ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 3, Summer 1993

MULTIPLIERS OF SEQUENCE SPACES

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0. Introduction. Let $A = (a_{nk})$ be a triangular nonnegative regular summation matrix, that is, the elements a_{nk} of A satisfy the conditions

(1) $$a_{nk} = 0, \qquad k > n,$$

(2) $$a_{nk} \geq 0, \qquad k = 0, 1, \dots, n;\ n = 0, 1, \dots,$$

(3) $$\lim_{n\to\infty} a_{nk} = 0, \qquad k = 0, 1, \dots,$$

(4) $$\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk} = 1,$$

of [**4**, p. 43]. We denote by $\tilde{m}_A$ the linear space of sequences $s = \{s_n\}$ such that the A-transform

$$As = \{As\}_n = \left\{ \sum_{h=0}^{n} a_{nk} s_k \right\}$$

is bounded. We assume also:

(5) each column of A has at least one nonzero element.

Under the semi-norms p_n, q:

$$p_n = |s_n|, \qquad q = ||As||_\infty = \text{LUB}_n \left| \sum_{k=0}^{n} a_{nk} s_k \right|,$$

Received by the editors on August 8, 1989, and in revised form on October 1, 1991.
AMS *MOS Classification.* 40C05, 46A45.
This research was supported by a grant from NSFG 7686.

$\tilde{m}_A$ becomes an $F - K$ space, that is, a locally convex sequence space in which the coordinate functionals are continuous. Let $(c_0)_A$ be the closed subspace of sequences $t = \{t_n\}$ which are evaluated to 0 by A, that is, $(c_0)_A$ is the space of sequence $t = \{t_n\}$ such that $\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk}t_k = 0$. Under the norm

$$\begin{aligned} ||s||_A &= \text{GLB}\,\{q(s+t)|\, t \in (c_0)_A\} \\ &= \text{GLB}_{t\in(c_0)_A}\text{LUB}_n \left| \sum_{k=0}^{n} a_{nk}(s_k + t_k) \right| \end{aligned}$$

$\tilde{m}_A/(c_0)_A$ is a Banach space which we denote by m_A. We shall not distinguish between a sequence s in $\tilde{m}_A$ and its coset in m_A; we shall denote both by the symbol s. We will denote the norm of a sequence in m_A by $||s||_A$ or simply by $||s||$ when the meaning is clear.

Evidently, $||s||_A \geq \limsup |As|$. We will show that equality holds. There is an integer n_0 such that for each positive number $\varepsilon |(As)_n| < \limsup |As| + \varepsilon$ when $n > n_0$. Let the sequence t be defined by the equations

$$\begin{aligned} t_n &= s_n, \qquad n \leq n_0, \\ t_n &= 0, \qquad n > n_0. \end{aligned}$$

Since t is a null sequence, t is in $(c_0)_A$. Hence $t = 0$ and $s - t = s$ in m_A. Also

$$\begin{aligned} q(s-t) &= \text{LUB}_{n>n_0} \left| \sum_{k>n_0} a_{nk}(s_k - t_k) \right| \\ &\leq \limsup As + \varepsilon \end{aligned}$$

(note that $(A(s-t))_n = 0$ for $n \leq n_0$). Since ε is arbitrary $||s||_A \leq \limsup |As|$. Thus

$$||s||_A = \limsup As.$$

This formula will be used in the sequel.

We study the space $M(A)$ of multipliers of the space m_A into itself, that is, the space of sequences $\mu = \{\mu_n\}$ such that if $s = \{s_n\}$ is a sequence in m_A then the sequence μs is in m_A; and, moreover, if $s = 0$ in m_A, then $\mu s = 0$ in m_A, that is, whenever the matrix A evaluates the sequence s to 0, then it evaluates the sequence μs to 0. We will assume throughout that the matrix A satisfies conditions (1)–(5).

The space $M(A)$ is a commutative normed ring of operators on m_A with the usual operator norm

$$||\mu|| = \mathrm{LUB}_{||s||_A \leq 1} ||\mu s||_A.$$

Two operators μ and μ' are identified in $M(A)$ if $||\mu - \mu'|| = 0$, that is, if the matrix A evaluates each sequence $\{(\mu_n - \mu'_n)s_n\}$ with s in m_A to 0.

It follows from the uniform limitedness theorem that if μ is in $M(A)$, then $||\mu|| < \infty$.

It will be seen that if μ is in $M(A)$, then the sequence $\{\mu_n\}$ is bounded. As a bounded continuous function on the discrete space N of natural numbers the sequence $\{\mu_n\}$ has a continuous extension to βN, the Stone-Čech compactification of N. We will say a few words about the Stone-Čech compactification. A completely regular space X can be imbedded densely in a compact space βX such that every bounded continuous function on X has a continuous extension to βX; for a description of the Stone-Čech compactification we refer the reader to [**3**, pp. 82–93]. If μ is a sequence in $M(A)$, thus a bounded continuous function on N, we denote by μ^β its continuous extension to βN; if ν is a point in βN, the symbol μ^β_ν will express the fact that the function μ^β is evaluated at ν. If E is a subset of N the symbol E^β will denote the intersection of the closure of the set E in βN with $\beta N - N$.

For many matrices we will find that the condition:

$(*)$ if μ is in $M(A)$ and $\mathrm{GLB}|\mu_n| > 0$, then $1/\mu$ is in $M(A)$

holds. If $(*)$ holds, then we can give some information regarding the maximal ideal space $\Delta(A)$ of $M(A)$. We recall that $\Delta(A)$ is defined to be the space of continuous homomorphisms of $M(A)$ into the complex numbers. If h is such a homomorphism and μ is in $M(A)$, we let $\hat{\mu}(h) = h(\mu)$; $\Delta(A)$ is the space of these homomorphisms h with the weakest topology which makes all functions $\hat{\mu}$, with μ in $M(A)$, continuous. In other words, $\Delta(A)$, as a subset of the unit sphere of the dual of $M(A)$, is given the weak $*$ topology. For more information on maximal ideal spaces we refer the reader to [**5**, pp. 50–51]. If $(*)$ holds, then $\Delta(A) = \beta N - N$ with two points ν_1, ν_2 of $\beta N - N$ identified if and only if $\mu^\beta_{\nu_1} = \mu^\beta_{\nu_2}$ for all μ in $M(A)$ and with $\beta N - N$ given the weakest topology making all functions μ^β continuous.

