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BIFURCATION OF SYNCHRONIZED PERIODIC SOLUTIONS IN SYSTEMS OF COUPLED OSCILLATORS II: GLOBAL BIFURCATION IN COUPLED PLANAR OSCILLATORS

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ABSTRACT. We continue the study of a class of differential equations that govern the evolution of indirectly coupled oscillators. In a previous paper we established the existence of synchronized periodic solutions for weak and strong coupling. In this paper we present an example that shows an interesting behavior of the solutions for intermediate coupling strength. We analyze a two-parameter family of branches of periodic solutions and show when a branch has Hopf bifurcation points and/or turning points. We also study the stability of the periodic solutions.

1. Introduction. Many problems in physics, chemistry and biology involve systems of ordinary differential equations that govern the evolution of oscillatory subunits coupled indirectly through a passive medium [**10**]. In this paper we study the following system of ordinary differential equations, in which the oscillators that govern the states of the uncoupled subunits are all identical.

$$\begin{aligned} \frac{dx_i}{dt} &= f(x_i) + \delta P(x_0 - x_i), \qquad i = 1, \dots, N, \\ \frac{dx_0}{dt} &= \varepsilon \delta P\bigg(\frac{1}{N}\sum_{i=1}^{N} x_i - x_0\bigg). \end{aligned} \tag{1}$$

Here the variable x_0 represents the state of the coupling medium through which the subunits are coupled. P is an $n \times n$ constant matrix of permeability coefficients or conductances, and the parameters ε^{-1} and δ measure the relative capacity of the coupling medium and the coupling strength, respectively [**4, 5**]. In the absence of coupling, the

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evolution in the i^{th} subunit is governed by the n-dimensional system

$$\frac{dx_i}{dt} = f(x_i)$$

and it is assumed that this system has a nonconstant periodic solution.

We call a solution of (1) synchronized when the evolutions of the subunits are all identical. In [**9**] we established the existence of synchronized periodic solutions for weak and strong coupling. In this paper we show, with a particular choice of f and P in (1), how these solutions behave as the coupling strength varies. In the following sections we present results obtained in [**8**], in which a two-parameter family of global branches of synchronized periodic solutions is constructed when f is a truncated normal form of a planar oscillator near a Hopf bifurcation point and P is a multiple of the identity matrix. Specifically, we assume that $f : \mathbf{R}^2 \to \mathbf{R}^2$ is given by

$$f(y,z) = \begin{pmatrix} y + \beta z - y(y^2+z^2) \\ -\beta y + z - z(y^2+z^2) \end{pmatrix}, \quad \beta > 0, \tag{2}$$

and $P = 4I_{2\times 2}$. Then the results in [**9**] lead to the following conclusion. When $|\varepsilon\delta|$ and $|\delta|$ are both sufficiently small or when $|\varepsilon\delta|$ is sufficiently large and $|\delta|$ is sufficiently small, (1) has a synchronized periodic solution $x_i = \bar{\phi}(t,\varepsilon,\delta)$, $i = 1,\dots,N$, $x_0 = \phi_0(t,\varepsilon,\delta)$ whose period equals $T(\varepsilon,\delta)$. Moreover, as $|\varepsilon\delta| \to 0$ and $|\delta| \to 0$, or as $|\varepsilon\delta| \to \infty$ and $|\delta| \to 0$,

$$\begin{aligned} \bar{\phi}(t,\varepsilon,\delta) &\to \begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}, \\ \phi_0(t,\varepsilon,\delta) &\to \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ T(\varepsilon,\delta) &\to 2\pi/\beta. \end{aligned} \tag{3}$$

For a fixed $\varepsilon \neq -1, 0$, these solutions also exist for all sufficiently large $|\delta|$ and

$$\begin{aligned} \bar{\phi}(t,\varepsilon,\delta) &\to \begin{pmatrix} \cos(\varepsilon\beta t/(1+\varepsilon)) \\ -\sin(\varepsilon\beta t/(1+\varepsilon)) \end{pmatrix}, \\ \phi_0(t,\varepsilon,\delta) &\to \begin{pmatrix} \cos(\varepsilon\beta t/(1+\varepsilon)) \\ -\sin(\varepsilon\beta t/(1+\varepsilon)) \end{pmatrix}, \\ T(\varepsilon,\delta) &\to \frac{2(1+\varepsilon)\pi}{\varepsilon\beta}, \end{aligned} \tag{4}$$

as $|\delta| \to \infty$. In this paper we describe the global behavior of these solutions.

The construction of synchronized periodic solutions is done in Section 2. Let Ω denote the positive quadrant of the ε-β plane, i.e.,

$$\Omega = \{(\varepsilon, \beta) : \varepsilon > 0, \beta > 0\}.$$

For each $(\varepsilon, \beta) \in \Omega$, we construct synchronized periodic solutions of (1) for all admissible values of δ. One finds that the global behavior of periodic solutions is dependent on ε and β. Ω is divided by the curves defined by $\beta = 2\sqrt{\varepsilon(1+\varepsilon)}$ and $\varepsilon = 1/8$. If (ε, β) lies below the curve $\beta = 2\sqrt{\varepsilon(1+\varepsilon)}$, for a given $\delta \in \mathbf{R}$, there exists at least one periodic solution. The branches of solutions in this case are shown in Figures 1(a) and (c). If (ε, β) lies on or above the curve, there are no periodic solutions for certain values of δ and they disappear via Hopf bifurcation from the steady state $\bar{x} = 0$, $x_0 = 0$ (cf. Figure 1(b), Figure 1(d)). When $\varepsilon \geq 1/8$, the periodic solution for a given $\delta \in \mathbf{R}$ is unique whenever it exists (cf. Figure 1(a), Figure1(b)). However, when $0 < \varepsilon < 1/8$, there can be more than one periodic solution for certain values of δ, i.e., the branch of periodic solutions can have turning points (cf. Figure 1(c), Figure 1(d)). We also discuss the stability of these periodic solutions in Section 3.

Studies related to (1) are found in a number of publications including [**5, 4, 8, 10, 11, 7, 1** and **3**]. Bifurcations of periodic solutions of directly coupled oscillators are also studied in numerous papers including [**2**] and [**6**]. The relationship between the results obtained in these references and ours is discussed in [**9**]. Preliminary results of this paper have appeared in [**8**].

2. Global branches of synchronized periodic solutions. To construct synchronized periodic solutions of (1), we first recall some results in [**5**] and [**9**] concerning the reduction of the system. Let

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i,$$
$$w_i = x_i - \bar{x}, \qquad i = 1, \dots, N-1.$$

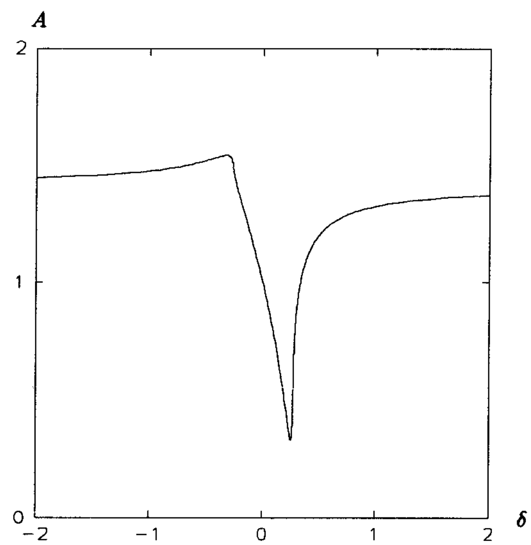

FIGURE 1 (a). Behavior of ω_0 for $(\varepsilon, \beta) \in \Omega_1$. $(\varepsilon, \beta) = (0.150000, 0.747596) \in \Omega_1$. There exist unique periodic solutions defined by (16) for all $\delta \in \mathbf{R}$.

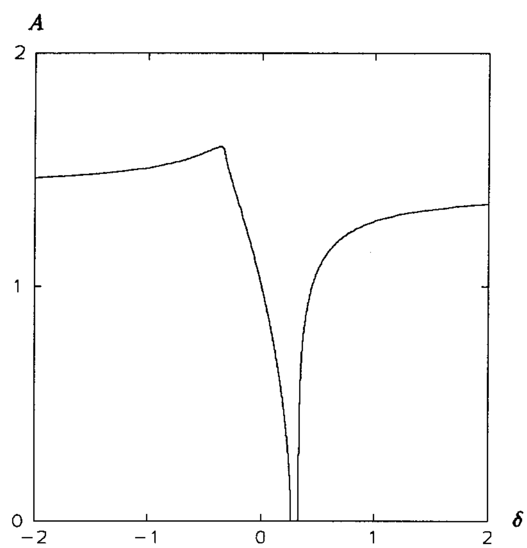

FIGURE 1 (b). Behavior of ω_0 for $(\varepsilon, \beta) \in \Omega_2$. $(\varepsilon, \beta) = (0.150000, 0.913729) \in \Omega_2$. A branch of the unique periodic solutions defined by (16) bifurcates from the steady state at $\delta_3 = 0.263429$ and $\delta_4 = 0.329179$ via Hopf bifurcation.

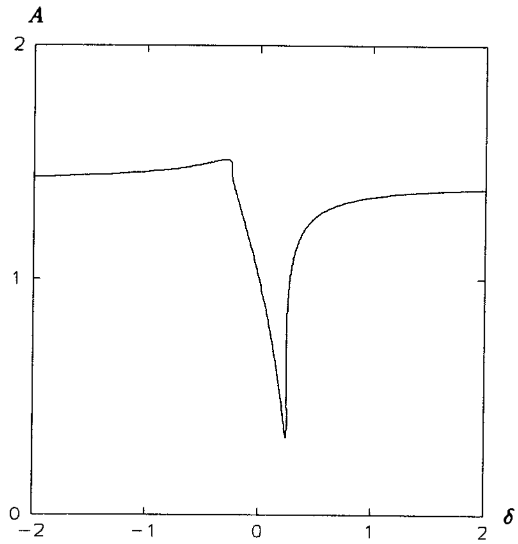

FIGURE 1 (c). Behavior of ω_0 for $(\varepsilon, \beta) \in \Omega_3$. $(\varepsilon, \beta) = (0.103553, 0.608487) \in \Omega_3$. ω_0 has turning points at $\pm\delta_1 = \pm 0.252044$ and $\pm\delta_2 = \pm 0.246113$.

