar{x}) = 1$; thus (2) does not hold.

We prove (2) $\Rightarrow$ (4) also by the contrapositive. Let X not satisfy (4); then there exists $\bar{x} \in S$ and a sequence $x^*_n \in \mathcal{E}(B^*)$ such that, for all $n \geq 1$, $x^*_n(\bar{x}) > 1-1/n$ and $x^*_{n+1} \notin \{x^*_1, \dots, x^*_n\}$. We conclude as above that (2) does not hold.

We prove (4) $\Rightarrow$ (5) by using the upper semi-continuity of the subdifferential. Let X satisfy (4), and let $x \in S$; we choose k such that $\sup\{x^*(x)/x^* \in \mathcal{E}(B^*)\backslash\partial\gamma_e(x)\} < k < 1$. Then every $x^* \in \mathcal{E}(B^*)$ such that $x^*(x) > k$ is included in $\partial\gamma_e(x)$, and thus $\partial\gamma(x)$ is included in the w^*-open half space $\{y^* \in X^*/y^*(x) > k\}$. There exists some $V \in \mathcal{V}(x)$ such that $y^*(x) > k$ if $y \in V$ and $y^* \in \partial\gamma(y)$. Thus if $y^* \in \partial\gamma_e(y)$, then $y^* \in \mathcal{E}(B^*)$ and $y^*(x) > k$. Therefore, $\partial\gamma_e(y) \subset \partial\gamma_e(x)$ for every $y \in V$. This is property (b'_1) of Theorem 3.

By using property (b_1) of Theorem 3, we establish: (5) $\Rightarrow$ (6). Indeed, let $v \neq 0$ and $x \notin U_v$, what amounts to $\partial\gamma(x) \subset \{x^* \in X^*/x^*(v) \geq 0\}$; let $V \in \mathcal{V}(x)$ such that, for $y \in V$, $\partial\gamma(y) \subset \partial\gamma(x)$, then $\partial\gamma(y) \subset \{x^* \in X^*/x^*(v) \geq 0\}$. Thus, V is included in the complementary set of U_v, which is open.

To obtain (6) $\Rightarrow$ (7), we use Proposition 1 and (K_5) of Theorem 2: if U_v is closed, then $U_v = Q_v$, for any $v \neq 0$.

Finally, (7) $\Rightarrow$ (8). If X satisfies (7), then each $C(x)$, for $x \in S$, is closed in the norm-topology (Property (K_2) of Theorem 2); whence it is τ-closed and the same is true for the ball B. Conversely, if $C(x)$ is τ-closed for each $x \in B$, then the section of B with any finite dimensional subspace, which is naturally τ-closed, has the same property. Since there is only one locally convex topology compatible with the linear structure on a finite dimensional space, the section of B with such a subspace is a polytope [**14**, Proposition 5.8]. Thus, X satisfies (7). □

Theorem 1 leaves open some questions. We give below counterexamples which show that (2) or (3) does not imply (1), and (4) does not imply (2) or (3). Answers concerning the hypothetical implications: (7) or (8) $\Rightarrow$ (6), (6) $\Rightarrow$ (5) and (5) $\Rightarrow$ (4) are not known. Particularly, (7) $\Rightarrow$ (5) is an old open problem [**7**]. Note that (K_2) of Theorem 2 and (a_3) of Theorem 3, as well as (K_5) of Theorem 2 and (a_4) of Theorem 3, might perhaps be useful to attack this question. We add only a simple remark. If we define the new condition:

(H) each $\partial\gamma_e(x)$, for $x \neq 0$, is finite,

then it is easy to verify that:

(2) or (3) $\Rightarrow$ (H); (4) and (H) $\Rightarrow$ (3) or (2); (6) and (H) $\Rightarrow$ (5).

Section 7 describes a general setting in which all properties (2), (3), (4), (5), (6), (7), and (8) are equivalent.

Counterexamples. We denote briefly by c_0 (respectively, l_1) the space of real sequences $x = (x_n)_{n\geq 1}$ which converge to 0 (respectively, for which $\sum_{n\geq 1} |x_n|$ is finite). For $n \geq 1$, ε_n denotes the sequence $(0, \dots, 0, 1, 0, \dots)$, with 1 in the n^{th} place.

First example. Let X be the space of all the convergent sequences $(x_n)_{n\geq 1}$, such that $\lim_n x_n = (1/2)x_1 + (1/3)x_2$, endowed with the sup norm [**17**]. Then $X^* = l_1$ and the extreme points of B^* are the $\pm\varepsilon_n$ $(n \geq 1)$. The only w^*-accumulation points of $\mathcal{E}(B^*)$ are $\pm((1/2)\varepsilon_1 + (1/3)\varepsilon_2)$, both of norm less than one. Thus, this space X satisfies (2) but not (1).