In case A is a normal matrix, that is, A is triangular with no zero elements on the main diagonal, then A has a reciprocal A^{-1}. In this case the sequence μ is in $M(A)$ if and only if the matrix $T = A\tilde{\mu}A^{-1} = (t_{nk})$ is regular on null sequences, that is, if T evaluates each null sequence to zero, where $\tilde{\mu}$ denotes the diagonal matrix with the sequence $\{\mu_n\}$ on its main diagonal. In order that the matrix $T = (t_{nk})$ be regular on null sequences it is necessary and sufficient that

$$\lim_{n\to\infty} t_{n,k} = 0$$

$$||T|| = \limsup \sum_{k=0}^{\infty} |t_{nk}| < \infty.$$

In fact $||\mu|| = ||T||$. The elements μ in $M(A)$ such that the matrix $T = A\tilde{\mu}A^{-1}$ is regular form an Abelian semigroup $\mathcal{S}$. In [**1**] at the suggestion of Professor George Piranian I studied $\mathcal{S}$ for various Hausdorff matrices A; here we will rarely deal with $\mathcal{S}$.

Section 1. In this section we obtain some conditions on sequences μ in $M(A)$ and draw conclusions about the maximal ideal space $\Delta(A)$. We begin with some general remarks about $\Delta(A)$. If $M(A)$ contains idempotents other than the zero and the unit element, then $M(A)$ is not connected. More generally, suppose that condition $(*)$ holds for a matrix A; for two infinite subsets E_1 and E_2 of N the sets E_1^β and E_2^β are separated in $\Delta(A)$ if and only if $\mathrm{GLB}_{\mu\in M(A),\nu_1,\nu_2\in\beta N-N}|\mu^\beta_{\nu_1} - \mu^\beta_{\nu_2}| > 0$. If $(*)$ holds for the matrix A and $\Delta\mu_n = o(1)$ for each sequence μ in $M(A)$, then $\Delta(A)$ is connected; if μ_n converges for each $\mu \in M(A)$, then $\Delta(A)$ is a point.

Theorem 1.1. *If μ is in $M(A)$, then the sequence $\{\mu_n\}$ is bounded and $||\mu|| \geq \limsup |\mu_n|$.*

Proof. We show first that if μ is a multiplier on $\tilde{m}_A$, then the sequence $\{\mu_n\}$ is bounded. For each m let $\delta^{(m)}$ denote the sequence with mth entry 1, all others 0, that is,

$$\delta^{(m)}_m = 1, \qquad \delta^{(m)}_n = 0, \qquad n \neq n.$$

The collection of sequences $\{\delta^{(m)}/||A\delta^{(m)}||_\infty\}$, $m = 0, 1, \dots$ is a bounded set in $\tilde{m}_A$. If μ is a multiplier on $\tilde{m}_A$, then the collection of sequences $\{\mu_n(\delta^{(m)})_n/||A\delta^{(m)}||_\infty\}$ is bounded in $\tilde{m}_A$, that is, there is a number M, such that for all m

$$q(\mu_n(\delta^{(m)})_n)/||A\delta^{(m)}||_\infty \le M$$

or

$$\text{LUB}_{m,n}(|\mu_n|a_{nm}/\text{LUB}_j a_{jm}) \le M.$$

Thus, if μ is a multiplier on $\tilde{m}_A$, μ is bounded. The same holds if μ is a multiplier on m_A.

To show that $||\mu|| \ge \limsup |\mu_n|$ it suffices to show that if λ is a cluster value of the sequence $\{\mu_n\}$, then λ is an eigenvalue of the operator μ. If λ is a cluster value of $\{\mu_n\}$, then there exists a sequence of integers $\{n_j\}$ increasing to infinity such that μ_{n_j} tends to λ; moreover, by passing to a subsequence if necessary we may take the numbers $\{n_j\}$ so that $\sum_j |\mu_{n_j} - \lambda| < \infty$. Let $y^{(n_j)}$ denote the sequence $A\delta^{(n_j)}/||A\delta^{(n_j)}||_\infty$, $j = 1, 2, \ldots$. If the numbers n_j are chosen so that, moreover,

$$\sum_{r<j} a_{n,n_r}/||A\delta^{(n_r)}||_\infty < 1/j \tag{6}$$

for $n_j \le n < n_{j+1}$, then the series $\sum_{j=1}^\infty y^{(n_j)}$ converges elementwise to a sequence y in $m - c_0$, that is, y is a bounded nonnull sequence. The sequence $z = \sum_{j=1}^\infty \delta^{(n_j)}/||A\delta^{(n_j)}||_\infty$ is in $m_A - (c_0)_A$. On the other hand, the sequence $(\mu - \lambda)z$ is in $(c_0)_A$; it is equal to the zero element in m_A. This shows that λ is an eigenvalue of the operator μ regarded as an operator on m_A. This completes the proof. □

Suppose that the matrix A has a reciprocal A^{-1}. If μ is in $\mathcal{S}$ and 0 is a cluster value of the sequence $\{\mu_n\}$, then the matrix $A\tilde{\mu}A^{-1}$ evaluates a bounded divergent sequence, namely, the sequence Az for the sequence z of the preceding paragraph. On the other hand, if the sequence $\{\mu_n\}$ is bounded away from 0 and $(*)$ holds, then $1/\mu$ is in $M(A)$, that is, $||T^{-1}|| = ||A\tilde{\mu}^{-1}A^{-1}|| < \infty$ and the matrix T transforms no unbounded sequence into a bounded sequence. A theorem of Darevsky [**2**] asserts that if a regular summation matrix evaluates a divergent sequence, then it evaluates an unbounded sequence. Hence, if $(*)$ holds, μ is in $\mathcal{S}$ and μ_n is bounded away from zero, then the matrix T evaluates precisely the convergent sequences.