Then (1) becomes

(5)

$$\frac{d\bar{x}}{dt} = \frac{1}{N}\left[\sum_{j=1}^{N-1} f(w_j+\bar{x}) + f\left(\bar{x} - \sum_{j=1}^{N-1} w_j\right)\right] - \delta P(\bar{x} - x_0),$$

$$\frac{dx_0}{dt} = \varepsilon\delta P(\bar{x} - x_0),$$

$$\frac{dw_i}{dt} = f(w_i+\bar{x}) - \frac{1}{N}\left[\sum_{j=1}^{N-1} f(w_j+\bar{x}) + f\left(\bar{x} - \sum_{j=1}^{N-1} w_j\right)\right] - \delta P w_i,$$

$$i = 1, \ldots, N-1.$$

When $w_i = 0$, $i = 1, \ldots, N-1$, this becomes

$$\frac{d\bar{x}}{dt} = f(\bar{x}) + \delta P(x_0 - \bar{x}), \qquad \frac{dx_0}{dt} = \varepsilon\delta P(\bar{x} - x_0). \tag{6}$$

If

$$\bar{x} = \bar{\phi}(t, \varepsilon, \delta), \qquad x_0 = \phi_0(t, \varepsilon, \delta) \tag{7}$$

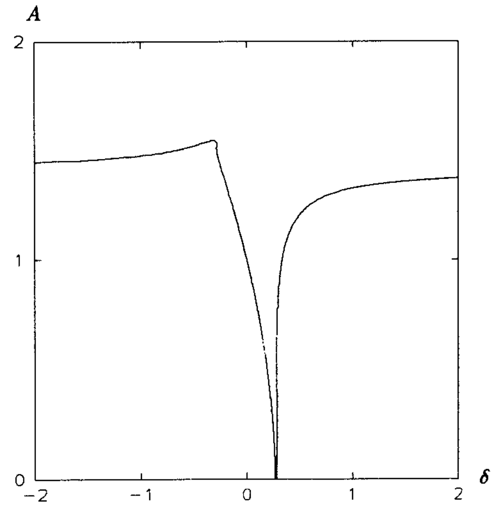

FIGURE 1 (d). Behavior of ω_0 for $(\varepsilon, \beta) \in V_1$. $(\varepsilon, \beta) = (0.114277, 0.718594) \in V_1$. ω_0 has turning points at $\pm\delta_1 = \pm 0.286025$ and $\pm\delta_2 = \pm 0.283775$. It has Hopf bifurcation points at $\delta_3 = 0.270280$ and $\delta_4 = 0.282415$. $\Theta_1 = -1.136857$, $\Theta_2 = -0.966082$, $\Theta_3 = -1.293328$, and $\Theta_4 = -1.225169$. In general, $\Theta_3 < \Theta_4 < \Theta_1 < \Theta_2$ and $\delta_3 < \delta_4 < \delta_2 < \delta_1$ for all $(\varepsilon, \beta) \in V_1$.

is a solution of (6), then

$$\begin{aligned} &\bar{x} = \bar{\phi}(t, \varepsilon, \delta), \qquad x_0 = \phi_0(t, \varepsilon, \delta), \\ &w_i = 0, \qquad i = 1, \dots, N-1 \end{aligned} \tag{8}$$

is a solution of (5) and

$$x_0 = \phi_0(t, \varepsilon, \delta), \qquad x_i = \bar{\phi}(t, \varepsilon, \delta), \qquad i = 1, \dots, N \tag{9}$$

is a synchronized solution of (1). The variational equation of (6) with respect to (7) is

$$\frac{d}{dt}\begin{pmatrix} \bar{x} \\ x_0 \end{pmatrix} = \begin{bmatrix} Df(\bar{\phi}(t, \varepsilon, \delta)) - \delta P & \delta P \\ \varepsilon\delta P & -\varepsilon\delta P \end{bmatrix} \begin{pmatrix} \bar{x} \\ x_0 \end{pmatrix}. \tag{10}$$

On the other hand, the variational equation of (5) with respect to (8) consists of (10) and the additional $N-1$ linear systems

$$\frac{dw_i}{dt} = [Df(\bar{\phi}(t, \varepsilon, \delta)) - \delta P]w_i, \qquad i = 1, \dots, N-1. \tag{11}$$

Note that the subspace of $\mathbf{R}^{(N+1)n}$ defined by $x_1 = x_2 = \cdots = x_N$ is an invariant subspace of (1), and that (9) belongs to this subspace. Note also that (10) determines the stability of (9) with respect to the solutions in the subspace, and that (11) determines its stability in the complement. In [**9**] we established the existence of periodic solutions of (6) for some extreme values of the parameters. This led to the existence of synchronized periodic solutions of (1). In this section we construct a two-parameter family of branches of periodic solutions.

We first convert (6) using the polar coordinates:

$$\bar{x} = r_1 \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \qquad x_0 = r_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix},$$

and let $\Theta = \theta_1 - \theta_0$. Then we obtain

$$\begin{aligned} \frac{dr_1}{dt} &= (1-4\delta)r_1 - r_1^3 + 4\delta r_0 \cos\Theta, \\ \frac{dr_0}{dt} &= -4\varepsilon\delta(r_0 - r_1\cos\Theta), \\ \frac{d\Theta}{dt} &= -\beta - \frac{4\delta(r_0^2 + \varepsilon r_1^2)}{r_1 r_0}\sin\Theta. \end{aligned} \tag{12}$$

Clearly, a steady state solution of (12) gives rise to a periodic solution of (6). Such solutions are called phase-locked solutions because the phase difference between the cells ($x_i, i = 1, \ldots, N$) and medium (x_0) is time-independent. One easily finds that (12) has a family of steady state solutions given by

$$\begin{gathered} r_1 = \sqrt{1 - 4\delta\sin^2\Theta}, \qquad r_0 = \cos\Theta\sqrt{1-4\delta\sin^2\Theta}, \\ \delta = \beta g(\Theta, \varepsilon), \end{gathered} \tag{13}$$

where

$$g(\Theta,\varepsilon) \equiv -\frac{\cot\Theta}{4(\cos^2\Theta + \varepsilon)}. \tag{14}$$

Define

$$\begin{aligned} \bar{\phi}(t,\Theta,\varepsilon,\beta) &\equiv \begin{pmatrix} \sqrt{1-4\delta\sin^2\Theta}\cos[(4\varepsilon\delta\tan\Theta)t + \Theta] \\ \sqrt{1-4\delta\sin^2\Theta}\sin[(4\varepsilon\delta\tan\Theta)t + \Theta] \end{pmatrix}, \\ \phi_0(t,\Theta,\varepsilon,\beta) &\equiv \begin{pmatrix} \cos\Theta\sqrt{1-4\delta\sin^2\Theta}\cos[(4\varepsilon\delta\tan\Theta)t] \\ \cos\Theta\sqrt{1-4\delta\sin^2\Theta}\sin[(4\varepsilon\delta\tan\Theta)t] \end{pmatrix}. \end{aligned} \tag{15}$$

It follows that

$$(16)\qquad \bar{x} = \bar{\phi}(t, \Theta, \varepsilon, \beta), \qquad x_0 = \phi_0(t, \Theta, \varepsilon, \beta), \qquad \delta = \beta g(\Theta, \varepsilon)$$

is a periodic solution of (6) with period

$$(17)\qquad T(\Theta, \varepsilon, \beta) \equiv \frac{2\pi}{|4\varepsilon\delta \tan\Theta|} = \frac{2\pi(\cos^2\Theta + \varepsilon)}{\varepsilon\beta},$$

provided that

$$h(\Theta, \varepsilon, \beta) \equiv 1 - 4\beta g(\Theta, \varepsilon)\sin^2\Theta = 1 + \frac{\beta\cos\Theta\sin\Theta}{\cos^2\Theta + \varepsilon} > 0.$$

Note that δ varies from 0 to ∞ as Θ increases from $-\pi/2$ to 0, and it varies from 0 to $-\infty$ as Θ decreases from $\pi/2$ to 0. Note also that, in view of (14) and (15), we may restrict ourselves to the case $\Theta \in [-\pi/2, 0) \cup (0, \pi/2]$.

We denote by $\omega_0 = \omega_0(\varepsilon, \beta)$ the branch of periodic solutions (16) in $(\bar{x}, x_0, \delta)$-space. It is easily seen that, when $\delta = 0$, (16) coincides with (3). Moreover, as $|\delta| \to \infty$, (16) tends to (4). Thus ω_0 passes through (3) at $\delta = 0$ and bifurcates from (4) at $\delta = \pm\infty$. To study the behavior of ω_0 for intermediate values of δ, we would like to express (15) in terms of δ using (13). Then we need to know where $g(\Theta, \varepsilon)$ is increasing or decreasing as a function of Θ. We also need to know when $h(\Theta, \varepsilon, \beta) > 0$ holds. The functions $g(\Theta, \varepsilon)$ and $h(\Theta, \varepsilon, \beta)$ are analyzed in [**8**], and the results are summarized in Properties 1–4.