Second example. Let X be the l_1-product of $\mathbf{R}$ and c_0, i.e., $X = (\mathbf{R} \times c_0)_{l_1}$ [**1**]. Then $X^* = (\mathbf{R} \times l_1)_\infty$ and $\mathcal{E}(B^*)$ is the set $\{(\pm 1, \pm\varepsilon_n)/n \geq 1\}$. For instance, $z^* = (1,0)$ is a point of $\mathcal{E}'(B^*)$ such that $z^*(z) = 1$ for $z = (1,0)$, a point of S. Thus, X does not satisfy (2). We prove now that X satisfies (4). To make explicit $\partial\gamma_e(z)$ for $z \in S$, we will use the notations: for $t \in \mathbf{R}$, $\sigma(t) = 1$ if $t > 0$, $\sigma(t) = -1$ if $t < 0$ and $\sigma(0) = \pm 1$. For $x \in c_0$, $x \neq 0$, set $J(x) = \{n \geq 1/|x_n| = ||x||_\infty = \sup_k |x_k|\}$. If $z = (\alpha, x)$ with $x \neq 0$, then $\partial\gamma_e(z)$ is the finite set $\{(\sigma(\alpha), \sigma(x_n)\varepsilon_n)/n \in J(x)\}$. If $z = (\alpha, 0)$ with $\alpha = +1$ or $\alpha = -1$, $\partial\gamma_e(z)$ is the infinite set $\{(\sigma(\alpha), \pm\varepsilon_n)/n \geq 1\}$. It is easy to verify that, for every $z \in S$, $\sup\{z^*(z)/z^* \in \mathcal{E}(B^*)\backslash\partial\gamma_e(z)\} < 1$, which amounts to (4).

Third example. Let X be c_0 with the norm $||x|| = \max((|x_i| + |x_j|)/2, i \neq j)$. This space is considered in [**19**] with another definition. Then $X^* = l_1$ with the norm $||x^*|| = \max\{\sum_{i=1}^\infty |x_i^*|, \max_i 2|x_i^*|\}$, and $\mathcal{E}(B^*)$ is the set $\{\pm(\varepsilon_i/2) \pm (\varepsilon_j/2)/i \neq j, i \geq 1, j \geq 1\}$. For instance, $z^* = \varepsilon_1/2$ is a point of $\mathcal{E}'(B^*)$ such that $x^*(x) = 1$ for $x = 2\varepsilon_1$, a point of S. Thus, X does not satisfy (2) (compare with [**19**]).

We prove now that X satisfies (4). To make explicit $\partial\gamma_e(x)$ for $x \in S$, we will use the notation:

$$L(x) = \{(i,j)/i \neq j, i \geq 1, j \geq 1, ||x|| = (|x_i| + |x_j|)/2\}.$$

$L(x)$ is infinite if $x = \pm\varepsilon_n$ $(n \geq 1)$, and finite in all other cases. If x is different from ε_n and $-\varepsilon_n$ for all n, then $\partial\gamma_e(x)$ is the finite set $\{(\sigma(x_i)\varepsilon_i + \sigma(x_j)\varepsilon_j)/2/(i,j) \in L(x)\}$ (σ is defined in the second example).

If $x = \pm\varepsilon_n$, $\partial\gamma_e(x)$ is the infinite set $\{(\sigma(x_n)\varepsilon_n \pm \varepsilon_m)/2/m \geq 1, m \neq n\}$. It is easy to verify that, for every $x \in S$, $\sup\{x^*(x)/x^* \in \mathcal{E}(B^*)\backslash\partial\gamma_e(x)\} < 1$, which is (4).

7. Polyhedral Lindenstrauss spaces. A Lindenstrauss space (abbreviated LS-space) is an L_1-predual, i.e., a space X such that X^*

is linearly isometric to an $L_1(\mu)$ space for some measure μ.

For an LS-space, conditions (2) through (8) are equivalent, as proved in [**13**]. We recall (see [**17**]) that, if an LS-space X satisfies one of these conditions, then $X^* = l_1(\Gamma)$ for a suitable set Γ. The class of LS-spaces satisfying (1) is strictly smaller (see, e.g., our first example). In fact, we have the following result:

Proposition 2. *If X is an LS-space, then X satisfies* (1) *if and only if $X = c_0(\Gamma)$.*

Proof. It is sufficient to prove that, if X is an LS-space and satisfies (1), then $X = c_0(\Gamma)$. Indeed, if an LS-space satisfies (1), then $\mathcal{E}(X^*) \cup \{0\}$ is w^*-closed; thus, according to [**16**, Proposition 4.6], it is a $C_\sigma(K)$ space (see, e.g., [**20**] for the definition). Moreover, in [**20**, p. 346], it is proved that a $C_\sigma(K)$ space satisfying (7) is a space $c_0(\Gamma)$, which concludes the proof. □

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