Theorem 1.2. *If the matrix $A = (a_{nk})$ has a reciprocal and $\mu \in M(A)$, then*

$$||\mu|| \geq \limsup |\mu_n| + a_{n,n-1}|\Delta\mu_{n-1}|/a_{n-1,n-1}.$$

In particular, if $\mu \in M(A)$, then

$$\Delta\mu_n = 0(a_{n,n}/a_{n+1,n}).$$

Here $\Delta\mu_n$ denotes the difference $\mu_n - \mu_{n+1}$.

Proof. If we denote the matrix A^{-1} by (α_{nk}), then we have

$$\alpha_{n,n} = 1/a_{n,n},$$
$$\alpha_{n,n-1} = -a_{n,n-1}/a_{n,n}a_{n-1,n-1}.$$

We denote the matrix $A\tilde{\mu}A^{-1}$ by $T = (t_{nk})$. We have

$$\begin{aligned} t_{n,n-1} &= a_{n,n-1}\mu_{n-1}\alpha_{n-1,n-1} + a_{n,n}\mu_n\alpha_{n,n-1} \\ &= a_{n,n-1}\Delta\mu_{n-1}/a_{n-1,n-1} \\ t_{n,n} &= \mu_n. \end{aligned}$$

Since $||\mu|| = ||T|| \geq \limsup |t_{n,n}| + |t_{n,n-1}|$ the result follows. □

Theorem 1.2 shows that in general not every bounded sequence is in $M(A)$. For example, for the Euler Knopp matrix E_α, where α is a number in $(0,1)$, with elements l_{nk} given by the equations

$$l_{nk} = \binom{n}{k}\alpha^k(1-\alpha)^{n-k}, \qquad n = 0,1,\dots,\ k = 0,1,\dots,n,$$

Theorem 1.2 shows that if $\mu \in M(E_\alpha)$ then $\Delta\mu_n = 0(1/n)$. Consequently, if $(*)$ holds for the Euler Knopp matrix E_α, then $\Delta(E_\alpha)$ is connected. More generally, if A is a Hausdorff matrix with elements a_{nk} given by the equations

$$a_{nk} = \binom{n}{k}\int_0^1 u^k(1-u)^{n-k}\,d\chi(u),$$

where χ is a nondecreasing function on $[0,1]$ such that $\chi(0) = \chi(0+) = 0$, $\chi(1) = 1$, then A satisfies our hypotheses; if, moreover, $\chi(u) = \chi(1) = 1$ on some interval $1 - \delta \le u \le 1$ with $\delta > 0$, then we see that $a_{n,n-1} \ge n\delta a_{n-1,n-1}$. In this case, by Theorem 1.2, $\Delta\mu_n = 0(1/n)$ for all μ in $M(A)$; if $(*)$ holds for A, then $\Delta(A)$ is connected.

If the matrix A evaluates only convergent sequences, then, as is easily seen, $M(A)$ is the set of bounded sequences and $\Delta(A)$ is the totally disconnected space $\beta N - N$. This situation holds for some matrices A which evaluate some (indeed very few) divergent sequences.

For a matrix A, c_A denotes the convergence field of A, that is, the set of sequence s such that

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk} s_k$$

exists.

Theorem 1.3. *In order that $M(A)$ consist of the set of bounded sequences it is necessary and sufficient that*

(a) *every sequence s in m_A satisfy the condition*

$$\limsup \sum_{k=0}^{n} |a_{nk} s_k| < \infty,$$

and

(b) *every sequence t in c_A satisfy the condition*

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk} |t_k - \tau| = 0$$

for some complex number τ.

Of course, τ is the number to which the matrix A evaluates the sequence t.

Proof. Let $b_{nk} = a_{nk} s_k$, $n = 0, 1, \dots$, $k = 0, 1, \dots$, where s is a sequence in m_A. The sequence

$$\left\{ \sum_{k=0}^{n} a_{nk} \mu_k s_k \right\} = \left\{ \sum_{k=0}^{n} b_{nk} \mu_k \right\}$$

is bounded for each bounded sequence $\{\mu_n\}$ if and only if $\sum_{k=0}^{n} |b_{nk}|$ is bounded, that is, if and only if $\sum_{k=0}^{n} |a_{nk}s_k|$ is bounded. Now suppose that $t = 0$ in m_A. We have

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk}\mu_k t_k = 0$$

for all bounded sequences μ if and only if condition (b) holds for $\tau = 0$. But condition (b) holds if and only if it holds with $\tau = 0$. This completes the proof. □

Theorem 1.4. *Let E be an infinite subset of N. Then E^β is separated from $(N - E)^\beta$ in $\Delta(A)$ if and only if for every sequence s in m_A,*

$$\text{(a)} \qquad \limsup \left| \sum_{k\in E} a_{nk}s_k \right| < \infty,$$

and for every sequence t in $(c_0)_A$

$$\text{(b)} \qquad \lim_{n\to\infty} \sum_{k\in E} a_{nk}t_k = 0.$$

If $(*)$ holds and there exists a real sequence $u \in m_A$ satisfying the conditions

$$\text{(a}'\text{)} \qquad \begin{gathered} u_n \geq 0 \textit{ when } n \textit{ is in } E, \\ u_n < 0 \textit{ when } n \textit{ is in } N - E, \\ \limsup \sum_{k\in E} a_{nk}u_k = \infty, \end{gathered}$$

or a sequence v in $(c_0)_A$ such that

$$\text{(b}'\text{)} \qquad \begin{gathered} v_n \geq 0 \textit{ when } n \textit{ is in } E, \\ v_n < 0 \textit{ when } n \textit{ is in } N - E, \\ \limsup_{n\to\infty} \sum_{k\in E} a_{nk}v_k > 0, \end{gathered}$$

then E^β is not separated from $(N-E)^\beta$ in $\Delta(A)$.

Proof. If every real sequence s in m_A satisfies condition (a), then $s1_E$ is in m_A whenever s is in m_A. (By 1_E we understand the sequence $\{\mu_n\}$ with μ_n equal to 1 or 0 according to whether n is or is not in E.) If every sequence t in $(c_0)_A$ satisfies condition (b), then $t1_E$ is in $(c_0)_A$ whenever t is in $(c_0)_A$. Hence $1_E \in M(A)$ and E^β is separated from $(N-E)^\beta$ in $\Delta(A)$.