For $0 < \varepsilon \le 1/8$, define

$$\Theta_1 = \Theta_1(\varepsilon) \equiv -\sin^{-1}\sqrt{\lambda_+(\varepsilon)}, \qquad \Theta_2 = \Theta_2(\varepsilon) \equiv -\sin^{-1}\sqrt{\lambda_-(\varepsilon)},$$

where

$$\lambda_\pm(\varepsilon) \equiv \frac{3 \pm \sqrt{1 - 8\varepsilon}}{4}.$$

Property 1. (a) *If $\varepsilon > 1/8$, then*

$$\frac{\partial g}{\partial\Theta}(\Theta, \varepsilon) > 0 \qquad \textit{for all } \Theta \in [-\pi/2, 0) \cup (0, \pi/2].$$

(b) *If* $\varepsilon = 1/8$, *then*

$$\frac{\partial g}{\partial \Theta}(\Theta, {}^1\!/\!_8) > 0 \qquad \textit{for all } \Theta \in [-{}^\pi\!/\!_2, -{}^\pi\!/\!_3) \cup (-{}^\pi\!/\!_3, 0) \cup (0, {}^\pi\!/\!_3) \cup ({}^\pi\!/\!_3, {}^\pi\!/\!_2],$$

$$\frac{\partial g}{\partial \Theta}(\pm {}^\pi\!/\!_3, {}^1\!/\!_8) = 0.$$

(c) *If* $0 < \varepsilon < 1/8$, *then*

$$\frac{\partial g}{\partial \Theta}(\Theta, \varepsilon) \begin{cases} > 0 & \textit{for all } \Theta \in [-{}^\pi\!/\!_2, \Theta_1) \cup (\Theta_2, 0) \cup (0, -\Theta_2) \cup (-\Theta_1, {}^\pi\!/\!_2], \\ = 0 & \textit{for } \Theta = \pm\Theta_1, \pm\Theta_2, \\ < 0 & \textit{for all } \Theta \in (\Theta_1, \Theta_2) \cup (-\Theta_2, -\Theta_1). \end{cases}$$

For $\beta \geq 2\sqrt{\varepsilon(1+\varepsilon)}$, define

$$\begin{aligned} \Theta_3 &= \Theta_3(\varepsilon, \beta) \equiv -\sin^{-1}\sqrt{\mu_+(\varepsilon, \beta)}, \\ \Theta_4 &= \Theta_4(\varepsilon, \beta) \equiv -\sin^{-1}\sqrt{\mu_-(\varepsilon, \beta)}, \\ \Theta^* &= \Theta^*(\varepsilon) \equiv \Theta_3(\varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)}) = \Theta_4(\varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)}) \\ &= -\sin^{-1}\sqrt{\frac{1+\varepsilon}{1+2\varepsilon}}, \end{aligned}$$

where

$$\mu_\pm(\varepsilon, \beta) = \frac{2(1+\varepsilon) + \beta^2 \pm \beta\sqrt{\beta^2 - 4\varepsilon(1+\varepsilon)}}{2(1+\beta^2)}.$$

Property 2. *Let* $(\varepsilon, \beta) \in \Omega$.

(a)

$$h(\Theta, \varepsilon, \beta) > 0 \qquad \textit{for all } \Theta \in (0, \pi/2].$$

(b) *If* $0 < \beta < 2\sqrt{\varepsilon(1+\varepsilon)}$, *then*

$$h(\Theta, \varepsilon, \beta) > 0 \qquad \textit{for all } \Theta \in [-\pi/2, 0).$$

(c)

$$h(\Theta, \varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)}) > 0 \qquad \textit{for all } \Theta \in [-\pi/2, \Theta^*) \cup (\Theta^*, 0),$$
$$h(\Theta^*, \varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)}) = 0.$$

(d) *If* $2\sqrt{\varepsilon(1+\varepsilon)} < \beta$*, then*

$$h(\Theta, \varepsilon, \beta) \begin{cases} > 0 & \textit{for all } \Theta \in [-\pi/2, \Theta_3) \cup (\Theta_4, 0), \\ = 0 & \textit{for } \Theta = \Theta_3, \Theta_4, \\ < 0 & \textit{for all } \Theta \in (\Theta_3, \Theta_4). \end{cases}$$

Using these properties we proceed as follows. For $0 < \varepsilon \le 1/8$, define

$$\begin{aligned} \delta_1 &= \delta_1(\varepsilon, \beta) \equiv \beta g(\Theta_1(\varepsilon), \varepsilon) \\ &= \frac{\sqrt{2}\beta[1 + 10\varepsilon + (1-2\varepsilon)\sqrt{1-8\varepsilon}]\sqrt{1 + 4\varepsilon - \sqrt{1-8\varepsilon}}}{64\varepsilon(1+\varepsilon)^2}, \\ \delta_2 &= \delta_2(\varepsilon, \beta) \equiv \beta g(\Theta_2(\varepsilon), \varepsilon) \\ &= \frac{\sqrt{2}\beta[1 + 10\varepsilon - (1-2\varepsilon)\sqrt{1-8\varepsilon}]\sqrt{1 + 4\varepsilon + \sqrt{1-8\varepsilon}}}{64\varepsilon(1+\varepsilon)^2}. \end{aligned} \tag{18}$$

Note that

$$1/2 < \lambda_-(\varepsilon) < \lambda_+(\varepsilon) < 1 \qquad \text{for all } \varepsilon \in (0, 1/8),$$
$$\lambda_-(1/8) = \lambda_+(1/8) = 3/4.$$

It follows that

$$\begin{gathered} -\pi/2 < \Theta_1(\varepsilon) < \Theta_2(\varepsilon) < -\pi/4 \qquad \text{for all } \varepsilon \in (0, 1/8), \\ \Theta_1(1/8) = \Theta_2(1/8) = -\pi/3, \end{gathered} \tag{19}$$

and

$$\begin{gathered} 0 < \delta_2(\varepsilon, \beta) < \delta_1(\varepsilon, \beta) \qquad \text{for all } \varepsilon \in (0, 1/8), \\ \delta_2(1/8, \beta) = \delta_1(1/8, \beta) = 2\sqrt{3}\beta/9. \end{gathered} \tag{20}$$

Next, for $\beta \geq 2\sqrt{\varepsilon(1+\varepsilon)}$, define

$$
\begin{aligned}
\delta_3 &= \delta_3(\varepsilon,\beta) \equiv \beta g(\Theta_3(\varepsilon,\beta),\varepsilon) \\
&= \frac{2(1+\varepsilon)+\beta^2-\beta\sqrt{\beta^2-4\varepsilon(1+\varepsilon)}}{8(1+\varepsilon)^2}, \\
\delta_4 &= \delta_4(\varepsilon,\beta) \equiv \beta g(\Theta_4(\varepsilon,\beta),\varepsilon) \\
&= \frac{2(1+\varepsilon)+\beta^2+\beta\sqrt{\beta^2-4\varepsilon(1+\varepsilon)}}{8(1+\varepsilon)^2}, \\
\delta^* &= \delta^*(\varepsilon) \equiv 2\sqrt{\varepsilon(1+\varepsilon)}g(\Theta^*(\varepsilon),\varepsilon) = \frac{1+2\varepsilon}{4(1+\varepsilon)}.
\end{aligned}
\tag{21}
$$

By a simple calculation, one finds that

$$
\begin{gathered}
0 < \mu_-(\varepsilon,\beta) < \mu_+(\varepsilon,\beta) < 1 \qquad \text{for all } \beta > 2\sqrt{\varepsilon(1+\varepsilon)}, \\
\mu_-(\varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)}) = \mu_+(\varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)}) = \frac{1+\varepsilon}{1+2\varepsilon}.
\end{gathered}
$$

It follows that

$$
-\pi/2 < \Theta_3(\varepsilon,\beta) < \Theta_4(\varepsilon,\beta) < 0 \qquad \text{for all } \beta > 2\sqrt{\varepsilon(1+\varepsilon)},
$$

and that

$$
\begin{gathered}
0 < \delta_3(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) \qquad \text{for all } \beta > 2\sqrt{\varepsilon(1+\varepsilon)}, \\
\delta^*(\varepsilon) = \delta_3(\varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)}) = \delta_4(\varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)}.
\end{gathered}
\tag{22}
$$

Now consider the following subsets of Ω defined by the curves $\varepsilon = 1/8$ and $\beta = 2\sqrt{\varepsilon(1+\varepsilon)}$.

$$
\begin{aligned}
\Omega_1 &\equiv \{(\varepsilon,\beta) : \varepsilon \geq 1/8, 0 < \beta < 2\sqrt{\varepsilon(1+\varepsilon)}\}, \\
L_1 &\equiv \{(\varepsilon,\beta) : \varepsilon \geq 1/8, \beta = 2\sqrt{\varepsilon(1+\varepsilon)}\}, \\
\Omega_2 &\equiv \{(\varepsilon,\beta) : \varepsilon \geq 1/8, \beta > 2\sqrt{\varepsilon(1+\varepsilon)}\}, \\
\Omega_3 &\equiv \{(\varepsilon,\beta) : 0 < \varepsilon < 1/8, 0 < \beta < 2\sqrt{\varepsilon(1+\varepsilon)}\}, \\
L_2 &\equiv \{(\varepsilon,\beta) : 0 < \varepsilon < 1/8, \beta = 2\sqrt{\varepsilon(1+\varepsilon)}\}, \\
\Omega_4 &\equiv \{(\varepsilon,\beta) : 0 < \varepsilon < 1/8, \beta > 2\sqrt{\varepsilon(1+\varepsilon)}\}.
\end{aligned}
$$

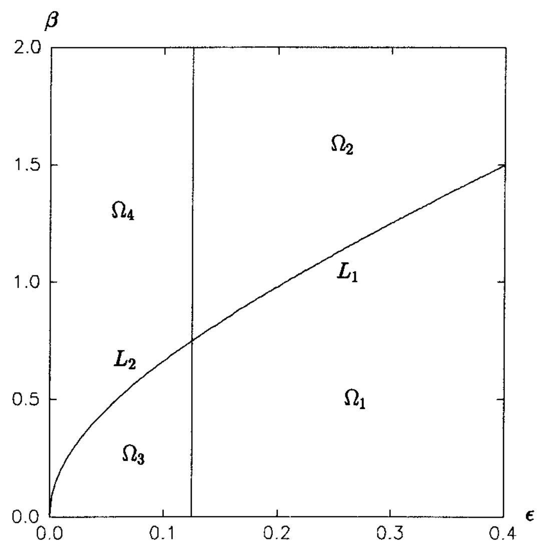

FIGURE 2 (a). Parameter Region Ω. $\Omega = \{(\varepsilon,\beta) : \varepsilon > 0, \beta > 0\}$ is divided by curves defined by $\varepsilon = 1/8$ and $\beta = 2\sqrt{\varepsilon(1+\varepsilon)}$. If $(\varepsilon,\beta) \in \Omega_1 \cup L_1 \cup \Omega_2$, $g(\Theta,\varepsilon)$ is a strictly increasing function of Θ. However, when $(\varepsilon,\beta) \in \Omega_3 \cup L_2 \cup \Omega_4$, $g(\Theta,\varepsilon)$ is a strictly decreasing function of Θ on $(\Theta_1,\Theta_2) \cup (-\Theta_2,-\Theta_1)$. When $(\varepsilon,\beta) \in \Omega_1 \cup \Omega_3$, $h(\Theta,\varepsilon,\beta)$ is always positive. When $(\varepsilon,\beta) \in L_1 \cup L_2$, $h(\Theta,\varepsilon,\beta)$ vanishes at $\Theta = \Theta^*$. When $(\varepsilon,\beta) \in \Omega_2 \cup \Omega_4$, $h(\Theta,\varepsilon,\beta)$ is nonpositive for $\Theta_3 \le \Theta \le \Theta_4$.