To show that E^β is not separated from $(N-E)^\beta$ we need only show that $\mathrm{GLB}_{\nu_1,\nu_2\in\beta N-N, v_1\neq v_2}|\mu^\beta_{\nu_1}-\mu^\beta_{\nu_2}|=0$ for every real sequence μ in $M(A)$. If there is a sequence u in $M(A)$ such that (a′) holds, then for every real sequence $\mu=\{\mu_n\}$ such that for some positive number δ, $\mu_n\geq\delta$ when n is in E, $\mu_n<0$ when n is in $N-E$, we have

$$\begin{aligned}\limsup_n\left|\sum_{k=0}^n a_{nk}\mu_k u_k\right| &\geq \limsup\sum_{k\in E} a_{nk}\mu_k u_k\\ &\geq \delta\limsup\sum_{k\in E} a_{nk}u_k=\infty;\end{aligned}$$

hence $\mu\notin M(A)$. If there exists a sequence v, in $(c_0)_A$ which is nonnegative on E, negative on $N-E$, for which (b′) holds, then

$$\limsup\left|\sum_{k=0}^n a_{nk}\mu_k v_k\right|\geq\delta\limsup\sum_{k\in E}a_{nk}v_k>0$$

by (b′). Thus $v=0$ in m_A, but $\mu v\neq 0$. Hence $\mu\notin M(A)$. Since δ is an arbitrary positive number E^β is not separated from $(N-E)^\beta$ in $\Delta(A)$. □

Corollary. *Suppose that $(*)$ holds. If E is a subset of N such that*

$$\lim_{n\to\infty}\sum_{k\in E}a_{nk}$$

exists and is different from 0 and 1, then E^β is not separated from $(N-E)^\beta$ in $\Delta(A)$.

We will say that the summation matrix A includes the summation matrix B if and only if every sequence evaluated by B is evaluated by A to the same value.

Suppose that $(*)$ holds for two matrices A and B satisfying our hypotheses and that A includes B. It is not true, in general, that the identity map of $\beta N - N$ induces a continuous map of $\Delta(B)$ into $\Delta(A)$; for a simple example we take A as the Cesàro matrix of order 1 with elements given by the equations

$$a_{nk} = 1/(n+1), \qquad k \le n, \qquad a_{nk} = 0, \qquad k > n.$$

The elements of α_{nk} of A^{-1} are given by the equations

$$\begin{aligned} \alpha_{n,n} &= n+1, \qquad \alpha_{n,n-1} = -n, \\ \alpha_{nk} &= 0, \qquad k \ne n,\ k \ne n-1 \end{aligned}$$

and the elements t_{nk} of the matrix $A\tilde{\mu}A^{-1}$ are given by the equations

$$\begin{aligned} t_{n,k} &= (k+1)\Delta\mu_k/(n+1), \qquad 0 \le k \le n-1, \\ t_{nn} &= \mu_n, \quad t_{n,k} = 0, \qquad k > n. \end{aligned}$$

For B we take the Nörlund matrix, with elements b_{nk} given by the equations

$$\begin{aligned} b_{n,n} &= b_{n,n-1} = 1/2, \qquad n \ge 1, \\ b_{n,k} &= 0, \qquad k \ne n,\ k \ne n-1. \end{aligned}$$

The elements β_{nk} of the matrix B^{-1} are given by the equation

$$\begin{aligned} \beta_{nk} &= 2(-1)^{n-k}, \qquad k \le n \\ \beta_{n,k} &= 0, \qquad k > n,\ n \ge 1, \end{aligned}$$

while the elements (s_{nk}) are the matrix $S = B\tilde{\mu}B^{-1}$ are given by the equations

$$\begin{aligned} s_{n,k} &= (-1)^{n-k-1}\Delta\mu_{n-1}, \qquad k < n \\ s_{n,n} &= \mu_n \\ s_{n,k} &= 0, \qquad k > n. \end{aligned}$$

The matrix T is regular on null sequence if and only if the quantities $\sum_{k=0}^{n-1}(k+1)|\Delta\mu_k|/(n+1) + |\mu_n|$ are bounded; this condition can be satisfied by many divergent sequences of zeros and ones. Thus $M(A)$ contains divergent sequences of zeros and ones and $\Delta(A)$ is not connected. On the other hand, if the matrix S is regular on null sequences, then $\Delta\mu_n = O(1/n)$, hence $\Delta(B)$ is connected (in fact $\Delta(B)$ is a nontrivial continuum). Hence, the identity map of $\Delta(B)$ into $\Delta(A)$ is not continuous. On the other hand, AB^{-1} is a regular matrix, that is, the matrix A includes the matrix B.

Conjecture. *Suppose that* $(*)$ *holds for two matrices* A *and* B *satisfying our hypotheses, that* A *includes* B*, and that whenever* A *evaluates a real sequence to its limit superior or to its limit inferior, then* B *evaluates the sequence to the same number. Then the identity map of* $\beta N - N$ *into itself induces a continuous map of* $\Delta(B)$ *into* $\Delta(A)$*.*

For the conjecture to hold, two points ν_1, ν_2 of $\beta N - N$ which are identified in $\Delta(B)$ must be identified in $\Delta(A)$.

Theorem 1.5. *Let* $\{\omega_n\}$ *be a sequence which increases to infinity. If the matrix* A *has the property that all sequences* s *in* m_A *satisfy the condition*

$$s_n = O(\omega_n), \tag{7}$$

while all sequences t *in* $(c_0)_A$ *satisfy the condition*

$$t_n = o(\omega_n), \tag{8}$$

then all sequences μ *such that* $\mu_n - L = O(1/\omega_n)$ *for some number* L *are in* $M(A)$*.*

Proof. The constant sequence $\{L, L\dots\}$ is in $M(A)$. Let $\varepsilon_n = L - \mu_n$; we must show that $\varepsilon = \{\varepsilon_n\}$ is in $M(A)$. If $s \in m_A$, then $s_n = O(\omega_n)$ and since $\varepsilon_n = O(1/\omega_n)$, the sequence $\{\varepsilon_n s_n\}$ is bounded; hence $\sum_{k=0}^{n} a_{nk}\varepsilon_k s_k$ is bounded. Also if a sequence t is in $(c_0)_A$, then by (8), $t_n = o(\omega_n)$ and hence $\{\varepsilon_n t_n\}$ is a null sequence and consequently in $(c_0)_A$. Consequently, A evaluates $\{\varepsilon_n t_n\}$ to 0. Consequently, ε is in $M(A)$. This completes the proof. □

We note that if μ_n is bounded away from 0 then $1/\mu_n - 1/L = 0(1/\omega_n)$ and thus $1/\mu \in M(A)$.

Theorem 1.6. *Suppose that the matrix A has a reciprocal $A^{-1} = (\alpha_{nk})$ with the property that α_{nk} is nonnegative or nonpositive according to whether $n-k$ is even or odd. Then if μ is a real sequence in $M(A)$ which is bounded away from 0 and such that for some number L, μ_n is alternately not less than L and not greater than L, then $1/\mu$ is in $M(A)$.*

Proof. Let $\varepsilon = \{\varepsilon_n\}$ denote the sequence $\{\mu_n - L\}$. The numbers ε_n alternate between being nonnegative and being nonpositive. The elements of the matrix $A\tilde{\varepsilon}A^{-1} = S = (s_{nk})$ have the property

$$|s_{nk} = \left|\sum_{j=k}^{n} a_{nj}\varepsilon_j\alpha_{jk}\right| = \sum_{j=k}^{n} a_{nj}|\varepsilon_j|\alpha_{jk}|.$$

If $\mu \in M(A)$, then since the constant sequence $\{L, L, \dots\}$ is in $M(A), \varepsilon \in M(A)$, that is, the matrix S is regular on the space of null sequences. Hence

$$\lim s_{nk} = 0, \qquad k = 0, 1, \dots . \tag{9}$$

$$||\varepsilon|| = ||S|| = \limsup \sum_{k=0}^{n} |s_{nk}| < \infty. \tag{10}$$

Since μ_n never vanishes, $1/\mu_n$ exists for all n and it is equal to $1/(L+\varepsilon_n)$; we denote this quantity by δ_n. We denote the matrix $T^{-1} = A\widetilde{(1/\mu)}A^{-1} = A\tilde{\delta}A^{-1}$ by (τ_{nk}); we have, for $k < n$,

$$\begin{aligned}\tau_{nk} &= \sum_{j=k}^{n} a_{nj}\alpha_{jk}/(L+\varepsilon_j)\\ &= \sum_{j=k}^{n} a_{nj}\alpha_{jk}(1/L - \varepsilon_j/L(L+\varepsilon_j))\\ &= -\sum_{j=k}^{n} a_{nj}\varepsilon_j\alpha_{jk}/L\mu_j\end{aligned}$$

since $\sum_{j=k}^{n} a_{nj}\alpha_{jk} = 0$ for $k < n$. Let GLB$|\mu_n|$ be denoted by η; $\eta > 0$. For $k < n$

$$\begin{aligned}|\tau_{nk}| &\le \left|\sum_{j=k}^{n} a_{nj}\varepsilon_j\alpha_{jk}\right|/|L|\eta \\ &\le |s_{nk}|/|L|\eta.\end{aligned}$$

By (9), $\lim_{n\to\infty}\tau_{nk} = 0$ for each k. Also

$$\sum_{k=0}^{n}|\tau_{nk}| \le \sum_{k=0}^{n-1}|s_{nk}|/|L|\eta + 1/\eta.$$

Thus $||T|| \le ||S||/|L|\eta < \infty$ and $1/\mu \in M(A)$. □

Theorem 1.7. *Suppose that there exist two infinite disjoint subsets E_1 and E_2 of N and two sequences $\{n_j\}$ and $\{k_j\}$ of integers increasing to infinity such that $n_j \le k_j < n_{j+1}$ and*

$$\max_{n_j\le n<n_{j+1},\,k\in E_1\cap[k_j,k_{j+1}]} a_{nk} \ne 0, \qquad j = 1, 2, \ldots$$

$$\left|\sum_{\substack{k\in E_1\\ k_j\le k<k_{j+1}}} a_{nk} - \sum_{\substack{k\in E_2\\ k_j\le k<k_{j+1}}} a_{nk}\right| \le \varepsilon_j \sum_{\substack{k\in E_1\\ k_j\le k<k_{j+1}}} a_{nk}, \tag{11}$$

and

$$\sum_{q=1}^{j-1}\sum_{\substack{k\in E_1\cup E_2\\ k_q\le k<k_{q+1}}} a_{nk}\Big/\max_{n_q\le m<n_{q+1}}\sum_{\substack{k\in E_1\cup E_2\\ k_q\le k<k_{q+1}}} a_{nk} < \varepsilon'_j \tag{12}$$

for $n_j \le n < n_{j+1}$, $j = 1, 2, \ldots$, where $\{\varepsilon_j\}$ and $\{\varepsilon'_j\}$ are positive null sequences. Then if $\mu \in M(A)$

$$\liminf_{j\to\infty} \max_{k\in E_\alpha\cap[k_j,k_{j+1}]}\mu_k - \min_{k'\in E_\beta\cap[k_j,k_{j+1}]}\mu_{k'} \ge 0$$
$$\alpha, \beta = 1, 2, \qquad \alpha \ne \beta.$$

Proof. Let A_j denote the quantity $\max\sum_{k\in E_j\cap[k_j,k_{j+1}]} a_{nk}$, where the maximum is over all n in $[n_j, n_{j+1}]$. We note that all A_j are nonzero.