These regions are shown in Figure 2(a). In view of Properties 1 and 2, the behavior of ω_0 depends on which subset (ε,β) belongs to. We summarize this result in the following proposition.

Proposition 1. (a) *When $(\varepsilon,\beta) \in \Omega_1 \cup L_1 \cup \Omega_2$, $g(\Theta,\varepsilon)$ is a strictly increasing function of Θ and ω_0 has no turning points.*

(b) *When $(\varepsilon,\beta) \in \Omega_3 \cup L_2 \cup \Omega_4$, $g(\Theta,\varepsilon)$ is a strictly increasing function of Θ on $[-\pi/2,\Theta_1) \cup (\Theta_2,0) \cup (0,-\Theta_2) \cup (-\Theta_1,\pi/2]$ and is a strictly decreasing function of Θ on $(\Theta_1,\Theta_2) \cup (-\Theta_2,-\Theta_1)$, and ω_0 has turning points at $\delta = \pm\delta_i$, $i = 1,2$, whenever $h(\pm\Theta_i,\varepsilon,\beta) > 0$.*

(c) *When $(\varepsilon,\beta) \in \Omega_1 \cup \Omega_3$, $h(\Theta,\varepsilon,\beta) > 0$ for all $\Theta \in [-\pi/2,\pi/2]$ and ω_0 extends from $\delta = -\infty$ to $\delta = \infty$.*

(d) *When $(\varepsilon,\beta) \in L_1 \cup L_2$, $h(\Theta,\varepsilon,2\sqrt{\varepsilon(1+\varepsilon)}) > 0$ for all $\Theta \in$*

$[-\pi/2, \Theta^*) \cup (\Theta^*, \pi/2]$ and $h(\Theta^*, \varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)}) = 0$. Thus, ω_0 vanishes at $\Theta = \Theta^*$.

(e) *When $(\varepsilon, \beta) \in \Omega_2 \cup \Omega_4$, $h(\Theta, \varepsilon, \beta) > 0$ for all $\Theta \in [-\pi/2, \Theta_3) \cup (\Theta_4, \pi/2]$, $h(\Theta_i, \varepsilon, \beta) = 0$, $i = 1, 2$, and $h(\Theta, \varepsilon, \beta) < 0$ for all $\Theta \in (\Theta_3, \Theta_4)$. Thus ω_0 vanishes for $\Theta_3 \leq \Theta \leq \Theta_4$.*

It is shown in [**8**] that ω_0 disappears via Hopf bifurcation from the steady state $\bar{x} = x_0 = 0$ at $\delta = \delta_3, \delta_4$. As (ε, β) approach $L_1 \cup L_2$, δ_3 and δ_4 approach δ^*. Some examples that illustrate the behavior of ω_0 for various ε and β are shown in Figures 1 (a)–(d), where the curves

$$\begin{aligned} \delta &= \beta g(\Theta, \varepsilon), \\ A &= \left(\frac{1}{T(\theta, \varepsilon, \beta)} \int_0^{T(\theta, \varepsilon, \beta)} \left\| \begin{pmatrix} \bar{\phi} \\ \phi_0 \end{pmatrix} \right\|^2 dt \right)^{1/2} \\ &= (1 + \cos^2 \Theta) h(\Theta, \varepsilon, \beta) \end{aligned}$$

in (δ, A)-plane are sketched.

Proposition 1 leads to the following results concerning the behavior of ω_0. If $(\varepsilon, \beta) \in \Omega_1$, ω_0 extends from $\delta = -\infty$ to $\delta = \infty$ without turning points (cf. Figure 1(a)), and there is a unique periodic solution (16) of (6) for each $\delta \in \mathbf{R}$. If $(\varepsilon, \beta) \in L_1$, ω_0 has no turning points. However, it vanishes at $\delta = \delta^*$. It follows that there is a unique periodic solution (16) of (6) for each $\delta \in \mathbf{R} - \{\delta^*\}$. If $(\varepsilon, \beta) \in \Omega_2$, ω_0 has no turning points but vanishes for $\delta_3 \leq \delta \leq \delta_4$ (cf. Figure 1(b)), and there is a unique periodic solution (16) of (6) for $\delta \in (-\infty, \delta_3) \cup (\delta_4, \infty)$. When $(\varepsilon, \beta) \in \Omega_3$, ω_0 exists for all $\delta \in \mathbf{R}$. However, it has turning points at $\delta = \delta_1$ and $\delta = \delta_2$ (cf. Figure 1(c)). It follows that (6) has one periodic solution when $\delta \in (-\infty, -\delta_1) \cup (-\delta_2, \delta_2) \cup (\delta_1, \infty)$, two periodic solutions when $\delta \in \{\pm\delta_1, \pm\delta_2\}$, three periodic solutions when $\delta \in (-\delta_1, -\delta_2) \cup (\delta_2, \delta_1)$ given by (16).

When $(\varepsilon, \beta) \in L_2$, the behavior of ω_0 depends on the relative positions of δ_1, δ_2 and δ^*, which are defined by (18) and (21). We summarize the results obtained in [**8**] concerning Θ_1, Θ_2, and Θ^* in Property 3.

TABLE 1. The relations between Θ_i and δ_i, $i = 1, \ldots, 4$, for $(\varepsilon, \beta) \in \Omega_4$ are shown. The entries in the first column indicate the regions in Ω_4 to which (ε, β) belongs. The entries in the second and the third columns indicate the relations between Θ_i, $i = 1, \ldots, 4$ and δ_i, $i = 1, \ldots, 4$, respectively.

$(\varepsilon, \beta) \in$	relation between Θ_i	relation between δ_i
V_1	$\Theta_3(\varepsilon,\beta) < \Theta_4(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon)$	$\delta_3(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) < \delta_2(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
l_5	$\Theta_3(\varepsilon,\beta) < \Theta_4(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon)$	$\delta_3(\varepsilon,\beta) < \delta_2(\varepsilon,\beta) = \delta_4(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
V_2	$\Theta_3(\varepsilon,\beta) < \Theta_4(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon)$	$\delta_3(\varepsilon,\beta) < \delta_2(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
γ_1	$\Theta_3(\varepsilon,\beta) < \Theta_1(\varepsilon) = \Theta_4(\varepsilon,\beta) < \Theta_2(\varepsilon)$	$\delta_3(\varepsilon,\beta) < \delta_2(\varepsilon,b) < \delta_1(\varepsilon,\beta) = \delta_4(\varepsilon,\beta)$
γ_2	$\Theta_3(\varepsilon,\beta) < \Theta_4(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon)$	$\delta_2(\varepsilon,\beta) = \delta_3(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
$(\hat{\varepsilon},\hat{\beta})$	$\Theta_3(\hat{\varepsilon},\hat{\beta}) < \Theta_1(\hat{\varepsilon}) = \Theta_4(\hat{\varepsilon},\hat{\beta}) < \Theta_2(\hat{\varepsilon})$	$\delta_2(\hat{\varepsilon},\hat{\beta}) = \delta_3(\hat{\varepsilon},\hat{\beta}) < \delta_1(\hat{\varepsilon},\hat{\beta}) = \delta_4(\hat{\varepsilon},\hat{\beta})$
V_3	$\Theta_3(\varepsilon,\beta) < \Theta_4(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon)$	$\delta_2(\varepsilon,\beta) < \delta_3(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
γ_3	$\Theta_3(\varepsilon,\beta) < \Theta_1(\varepsilon) = \Theta_4(\varepsilon,\beta) < \Theta_2(\varepsilon)$	$\delta_2(\varepsilon,\beta) < \delta_3(\varepsilon,\beta) < \delta_1(\varepsilon,\beta) = \delta_4(\varepsilon,\beta)$
V_4	$\Theta_3(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_4(\varepsilon,\beta) < \Theta_2(\varepsilon)$	$\delta_2(\varepsilon,\beta) < \delta_3(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
γ_4	$\Theta_3(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_4(\varepsilon,\beta) < \Theta_2(\varepsilon)$	$\delta_2(\varepsilon,\beta) = \delta_3(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
V_5	$\Theta_3(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_4(\varepsilon,\beta) < \Theta_2(\varepsilon)$	$\delta_3(\varepsilon,\beta) < \delta_2(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
l_4	$\Theta_3(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon) = \Theta_4(\varepsilon,\beta)$	$\delta_3(\varepsilon,\beta) < \delta_2(\varepsilon,\beta) = \delta_4(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
V_6	$\Theta_3(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon) < \Theta_4(\varepsilon,\beta)$	$\delta_3(\varepsilon,\beta) < \delta_2(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$
l_2	$\Theta_3(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon) < \Theta_4(\varepsilon,\beta)$	$\delta_3(\varepsilon,\beta) < \delta_2(\varepsilon,\beta) < \delta_1(\varepsilon,\beta) = \delta_4(\varepsilon,\beta)$
V_7	$\Theta_3(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon) < \Theta_4(\varepsilon,\beta)$	$\delta_3(\varepsilon,\beta) < \delta_2(\varepsilon,\beta) < \delta_1(\varepsilon,\beta) < \delta_4(\varepsilon,\beta)$

Property 3. (a)

$$\Theta^*(\varepsilon) < \Theta_1(\varepsilon) \quad \textit{and} \quad g(\Theta^*(\varepsilon),\varepsilon) < g(\Theta_1(\varepsilon),\varepsilon)$$
$$\textit{for all } \varepsilon \in (0, 1/8).$$

(b) *There is a unique* $\varepsilon^* \in (0, 1/8)$ *such that*

$$g(\Theta^*(\varepsilon),\varepsilon) \begin{cases} > g(\Theta_2(\varepsilon),\varepsilon) & \textit{for all } \varepsilon \in (0,\varepsilon^*), \\ = g(\Theta_2(\varepsilon),\varepsilon) & \textit{for } \varepsilon = \varepsilon^*, \\ < g(\Theta_2(\varepsilon),\varepsilon) & \textit{for all } \varepsilon \in (\varepsilon^*, 1/8). \end{cases}$$

According to Property 3(a), $\delta^*(\varepsilon) < \delta_1(\varepsilon, \beta)$ for all $\varepsilon \in (0, 1/8)$. As δ increases from 0, ω_0 meets the steady state at $\delta = \delta^*(\varepsilon)$, and then turns the direction at $\delta = \delta_1(\varepsilon, \beta)$. It changes the direction again at $\delta = \delta_2(\varepsilon, \beta)$ and then extends over (δ_2, ∞) without any turning points. Property 3(b) gives us the information concerning the relationship between $\delta^*(\varepsilon)$ and $\delta_2(\varepsilon, \beta)$, that there is a unique $\varepsilon^* \in (0, 1/8)$ for which $\delta^*(\varepsilon) = \delta_2(\varepsilon, \beta)$. A numerical computation shows that

$$\varepsilon^* \approx 0.103554.$$

Define the subsets ℓ_1 and ℓ_2 of L_2 by

$$\ell_1 \equiv \{(\varepsilon, \beta) : 0 < \varepsilon < \varepsilon^*, \beta = 2\sqrt{\varepsilon(1+\varepsilon)}\},$$

$$\ell_2 \equiv \{(\varepsilon, \beta) : \varepsilon^* < \varepsilon < 1/8, \beta = 2\sqrt{\varepsilon(1+\varepsilon)}\}.$$

Then $L_2 = \ell_1 \cup \{(\varepsilon^*, \beta^*)\} \cup \ell_2$, where $\beta^* = 2\sqrt{\varepsilon^*(1+\varepsilon^*)}$. When $(\varepsilon, \beta) \in \ell_1$, $\delta^*(\varepsilon) > \delta_2(\varepsilon, \beta)$. It follows that (6) has one periodic solution when $\delta \in (-\infty, -\delta_1) \cup (-\delta_2, \delta_2) \cup (\delta_1, \infty)$, two periodic solutions when $\delta \in \{\pm\delta_1, \pm\delta_2, \delta^*\}$, three periodic solutions when $\delta \in (-\delta_1, -\delta_2) \cup (\delta_2, \delta^*) \cup (\delta^*, \delta_1)$ given by (16). The other two cases $(\varepsilon, \beta) = (\varepsilon^*, \beta^*)$ and $(\varepsilon, \beta) \in \ell_2$ are analyzed similarly. The relationship between the number of the periodic solutions and these sets is summarized in Table 2.

When $(\varepsilon, \beta) \in \Omega_4$, the behavior of ω_0 depends on the relative positions of $\delta_1(\varepsilon, \beta)$, $\delta_2(\varepsilon, \beta)$, $\delta_3(\varepsilon, \beta)$, and $\delta_4(\varepsilon, \beta)$. From (20) and (22), we already know that

$$0 < \delta_2(\varepsilon, \beta) < \delta_1(\varepsilon, \beta), \qquad 0 < \delta_3(\varepsilon, \beta) < \delta_4(\varepsilon, \beta).$$

The results obtained in [**8**], which concern the behavior of Θ_3 and Θ_4 as β increases, are summarized in Property 4.

Property 4. *For every* $(\varepsilon, \beta) \in \Omega_2 \cup \Omega_4$,

$$\frac{\partial \Theta_3}{\partial \beta}(\varepsilon, \beta) < 0, \qquad \frac{\partial \Theta_4}{\partial \beta}(\varepsilon, \beta) > 0.$$

TABLE 2. The numbers of synchronized periodic solutions of (1) given by (16) are listed. The entries in the first column indicate the subsets of Ω to which (ε, β) belongs. The entries in the second, third, and fourth columns indicate the ranges of δ for which (6) has one periodic solution, two periodic solutions, and three periodic solutions, respectively.

$(\varepsilon, \beta) \in$	one solution	two solutions	three solutions
Ω_1	$(-\infty, \infty)$	$\varnothing$	$\varnothing$
L_1	$(-\infty, \delta^*) \cup (\delta^*, \infty)$	$\varnothing$	$\varnothing$
Ω_2	$(-\infty, \delta_3) \cup (\delta_4, \infty)$	$\varnothing$	$\varnothing$
Ω_3	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_2) \cup (\delta_1, \infty)$	$\{\pm\delta_1, \pm\delta_2\}$	$(-\delta_1, -\delta_2) \cup (\delta_2, \delta_1)$
ℓ_1	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_2) \cup (\delta_1, \infty)$	$\{\pm\delta_1, \pm\delta_2, \delta^*\}$	$(-\delta_1, -\delta_2) \cup (\delta_2, \delta^*) \cup (\delta^*, \delta_1)$
(ε^*, β^*)	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_2] \cup (\delta_1, \infty)$	$\{\pm\delta_1, -\delta_2\}$	$(-\delta_1, -\delta_2) \cup (\delta_2, \delta_1)$
ℓ_2	$(-\infty, -\delta_1) \cup (-\delta_2, \delta^*) \cup (\delta^*, \delta_2)$ $\cup(\delta_1, \infty)$	$\{\pm\delta_1, \pm\delta_2\}$	$(-\delta_1, -\delta_2) \cup (\delta_2, \delta_1)$
V_1	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup (\delta_4, \delta_2)$ $\cup(\delta_1, \infty)$	$\{\pm\delta_1, \pm\delta_2\}$	$(-\delta_1, -\delta_2) \cup (\delta_2, \delta_1)$
l_5	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup \{\delta_2\}$ $\cup(\delta_1, \infty)$	$\{\pm\delta_1, -\delta_2\}$	$(-\delta_1, -\delta_2) \cup (\delta_2, \delta_1)$
V_2	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup \{\delta_2\}$ $\cup(\delta_1, \infty)$	$\{\pm\delta_1, -\delta_2\} \cup (\delta_2, \delta_4]$	$(-\delta_1, -\delta_2) \cup (\delta_4, \delta_1)$
γ_1	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup \{\delta_2\}$ $\cup[\delta_1, \infty)$	$\{-\delta_1, -\delta_2\} \cup (\delta_2, \delta_1)$	$(-\delta_1, -\delta_2)$
γ_2	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_2] \cup (\delta_1, \infty)$	$\{\pm\delta_1, -\delta_2\} \cup (\delta_2, \delta_4]$	$(-\delta_1, -\delta_2) \cup (\delta_4, \delta_1)$
$(\hat{\varepsilon}, \hat{\beta})$	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3] \cup [\delta_1, \infty)$	$\{-\delta_1, -\delta_2\} \cup (\delta_2, \delta_1)$	$(-\delta_1, -\delta_2)$
V_3	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_2) \cup (\delta_1, \infty)$	$\{\pm\delta_1, \pm\delta_2\} \cup [\delta_3, \delta_4]$	$(-\delta_1, -\delta_2) \cup (\delta_2, \delta_3) \cup (\delta_4, \delta_1)$
γ_3	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_2) \cup [\delta_1, \infty)$	$\{-\delta_1, \pm\delta_2\} \cup [\delta_3, \delta_1)$	$(-\delta_1, -\delta_2) \cup (\delta_2, \delta_3)$
V_4	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_2) \cup [\delta_4, \infty)$	$\{-\delta_1, \pm\delta_2\} \cup [\delta_3, \delta_4)$	$(-\delta_1, -\delta_2) \cup (\delta_2, \delta_3)$
γ_4	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_2] \cup [\delta_4, \infty)$	$\{-\delta_1, -\delta_2\} \cup (\delta_2, \delta_4)$	$(-\delta_1, -\delta_2)$
V_5	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup \{\delta_2\}$ $\cup[\delta_4, \infty)$	$\{-\delta_1, -\delta_2\} \cup (\delta_2, \delta_4)$	$(-\delta_1, -\delta_2)$
l_4	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup (\delta_2, \infty)$	$\{-\delta_1, -\delta_2\}$	$(-\delta_1, -\delta_2)$
V_6	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup (\delta_4, \infty)$	$\{-\delta_1, -\delta_2\}$	$(-\delta_1, -\delta_2)$
l_2	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup (\delta_4, \infty)$	$\{-\delta_1, -\delta_2\}$	$(-\delta_1, -\delta_2)$
V_7	$(-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup (\delta_4, \infty)$	$\{-\delta_1, -\delta_2\}$	$(-\delta_1, -\delta_2)$

Moreover, as $\beta \to \infty$*,*

$$\Theta_3(\varepsilon,\beta) \to -\pi/2, \qquad \Theta_4(\varepsilon,\beta) \to 0.$$

By Properties 3(a) and 4, for each $\varepsilon \in (0,1/8)$, there is a unique $\beta_1(\varepsilon) > 2\sqrt{\varepsilon(1+\varepsilon)}$ such that

$$\Theta_1(\varepsilon) \begin{cases} > \Theta_4(\varepsilon,\beta) & \text{for } 2\sqrt{\varepsilon(1+\varepsilon)} \le \beta < \beta_1(\varepsilon), \\ = \Theta_4(\varepsilon,\beta) & \text{for } \beta = \beta_1(\varepsilon), \\ < \Theta_4(\varepsilon,\beta) & \text{for } \beta > \beta_1(\varepsilon). \end{cases}$$

It follows that

$$\delta_1(\varepsilon,\beta) > \delta_4(\varepsilon,\beta) \qquad \text{for } 2\sqrt{\varepsilon(1+\varepsilon)} \le \beta < \beta_1(\varepsilon),$$
$$\delta_1(\varepsilon,\beta_1(\varepsilon)) = \delta_4(\varepsilon,\beta_1(\varepsilon)).$$

By Properties 1(c) and 4, for each $\varepsilon \in (0,1/8)$, there is a unique $\beta_2(\varepsilon) > \beta_1(\varepsilon)$ such that

$$\delta_1(\varepsilon,\beta) \begin{cases} > \delta_4(\varepsilon,\beta) & \text{for } \beta_1(\varepsilon) < \beta < \beta_2(\varepsilon), \\ = \delta_4(\varepsilon,\beta) & \text{for } \beta = \beta_2(\varepsilon), \\ < \delta_4(\varepsilon,\beta) & \text{for } \beta > \beta_2(\varepsilon). \end{cases}$$

$\beta = \beta_1(\varepsilon)$ and $\beta = \beta_2(\varepsilon)$ define curves l_1 and l_2 respectively, on which $\delta_1(\varepsilon,\beta) = \delta_4(\varepsilon,b)$. The left end point of l_1 is $(0,0)$ and its right end point is $(1/8,\sqrt{3}/2)$. l_2 is asymptotic to the β-axis and its right end point is $(1/8,\sqrt{3}/2)$.