We define the sequence s by the equations:

$$\begin{aligned} s_k &= 1/A_j && \text{if } k \in E_1, \\ s_k &= -1/A_j && \text{if } k \in E_2, \\ s_k &= 0 && \text{if } k \notin E_1 \cup E_2. \end{aligned}$$

We note that $s \in (c_0)_A$, i.e., $s = 0$ in m_A. For $n_j \le n < n_{j+1}$,

$$\begin{aligned} \left| \sum_{k=0}^{n} a_{nk}\mu_k s_k \right| \ge \left| \sum_{k \in E_1 \cap [k_j, k_{j+1}]} a_{nk}\mu_k/A_j \right| \\ - \left| \sum_{k' \in E_2 \cap [k_j, k_{j+1}]} a_{nk'}\mu_{k'}/A_j \right| + o(1). \end{aligned}$$

If n is in $[n_j, n_{j+1}]$ and maximizes $\sum_{k \in E_2 \cap [k_j, k_{j+1}]} a_{nk}$ over this interval, then by (11)

$$\sum_{k=0}^{n} a_{nk}\mu_k s_k \ge \min_{k \in E_1 \cap [k_j, k_{j+1}]} \mu_k - \max_{k' \in E_2 \cap [k_j, k_{j+1}]} \mu_{k'} - o(1).$$

If

$$\liminf_{j \to \infty} \Big(\max_{k' \in E_2 \cap [k_j, k_{j+1}]} \mu_{k'} - \min_{k \in E_1 \cap [k_j, k_{j+1}]} \mu_k \Big) < 0,$$

then $| \sum_{k=0}^{n} a_{nk}\mu_k s_k|$ is greater than some positive constant η, for some value of n between n_j and $n_{j=1}$ for arbitrarily large values of j; thus $||\mu s|| \ge \eta$ although $s = 0$. This contradicts the fact that μ is in $M(A)$. A similar argument rules out

$$\liminf \Big(\max_{k' \in E_1 \cap [k_j, k_{j+1}]} \mu_{k'} - \min_{k \in E_2 \cap [k_j, k_{j+1}]} \mu_k \Big) < 0.$$

This completes the proof. □

Corollary. *If there exist sequences of integers* $\{n_j\}$, $\{k_j\}$, $\{k'_j\}$ *with* $n_j > k'_j > k_j > n_{j-1}$, $j = 1, 2, \dots,$ *each sequence increasing to infinity, and*

$$\max_{n_j \le n < n_{j+1}} a_{n,k_j} \ne 0, \qquad \max_{n_j \le n < n_{j+1}} a_{n,k'_j} \ne 0$$

$$\sum_{q\le j-1} a_{n,k_q} \Big/ \max_{n_q\le m<n_{q+1}} a_m, k_q + \sum_{q\le j-1} a_{n,k_{q'}} \Big/ \max_{n\le m<n_{q+1}} a_{m,k_q} = o(1)$$

and

$$|a_{n,k_q} - a_{n,k'_q}| \le \varepsilon_j a_{n,k_q} \qquad n_j \le n < n_{j+1},$$

where $\{\varepsilon_j\}$ *is a decreasing null sequence, then* $\lim_{j\to\infty} \mu_{k_j} - \mu_{k'_j} = 0$. *Hence the sets* $\{k_j\}^\beta$, $\{k'_j\}^{(\beta)}$ *coincide in* $\Delta(A)$.

Theorem 1.8. *Suppose that* $E = \{n_j\}$ *is a sequence of natural numbers increasing to infinity in such a way that*

$$\max_{n_j\le n<n_{j+1}} a_{n,n_j} \ne 0,$$

$$\sum_{n<j} a_{n,n_r} \Big/ \max_{n_j\le m<n_{j+1}} a_{m,n_r} = o(1)$$

for $n_j \le n < n_{j+1}$, *and that every sequence* s *in* m_A *satisfies the condition*

$$s_{n_j} = O\Big(1\Big/ \max_{n_j\le m<n_{j+1}} a_{m,n_j}\Big)$$

while every sequence t *in* $(c_0)_A$ *satisfies the condition*

$$t_{n_j} = o\Big(1\Big/ \max_{n_j\le m<n_{j+1}} a_{m,n_j}\Big).$$

Then E^β *is separated from* $(N-E)^\beta$ *in* $\Delta(A)$. *If* $E^\beta \ne \beta N - N$, *then* $\Delta(A)$ *is not connected. If* $(*)$ *holds, then* E^β *is open and closed in* $\Delta(A)$.

Proof. The first part follows from Theorem 1.3 since every sequence s in m_A satisfies

$$\limsup \sum_{k\in E} |a_{nk}s_k| < \infty$$

while every sequence t in $(c_0)_A$ satisfies

$$\lim \sum_{k\in E} a_{nk}|t_k| = 0.$$

The proof of Theorem 1.3 shows that 1_E is in $M(A)$. If $E^\beta \neq \beta N - N$, then $\Delta(A)$ is not connected. Our remarks at the beginning of this section show that if $(*)$ holds E^β is open and closed in $\Delta(A)$.

We apply Theorem 1.8 to show that $\Delta(\overline{N}, p)$ is not connected, where $(\overline{N}, p)$ denotes the weighted means matrix generated by a sequence $\{p_n\}$ of positive numbers. This matrix $(\overline{N}, p)$ has elements a_{nk} given by the equations

$$a_{nk} = p_k/P_n, \qquad k \leq n,$$
$$a_{nk} = 0, \qquad k > n.$$

We assume that $P_n \to \infty$ and that $\lim_{n\to\infty} p_k/P_n = 0$; then the elements of $(\overline{N}, p_n)$ satisfy conditions (1), (2), (3), (4) and (5). Also the reciprocal $(\overline{N}, p_n)^{-1}$ exists. By considering the matrix $(N, p_n)^{-1}$ we see that each sequence s in $m_{(\bar{N}, p_n)}$ satisfies the condition $s_n = O(P_n/p_n)$ while each sequence t in $(c_0)_{(\overline{N}, p_n)}$ satisfies the condition $t_n = o(P_n/p_n)$. Since a set $E = \{n_j\} \neq N$ can be chosen so as to satisfy the conditions stated in Theorem 1.8, we conclude that $\Delta(\overline{N}, p_n)$ is not connected for each weighted means matrix with $p_n > 0$ for all n and $P_n \to \infty$.

In the case $p_n = 1$ for all n, the weighted means matrix reduces to the Cesàro matrix of order 1, $C^{(1)}$. We have seen earlier that $(*)$ holds for $C^{(1)}$ and that $\Delta(C^{(1)})$ is not connected.