By Properties 3(a) and 4, for all $(\varepsilon,\beta) \in \Omega_4$, $\Theta_1(\varepsilon) > \Theta_3(\varepsilon,\beta)$ and hence $\delta_1(\varepsilon,\beta) > \delta_3(\varepsilon,\beta)$. From (19), it follows that $\Theta_2(\varepsilon) > \Theta_3(\varepsilon,\beta)$ for all $(\varepsilon,\beta) \in \Omega_4$. However, by Properties 3(b) and 4 for each $\varepsilon \in (0,\varepsilon^*)$, there is a unique $\beta_3(\varepsilon) > 2\sqrt{3(1+\varepsilon)}$ such that

$$\delta_2(\varepsilon,\beta) \begin{cases} < \delta_3(\varepsilon,\beta) & \text{for } 2\sqrt{\varepsilon(1+\varepsilon)} \le \beta < \beta_3(\varepsilon), \\ = \delta_3(\varepsilon,\beta) & \text{for } \beta = \beta_3(\varepsilon), \\ > \delta_3(\varepsilon,\beta) & \text{for } \beta > \beta_3(\varepsilon). \end{cases}$$

$\beta = \beta_3(\varepsilon)$ defines a curve l_3 in Ω_4, on which $\delta_2(\varepsilon,\beta) = \delta_3(\varepsilon,\beta)$. The left end point of l_3 is $(0,1/2)$ and its right end point is (ε^*,β^*), where

$$\beta^* = 2\sqrt{\varepsilon^*(1+\varepsilon^*)}.$$

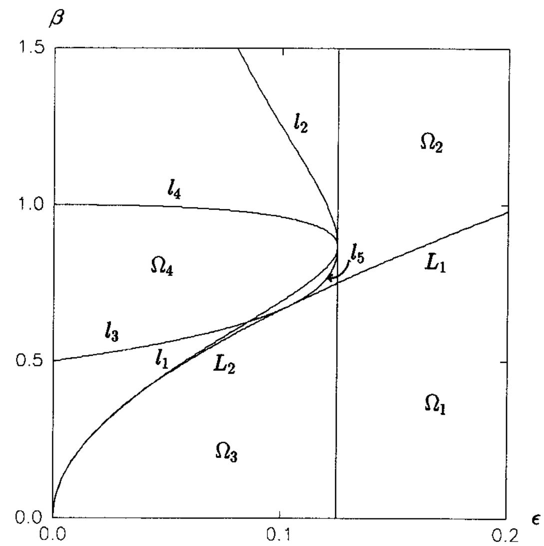

FIGURE 2(b). Curves $l_1, \ldots, l_5$. The curves $l_1, \ldots, l_5$ in Ω_4 are defined as follows.
$l_1 : \beta = \beta_1(\varepsilon), 0 < \varepsilon < 1/8$, $\Theta_1(\varepsilon) = \Theta_4(\varepsilon, \beta_1(\varepsilon))$,
$l_2 : \beta = \beta_2(\varepsilon)$, $0 < \varepsilon < 1/8$, $\delta_1(\varepsilon, \beta_2(\varepsilon)) = \delta_4(\varepsilon, \beta_2(\varepsilon))$, $\beta_1(\varepsilon) < \beta_2(\varepsilon)$,
$l_3 : \beta = \beta_3(\varepsilon)$, $0 < \varepsilon < \varepsilon^*$, $\delta_2(\varepsilon, \beta_3(\varepsilon)) = \delta_3(\varepsilon, \beta_3(\varepsilon))$,
$l_4 : \beta = \beta_4(\varepsilon)$, $0 < \varepsilon < 1/8$, $\Theta_2(\varepsilon) = \Theta_4(\varepsilon, \beta_4(\varepsilon))$,
$l_5 : \beta = \beta_5(\varepsilon)$, $\varepsilon^* < \varepsilon < 1/8$, $\delta_2(\varepsilon, \beta_5(\varepsilon)) = \delta_4(\varepsilon, \beta_5(\varepsilon)$,
$2\sqrt{\varepsilon(1+\varepsilon)} < \beta_5(\varepsilon) < \beta_4(\varepsilon)$.

By (19) and Properties 3(a) and 4, for each $\varepsilon \in (0, 1/8)$, there is a unique $\beta_4(\varepsilon) > 2\sqrt{\varepsilon(1+\varepsilon)}$ such that

$$\Theta_2(\varepsilon) \begin{cases} > \Theta_4(\varepsilon, \beta) & \text{for } 2\sqrt{\varepsilon(1+\varepsilon)} \le \beta < \beta_4(\varepsilon), \\ = \Theta_4(\varepsilon, \beta) & \text{for } \beta = \beta_4(\varepsilon), \\ < \Theta_4(\varepsilon, \beta) & \text{for } \beta > \beta_4(\varepsilon), \end{cases}$$

and it follows that

$$\delta_2(\varepsilon, \beta_4(\varepsilon)) = \delta_4(\varepsilon, \beta_4(\varepsilon)), \qquad \delta_2(\varepsilon, \beta) < \delta_4(\varepsilon, \beta) \qquad \text{for } \beta > \beta_4(\varepsilon).$$

If $0 < \varepsilon < \varepsilon^*$, then it follows from Properties 3(b) and 4 that, for $2\sqrt{\varepsilon(1+\varepsilon)} \le \beta < \beta_4(\varepsilon)$, $\delta_4(\varepsilon, \beta) > \delta_2(\varepsilon, \beta)$. However, for each $\varepsilon \in (\varepsilon^*, 1/8)$, there is a unique $\beta_5(\varepsilon)$ such that $2\sqrt{\varepsilon(1+\varepsilon)} < \beta_5(\varepsilon) < \beta_4(\varepsilon)$

and

$$\delta_2(\varepsilon,\beta)\begin{cases} > \delta_4(\varepsilon,\beta) & \text{for } 2\sqrt{\varepsilon(1+\varepsilon)} \le \beta < \beta_5(\varepsilon),\\ = \delta_4(\varepsilon,\beta) & \text{for } \beta = \beta_5(\varepsilon),\\ < \delta_4(\varepsilon,\beta) & \text{for } \beta_5(\varepsilon) < \beta < \beta_4(\varepsilon).\end{cases}$$

$\beta = \beta_4(\varepsilon)$ and $\beta = \beta_5(\varepsilon)$ define curves l_4 and l_5, respectively, on which $\delta_2(\varepsilon,\beta) = \delta_4(\varepsilon,\beta)$. l_4 lies below l_2 and lies above l_1 and l_3. l_5 lies below l_1. The left end point of l_4 is $(0,1)$ and its right end point is $(1/8, \sqrt{3}/2)$. The left end point of l_5 is (ε^*, β^*) and its right end point is $(1/8, \sqrt{3}/2)$. $l_1, \dots, l_5$ are numerically generated and shown in Figure 2(b). These curves divide Ω_4 into seven open sets $V_1, \dots, V_7$. A numerical computation shows that l_1 and l_3 have a unique intersection at $(\hat\varepsilon, \hat\beta)$ with

$$(\hat\varepsilon, \hat\beta) \approx (0.085582, 0.624811).$$

Let

$$\gamma_1 = \{(\varepsilon,\beta) \in l_1 : \hat\varepsilon < \varepsilon < 1/8\}, \qquad \gamma_2 = \{(\varepsilon,\beta) \in l_3 : \hat\varepsilon < \varepsilon < \varepsilon^*\},$$
$$\gamma_3 = \{(\varepsilon,\beta) \in l_1 : 0 < \varepsilon < \hat\varepsilon\}, \qquad \gamma_4 = \{(\varepsilon,\beta) \in l_3 : 0 < \varepsilon < \hat\varepsilon\}.$$

$V_1, \dots, V_7$, l_2, l_4, l_5, $\gamma_1, \dots, \gamma_4$, and $(\hat\varepsilon, \hat\beta)$ are shown in Figure 2(c). The relative positions of $\delta_1(\varepsilon,\beta), \delta_2(\delta,\beta), \delta_3(\varepsilon,\beta)$, and $\delta_4(\varepsilon,\beta)$ depend on which subset (ε,β) belongs to. For example, when $(\varepsilon,\beta) \in V_1$, $\varepsilon^* < \varepsilon < 1/8$ and $2\sqrt{\varepsilon(1+\varepsilon)} < \beta < \beta_5(\varepsilon)$. It follows that $\Theta_3(\varepsilon,\beta) < \Theta_4(\varepsilon,\beta) < \Theta_1(\varepsilon) < \Theta_2(\varepsilon)$ and $\delta_3(\varepsilon,\beta) < \delta_4(\varepsilon,\beta) < \delta_2(\varepsilon,\beta) < \delta_1(\varepsilon,\beta)$. Therefore, ω_0 vanishes for $\delta_3 < \delta < \delta_4$. Moreover, $\delta = \delta_1$ and $\delta = \delta_2$ are the turning points. It follows that (6) has one periodic solution given by (16) when $\delta \in (-\infty, -\delta_1) \cup (-\delta_2, \delta_3) \cup (\delta_4, \delta_2) \cup (\delta_1, \infty)$, two periodic solutions at $\delta = \pm\delta_1$ and $\delta = \pm\delta_2$, and three periodic solutions when $\delta \in (-\delta_1, -\delta_2) \cup (\delta_2, \delta_1)$ (cf. Figure 1(d)). The remaining cases can be treated similarly. We summarize the information about the relative positions of the Hopf bifurcation points and the turning points in Table 1. The relationship between the number of periodic solutions given by (16) and the subsets of Ω is also summarized in Table 2.

For each $(\varepsilon,\beta) \in \Omega_4$, Hopf bifurcation at $\delta = \delta_3(\varepsilon,\beta)$ is subcritical, i.e., the periodic solutions that bifurcate from the steady state exist for $\delta < \delta_3(\varepsilon,\beta)$. When $(\varepsilon,\beta) \in \Omega_4$ lies below l_1, i.e., $(\varepsilon,\beta) \in V_1 \cup l_5 \cup V_2 \cup \gamma_2 \cup V_3$, Hopf bifurcation at $\delta = \delta_4(\varepsilon,\beta)$ is supercritical, i.e., the periodic solutions that bifurcate from the steady state exist for $\delta > \delta_4(\varepsilon,\beta)$. However, when (ε,β) lies on or above l_1 and below l_4,

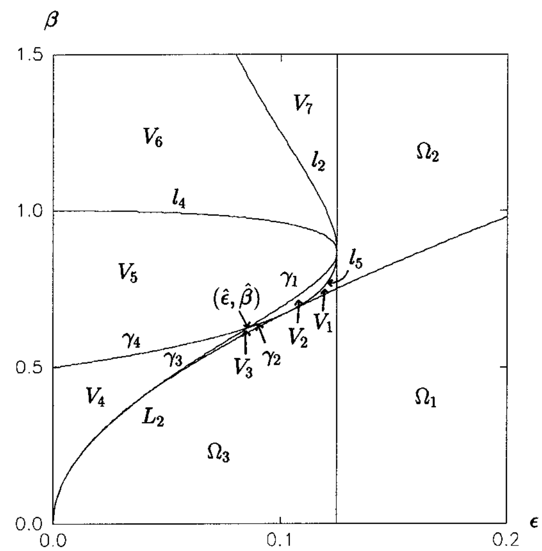

FIGURE 2(c). Subsets of Ω_4. $V_1, \dots, V_7$, l_2, l_4, l_5, and $\gamma_1, \dots, \gamma_4$ are shown.

i.e., $(\varepsilon, \beta) \in l_1 \cup V_4 \cup \gamma_4 \cup V_5$, Hopf bifurcation at $\delta = \delta_4(\varepsilon, \beta)$ becomes subcritical and the turning point at $\delta = \delta_1(\varepsilon, \beta)$ disappears from ω_0. When (ε, β) lies on or above l_4, i.e., $(\varepsilon, \beta) \in l_4 \cup V_6 \cup l_2 \cup V_7$, Hopf bifurcation at $\delta = \delta_4(\varepsilon, \beta)$ again becomes supercritical and now the turning point at $\delta = \delta_2(\varepsilon, \beta)$ disappears from ω_0.

3. Stability of the synchronized periodic solutions. In this section we present results in [**8**] concerning the stability of the periodic solutions (16) of (6). We analyze the multipliers of the variational system which consists of (10) and (11). We leave out some computational details given in [**8**]. We first analyze (10). It is shown in [**8**] that three multipliers of (10) are determined by the eigenvalues of the 3×3-matrix $B(\theta, \varepsilon, \beta)$ whose entries $B_{ij}(\Theta, \varepsilon, \beta)$, $i = 1, 2, 3$, $j = 1, 2, 3$, are defined by

$$B_{11}(\Theta, \varepsilon, \beta) = -2\left(1 + \frac{\beta \sin\Theta \cos\Theta}{\cos^2\Theta + \varepsilon}\right) + \frac{\beta \cos^2\Theta \cot\Theta}{\cos^2\Theta + \varepsilon},$$

$$B_{21}(\Theta, \varepsilon, \beta) = -\frac{\varepsilon\beta \cos\Theta \cot\Theta}{\cos^2\Theta + \varepsilon},$$

$$
\begin{aligned}
B_{31}(\Theta,\varepsilon,\beta) &= -\frac{\beta\cos\Theta(\cos^2\Theta-\varepsilon)}{(\cos^2\Theta+\varepsilon)\sqrt{1+\cos^2\Theta}},\\
B_{12}(\Theta,\varepsilon,\beta) &= -\frac{\beta\cos\Theta\cot\Theta}{\cos^2\Theta+\varepsilon},\\
B_{22}(\Theta,\varepsilon,\beta) &= \frac{\varepsilon\beta\cot\Theta}{\cos^2\Theta+\varepsilon},\\
B_{32}(\Theta,\varepsilon,\beta) &= \frac{\beta(\cos^2\Theta-\varepsilon)}{(\cos^2\Theta+\varepsilon)\sqrt{1+\cos^2\Theta}},\\
B_{13}(\Theta,\varepsilon,\beta) &= \frac{\beta\cos\Theta\sqrt{1+\cos^2\Theta}}{\cos^2\Theta+\varepsilon},\\
B_{23}(\Theta,\varepsilon,\beta) &= \frac{\varepsilon\beta\sqrt{1+\cos^2\Theta}}{\cos^2\Theta+\varepsilon},\\
B_{33}(\Theta,\varepsilon,\beta) &= \beta\cot\Theta.
\end{aligned}
$$

That is, if λ is an eigenvalue of $B(\Theta,\varepsilon,\beta)$, then $e^{\lambda T}$ is a multiplier of (10), where $T = T(\Theta,\varepsilon,\beta)$ is the period of (16) defined by (17). The characteristic equation for $B(\Theta,\varepsilon,\beta)$ is

$$p(\lambda,\Theta,\varepsilon,\beta) \equiv \lambda^3 + b_1(\Theta,\varepsilon,\beta)\lambda^2 + b_2(\Theta,\varepsilon,\beta)\lambda + b_3(\Theta,\varepsilon,\beta) = 0,$$

where

$$
\begin{aligned}
b_1(\Theta,\varepsilon,\beta) &= 2h(\Theta,\varepsilon,\beta) - 2\beta\cot\Theta,\\
b_2(\Theta,\varepsilon,\beta) &= -\frac{2\beta\cot\Theta(\cos^2\Theta+2\varepsilon)}{\cos^2\Theta+\varepsilon}h(\Theta,\varepsilon,\beta)\\
&\qquad + (\beta\cot\Theta)^2 + \left[\frac{\beta(\cos^2\Theta-\varepsilon)}{\cos^2\Theta+\varepsilon}\right]^2,\\
b_3(\Theta,\varepsilon,\beta) &= 8\varepsilon\beta^2\frac{\partial g}{\partial\Theta}(\Theta,\varepsilon)h(\Theta,\varepsilon,\beta).
\end{aligned}
$$

One finds that when Θ is in a neighborhood of $\Theta_0 = \pm\pi/2$, the eigenvalues of $B(\Theta,\varepsilon,\beta)$ have the form $\lambda = \lambda_0 + \lambda_1(\Theta-\Theta_0) + \mathcal{O}((\Theta-\Theta_0)^2)$, where $\lambda_0 = -2$, $\pm i\beta$. Moreover, when $\lambda_0 = \pm i\beta$, $\lambda_1 = -\beta$. It follows that, for each $(\varepsilon,\beta)\in\Omega$, there is a $\tau_1 = \tau_1(\varepsilon,\beta) > 0$ such that, for all $\Theta \in (-\pi/2, -\pi/2+\tau_1)$, $B(\Theta,\varepsilon,\beta)$ has three eigenvalues with negative real parts, and for all $\Theta \in (\pi/2-\tau_1, \pi/2)$, $B(\Theta,\varepsilon,\beta)$ has one eigenvalue with a negative real part and two with positive real parts.

One shows further that, for all small $|\Theta|$, an eigenvalue of $B(\Theta, \varepsilon, \beta)$ have the form

$$\lambda = \frac{1}{\sin\Theta}\left[-\frac{2\varepsilon}{1+\varepsilon}\Theta + \mathcal{O}(\Theta^2)\right],$$

and the other two eigenvalues have the form

$$\lambda = \frac{1}{\sin\Theta}[\beta + o(1)] \qquad \text{as } |\Theta| \to 0.$$

This shows that, for each $(\varepsilon, \beta) \in \Omega_4$, there is a $\tau_2 = \tau_2(\varepsilon, \beta) > 0$ such that, for all $\Theta \in (-\tau_2, 0)$, $B(\Theta, \varepsilon, \beta)$ has three eigenvalues with negative real parts, and for all $\Theta \in (0, \tau_2)$, $B(\Theta, \varepsilon, \beta)$ has one eigenvalue with a negative real part and two with positive parts.

Now we summarize these results obtained for the multipliers associated with (16) in Proposition 2 in terms of δ. Note that, given $(\varepsilon, \beta) \in \Omega$, there exists a unique periodic solution (16) of (6) when $|\delta|$ is sufficiently small or sufficiently large.