We apply Theorem 1.8 to certain Nörlund matrices (N, p), where $\{p_n\}$ is a sequence of nonnegative numbers, $p_0 \neq 0$, $p(z) = \sum_{n=0}^\infty p_n z^n$. The elements of the matrix (N, p) are given by the formulas

$$a_{nk} = p_{n-k}/P_n, \qquad k \leq n,$$
$$a_{nk} = 0, \qquad k > n,$$

where $P_n = \sum_{k=0}^n p_k$. We will always assume that p_n/P_n tends to 0 as n tends to infinity; the Nörlund matrix (N, p) then satisfies conditions (1)–(5).

The elements α_{nk} of $(N, p)^{-1}$ are given by the formulas

$$\alpha_{nk} = P_k q_{n-k}, \qquad k \leq n,$$
$$\alpha_{nk} = 0, \qquad k > n,$$

where $p(z) = \sum_{n=0}^{\infty} p_k z^k$, $q(z) = 1/p(z) = \sum_{n=0}^{\infty} q_n z^n$. The elements t_{nk} of $(N,p)\tilde{\mu}(N,p)^{-1}$ are given by the formulas

$$t_{nk} = \left(\sum_{j=k}^{n} p_{n-j}\mu_j q_{j-k}\right) P_k/P_n, \quad \text{if } k \le n,$$
$$t_{nk} = 0, \qquad \text{if } k > n.$$

We first consider Nörlund matrices where $P_n \to \infty$. Every sequence s in m_A satisfies the condition

$$s_n = O\left(\sum_{k=0}^{n} P_k |q_{n-k}|\right).$$

If $\lim_{n\to\infty} q_{n-k} = 0$ for each k, then every sequence in $(c_0)_{(N,p)}$ satisfies the condition

$$t_n = o\left(\sum_{k=0}^{n} P_k |q_{n-k}|\right).$$

These facts can be deduced by observing the matrix $(N,p)^{-1}$. Hence, in the case $P_n \to \infty$, $\lim_{n\to\infty} q_{n-k} = 0$, $k = 0, 1, \dots$, if there exists a sequence of integers $\{n_j\}$ increasing to infinity in such a way that

$$\sum_{k=0}^{n_j} P_k |q_{n_j-k}| = O(P_{n_j}/p_{n_j})$$

then by Theorem 1.8, $\Delta(N, p_n)$ is not connected.

In the case $p_r = \binom{r+m-1}{r}$, $r = 0, 1, \dots$; thus $p(z) = (1-z)^{-m}$ and we obtain the Cesäro matrix of order m. If m is an integer greater than 1, then by calculating the matrix $C_m \tilde{\mu} C_m^{-1}$ we can show that $\Delta\mu_n = O(1/n)$ for all μ in $M(C^{(m)})$, and hence $\Delta(C^{(m)})$ is connected if $m > 1$. □

We consider Nörlund matrices (N,p) in the case where P_n is bounded; without loss of generality we may take P_n tending to 1. If $p(z)$ has no zeros in or on the boundary of the unit disc D, then, by a famous theorem of Wiener, $q(z) = 1/p(z) = \sum_{n=0}^{\infty} q_n z^n$ has the property that $\sum_{n=0}^{\infty} |q_n| < \infty$; hence $||(N,p)^{-1}|| < \infty$. The matrix (N,P) evaluates no unbounded sequences; by Darevsky's theorem this matrix evaluates no divergent sequences. For this reason we restrict attention to zeros of $p(z)$ in $\overline{D}$. We note also that if $p(z_0) = 0$ for some point $z_0 \neq 0$, then the matrix (N, p) evaluates the sequence $\{z_0^{-n}\}$ to 0.

Theorem 1.9. *If the polynomial* $p(z) = \sum_{j=0}^{r} p_j z^j$ *has no zero coefficients and all zeros of* $p(z)$ *are distinct and on the boundary* ∂D *of the unit disc* D *and* μ *is in* $M(N,p)$, *then* $\Delta\mu_n = o(1)$ *and consequently* $\Delta(N,p)$ *is a continuum. If the polynomial* $p(z)$ *has no zero coefficients all its* r *zeros are distinct and interior to* D *and* μ *is in* $\Delta(N,p)$, *then* $\Delta\mu_n = O(\alpha^n)$, *where* α *is the largest modulus of the zeros of* $p(z)$; *consequently in this case* $\Delta(N,p)$ *is a point.*

Proof. Suppose first that $p(z)$ has zeros $z_1, \dots, z_r$ with $z_i \neq z_j$ for $i \neq j$ and $|z_j| = 1$ for all j. For each j let the sequence $s^{(j)}$ be defined by the equations:

$$s_n^{(j)} = z_j^{-n}, \qquad n = 0, 1, \dots,\ j = 1, 2, \dots, r.$$

Each sequence $s^{(j)}$ is in $(c_0)_{(N,p_n)}$. If $\mu \in M(N,p)$, then the numbers μ_{n-m}, $m = 0, 1, \dots, r$ must satisfy the equations

$$p_r z_j^r \mu_{n-r} + \cdots p_1 z_j \mu_{n-1} + p_0 \mu_n = o(1), \qquad j = 1, 2, \dots, r, \tag{13}$$

and since $p_0 = -p_1 z_j - p_2 z_j^2 + \cdots p_r z_j^r$ for each j, the numbers μ_m must satisfy the conditions

$$\begin{aligned} p_r z_j^r(\mu_n - \mu_{n-r}) &+ p_{r-1} z_j^{r-1}(\mu_n - \mu_{n-r-1}) \\ &+ p_1 z_j(\mu_n - \mu_{n-1}) = o(1). \end{aligned} \tag{14}$$

The absolute value of the determinant of the equations (14), with the numbers $\mu_n - \mu_{n-m}$ considered as unknowns is $C \prod_{1 \le i \le r, 1 \le j \le r, i \neq j} |z_i - z_j|$ for some positive constant C. Hence $\Delta\mu_n = o(1)$ if all points z_j are on ∂D; consequently, $\Delta(N,p)$ is a continuum which may be a point.

If the zeros of $p(z)$, $z_1, z_2, \dots, z_r$ are interior to D and $\mu \in M(A)$, we again obtain (13), and in this case it follows that

$$\begin{aligned} p_r z_j^r(\mu_n - \mu_{n-r}) + p_{r-1} z_j^{(r-1)}(\mu_n - \mu_{n-r-1}) + p_1 z_j(\mu_n - \mu_{n-r-1}) \\ = o(\alpha^n), \end{aligned}$$

$j = 1, 2, \dots, r$, where $\alpha = \max |z_i|$. The determinant of this system, again with the numbers $\mu_n - \mu_{n-m}$ as unknowns, is equal to some positive constant, since the numbers z_i are distinct. It follows that in this case $\Delta\mu_{n-1} = \mu_{n-1} - \mu_n = o(\alpha^n)$, and hence $\{\mu_n\}$ must converge if μ is in $M(A)$. This completes the proof. □

In the case $p_0 = \alpha$, $p_1 = 1 - \alpha$, $p_n = 0$ for $n \geq 2$, $p(z) = \alpha + (1 - \alpha)z$. If $0 < \alpha \leq 1$, (1), (2), (3), (4) and (5) are satisfied. The function $p(z)$ has $-\alpha/(1 - \alpha)$ as its only zero. If $\alpha > 1/2, p(z)$ has no zeros in $\overline{D}$, $M(N, p)$ is l^∞/c_0 and $\Delta(N, p) = \beta N - N$. If $\alpha = 1/2$ we found that $\Delta(N, p)$ is a nontrivial continuum. If $0 < \alpha < 1/2$ it follows from Theorem 1.9 that $\Delta(N, p_n)$ is a point.

It is clear that $(*)$ holds for all Nörlund matrices of the form (N, p) with $p(z) = \alpha + (1 - \alpha)z$, $0 < \alpha \leq 1$.

2. Nilpotents in $M(A)$. Only if the hypotheses (a) and (b) in Theorem 1.3 are fulfilled have we been able to show that $M(A)$ is semi-simple. In general, the ring $M(A)$ contains nontrivial nilpotents. For example, if μ is a null sequence in $M(\overline{N}, p)$, where $M(\overline{N}, p)$ is a weighted means matrix satisfying our assumptions, then $\mu^2 = 0$. Since for matrices $(\overline{N}, p)$ of the form that we have considered, $M(\overline{N}, p)$ contains null sequences which are not equal to the zero element, $M(\overline{N}, p)$ is not semi-simple; in particular, this holds for the Cesàro matrix of order 1. Also for each Nörlund matrix (N, p) with $p_0 = \alpha$, $p_1 = 1 - \alpha$, $p_n = 0$, $n \geq 2$, where α is a constant in $(0, 1/2]$, $M(N, p)$ contains elements $\mu \neq 0$ such that $\mu^2 = 0$. Our general result is:

Theorem 2.1. *If $\{\omega_n\}$ is a sequence of real numbers increasing to infinity such that each sequence s in m_A satisfies the conditions $s_n = O(\omega_n)$ while each sequence t in $(c_0)_A$ satisfies the condition $t_n = o(\omega_n)$ and there is a sequence u in m_A, $u \neq 0$ such that $|u_n| \geq \eta\omega_n$ when n is in a subset E of N such that*

$$\sum_{k \in E} a_{nk} \geq \lambda,$$

where η and λ are positive constants, then $M(A)$ contains elements μ such that $\mu \neq 0$, $\mu^2 = 0$.

Proof. Let the sequence μ be defined by the equations

$$\mu_n = \operatorname{sgn} u_n/\omega_n, \qquad n \in E,$$
$$\mu_n = 0 \qquad n \notin E.$$

Then μ is in $M(A)$, and for each s in m_A

$$\begin{aligned}\left|\sum_{k=0}^{n} a_{nk}\mu_k^2 s_k\right| &\le \sum_{k\in E} a_{nk}|\mu_k^2 s_k| \\ &\le M\sum_{k\in E} a_{nk}(1/\omega_k)^2\omega_k \\ &\le M\sum_{k\in E} a_{nk}/\omega_k,\end{aligned}$$

where M is a positive constant. The last sum tends to 0 because $\{1/\omega_n\}$ is a null sequence and the matrix A is regular. Hence $\mu^2 s = 0$ for all s in m_A, thus $\mu^2 = 0$. But $||\mu u||_A \ge \lambda\eta$, hence $\mu \ne 0$.

In the case of the Euler Knopp matrix E_α, $0 < \alpha < 1$, we see, by considering the elements of E_α^{-1} that every sequence s in m_{E_α} must satisfy the condition $s_n = O((2-\alpha)/\alpha)^n$ while every sequence t in $(c_0)_{E_\alpha}$ must satisfy the condition $t_n = o((2-\alpha)/\alpha)^n$. Also the sequence $\{(\alpha-2)/\alpha\}^n$ is transformed into the sequence $(-1)^n$ by E_α; thus, the sequence $\{(\alpha-2)/\alpha\}^n$ is a nonzero element of m_{E_α}. By Theorem 2.1, $M(E_\alpha)$ contains elements $\mu \ne 0$ such that $\mu^2 = 0$.

In the case of the Nörlund matrix (N,p) with $P_n = 1$ where the polynomial $p(z)$ has only simple zeros $z_1 \cdots z_r$, all on ∂D, we see that the Taylor coefficients of $1/p(z)$ are bounded and hence the elements of $(N,p)^{-1}$ are bounded. We see that every sequence s in $m(N,p)$ must satisfy the condition $s_n = O(n)$ while every sequence t in $(c_0)_{(N,p)}$ must satisfy the condition $t_N = o(n)$. Also the sequence $\{nz_0^{-n}\}$, where z_0 is a zero of $p(z)$, is in $m_{(N,p)} - (c_0)_{(N,p)}$. It follows that $M(N,p)$ contains elements μ such that $\mu \ne 0$, $\mu^2 = 0$. □

Conjecture. *If $M(A)$ is semi-simple and $(*)$ holds for the matrix A, then $\Delta(A)$ is totally disconnected.*

Acknowledgment. I am indebted to Professor Jésus Gil de La Madrid and Professor Wilhelm Klingenberg for many helpful suggestions. I am indebted also to the referee for his suggestions concerning

the definition of the space m_A and a simplification of the proof of Theorem 1.1.

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