Proposition 2. *For each* $(\varepsilon, \beta) \in \Omega$*, there are* $d_1 = d_1(\varepsilon, \beta) > 0$ *and* $d_2 = d_2(\varepsilon, \beta) > 0$ *such that if* $|\delta| < d_1$ *or* $|\delta| > d_2$*, a unique periodic solution* (16) *of* (6) *exists and* 1 *is a simple multiplier associated with it. Moreover, if* $\delta \in (0, d_1) \cup (d_2, \infty)$*, three multipliers have modulus less than* 1*, and if* $\delta \in (-\infty, -d_2) \cup (-d_1, 0)$*, one multiplier has modulus less than* 1 *and two have modulus greater than* 1*.*

Next, we consider the periodic solutions on ω_0 near turning points. Recall that ω_0 has a turning point at δ_1 if $(\varepsilon, \beta) \in \Omega_3 \cup L_2 \cup \Omega_4$ and lies below curve l_1, i.e., $(\varepsilon, \beta) \in U_1$, where

$$U_1 \equiv \Omega_3 \cup L_2 \cup V_1 \cup l_5 \cup V_2 \cup \gamma_2 \cup V_3.$$

It has a turning point at δ_2 if $(\varepsilon, \beta) \in \Omega_3 \cup L_2 \cup \Omega_4$ and lies below curve l_4, i.e., $(\varepsilon, \beta) \in U_2$, where

$$U_2 \equiv U_1 \cup l_1 \cup V_4 \cup \gamma_4 \cup V_5.$$

It can be shown that there is a $\tau_3 = \tau_3(\varepsilon, \beta) > 0$ such that, for all $\Theta \in (\Theta_1 - \tau_3, \Theta_1)$, $B(\Theta, \varepsilon, \beta)$ has three eigenvalues with negative real parts, and for all $\Theta \in (\Theta_1, \Theta_1 + \tau_3)$, $B(\Theta, \varepsilon, \beta)$ has two eigenvalues with negative real parts and one with a positive real part. Similarly, there is a $\tau_4 = \tau_4(\varepsilon, \beta) > 0$ such that, for all $\Theta \in (\Theta_2 - \tau_4, \Theta_2)$, $B(\Theta, \varepsilon, \beta)$ has two eigenvalues with negative real parts and one with a positive real part, and for all $\Theta \in (\Theta_2, \Theta_2 + \tau_4)$, $B(\Theta, \varepsilon, \beta)$ has three eigenvalues with negative real parts.

For each $(\varepsilon, \beta) \in \Omega_2 \cup L_2 \cup \Omega_4$, ω_0 also has a turning point at $-\delta_1$ and $-\delta_2$. One finds that there is a $\tau_5 = \tau_5(\varepsilon, \beta) > 0$ such that for all $\Theta \in (-\Theta_1, -\Theta_1 + \tau_5)$, $B(\Theta, \varepsilon, \beta)$ has one eigenvalue with a negative real part, and for all $\Theta \in (-\Theta_1 - \tau_5, -\Theta_1)$, $B(\Theta, \varepsilon, \beta)$ has two eigenvalues with negative real parts and one with a positive real part. Similarly, there is a $\tau_6 = \tau_6(\varepsilon, \beta) > 0$ such that for all $\Theta \in (-\Theta_2, -\Theta_2 + \tau_6)$, $B(\Theta, \varepsilon, \beta)$ has two eigenvalues with negative real parts and one with a positive real part, and for all $\Theta \in (-\Theta_2 - \tau_6, -\Theta_2)$, $B(\Theta, \varepsilon, \beta)$ has one eigenvalue with a negative real part and two with positive real parts.

We summarize the information about the relation between the multipliers associated with (16) and turning points of ω_0 in Proposition 3.

Proposition 3. (a) *Suppose* $(\varepsilon, \beta) \in U_1$. *Then there is* $\tau_3 = \tau_3(\varepsilon, \beta) > 0$ *such that* 1 *is a simple multiplier associated with* (16) *if* $0 < |\Theta - \Theta_1| < \tau_3$. *Moreover, if* $\Theta \in (\Theta_1 - \tau_3, \Theta_1)$, *three multipliers have modulus less than* 1, *and if* $\Theta \in (\Theta_1, \Theta_1 + \tau_3)$, *two multipliers have modulus less than* 1 *and one has modulus greater than* 1.

(b) *Suppose* $(\varepsilon, \beta) \in U_2$. *Then there is* $\tau_4 = \tau_4(\varepsilon, \beta) > 0$ *such that* 1 *is a simple multiplier associated with* (16) *if* $0 < |\Theta - \Theta_2| < \tau_4$. *Moreover, if* $\Theta \in (\Theta_2 - \tau_4, \Theta_2)$, *two multipliers have modulus less than* 1 *and one has modulus greater than* 1, *and if* $\Theta \in (\Theta_2, \Theta_2 + \tau_4)$, *three multipliers have modulus less than* 1.

(c) *Suppose* $(\varepsilon, \beta) \in \Omega_3 \cup L_2 \cup \Omega_4$. *Then there are* $\tau_5 = \tau_5(\varepsilon, \beta) > 0$ *and* $\tau_6 = \tau_6(\varepsilon, \beta) > 0$ *such that* 1 *is a simple multiplier associated with* (16) *if* $0 < |\Theta + \Theta_1| < \tau_5$ *or* $0 < |\Theta + \Theta_2| < \tau_6$. *Moreover, if* $\Theta \in (-\Theta_1, -\Theta_1 + \tau_5) \cup (-\Theta_2 - \tau_6, -\Theta_2)$ *one multiplier has modulus less than* 1 *and two have modulus greater than* 1, *and if* $\Theta \in (-\Theta_1 - \tau_5, -\Theta_1) \cup (-\Theta_2, -\Theta_2 + \tau_6)$, *two multipliers have modulus less than* 1 *and one has modulus greater than* 1.

Next, we discuss the stability of the synchronized periodic solutions near the bifurcation point of ω_0 from the steady state. Recall that when $(\varepsilon, \beta) \in L_1 \cup L_2$, ω_0 bifurcates from the steady state at $\delta = \delta^*(\varepsilon)$ and the bifurcation is transcritical. When $(\varepsilon, \beta) \in \Omega_2 \cup \Omega_4$, ω_0 bifurcates from the steady state at $\delta = \delta_3(\varepsilon, \beta)$ and $\delta = \delta_4(\varepsilon, \beta)$. The bifurcation at $\delta = \delta_3(\varepsilon, \beta)$ is always subcritical. Define

$$W_1 \equiv \Omega_2 \cup \Omega_4 - W_2, \qquad W_2 \equiv l_1 \cup V_4 \cup \gamma_4 \cup V_5.$$

Then when $(\varepsilon, \beta) \in W_1$, the bifurcation at $\delta = \delta_4(\varepsilon, \beta)$ is supercritical and when $(\varepsilon, \beta) \in W_2$, the bifurcation is subcritical.

Suppose $(\varepsilon, \beta) \in \Omega_2 \cup \Omega_4$. It is shown that there is a $\tau_7 = \tau_7(\varepsilon, \beta) > 0$ such that for all $\Theta \in (\Theta_3 - \tau_7, \Theta_3)$, $B(\Theta, \varepsilon, \beta)$ has three eigenvalues with negative real parts. It is also shown that, for each $(\varepsilon, \beta) \in W_1$, there is a $\tau_8 = \tau_8(\varepsilon, \beta) > 0$ such that, for all $\Theta \in (\Theta_4, \Theta_4 + \tau_8)$, $B(\Theta, \varepsilon, \beta)$ has three eigenvalues with negative real parts. Furthermore, for each $(\varepsilon, \beta) \in W_2$, there is a $\tau_9 = \tau_9(\varepsilon, \beta) > 0$ such that, for all $\Theta \in (\Theta_4, \Theta_4 + \tau_9)$, $B(\Theta, \varepsilon, \beta)$ has two eigenvalues with negative real parts and one with a positive real part. Finally, for $(\varepsilon, \beta) \in L_1 \cup L_2$, there is a $\tau_{10} = \tau_{10}(\varepsilon, \beta) > 0$ such that if $0 < |\Theta - \Theta^*| < \tau_{10}$, $B(\Theta, \varepsilon, 2\sqrt{\varepsilon(1+\varepsilon)})$ has three eigenvalues with negative real parts. We summarize the results obtained for the multipliers associated with (16) and the bifurcations at $\delta = \delta_3, \delta = \delta_4$, and $\delta = \delta^*$ in Proposition 4.

Proposition 4. (a) *If $(\varepsilon, \beta) \in \Omega_2 \cup \Omega_4$, then the bifurcation of the periodic solutions* (16) *from the steady state at $\delta = \delta_3$ is subcritical. Moreover, there is a $d_7 = d_7(\varepsilon, \beta) > 0$ such that if $\delta \in (\delta_3 - d_7, \delta_3)$,* 1 *is a simple multiplier associated with* (16) *and three multipliers have modulus less than* 1.

(b) *If $(\varepsilon, \beta) \in W_1$, then the bifurcation of* (16) *from the steady state at $\delta = \delta_4$ is supercritical. Moreover, there is a $d_8 = d_8(\varepsilon, \beta) > 0$ such that if $\delta \in (\delta_4, \delta_4 + d_8)$,* 1 *is a simple mulitplier associated with* (16) *and three multipliers have modulus less than* 1.

(c) *If $(\varepsilon, \beta) \in W_2$, then the bifurcation of* (16) *from the steady state at $\delta = \delta_4$ is subcritical. Moreover, there is a $d_9 = d_9(\varepsilon, \beta) > 0$ such that if $\delta \in (\delta_4 - d_9, \delta_4)$,* 1 *is a simple multiplier associated with* (16), *two multipliers have modulus less than* 1, *and one has modulus greater than* 1.

(d) *If* $(\varepsilon, \beta) \in L_1 \cup L_2$, *then the bifurcation of* ω_0 *from the steady state at* $\delta = \delta^*$ *is transcritical. Moreover, there is a* $d_{10} = d_{10}(\varepsilon, \beta) > 0$ *such that if* $0 < |\delta - \delta^*| < d_{10}$, 1 *is a simple multiplier associated with* (16) *and three multipliers have modulus less than* 1.

The behavior of periodic solutions that bifurcate from the steady state is closely related to its stability. The variational equation of (6) with respect to the steady state $\bar{x} = x_0 = 0$ is

$$\frac{d}{dt}\begin{pmatrix} \bar{x} \\ x_0 \end{pmatrix} = A(\delta, \varepsilon, \beta)\begin{pmatrix} \bar{x} \\ x_0 \end{pmatrix},$$

where

$$A(\delta, \varepsilon, \beta) = \begin{bmatrix} K - \delta P & \delta P \\ \varepsilon\delta P & -\varepsilon\delta P \end{bmatrix}, \qquad K = Df(0).$$

It is shown in [**5**] that the characteristic equation for $A(\delta, \varepsilon, \beta)$ has the form

$$\det\{\lambda^2 - [K - (1+\varepsilon)\delta P] - \varepsilon\delta P K\} = 0,$$

which is

$$\{\lambda^2 + [4(1+\varepsilon)\delta - 1]\lambda - 4\varepsilon\delta\}^2 + \beta^2(\lambda + 4\varepsilon\delta)^2 = 0. \tag{23}$$

(23) shows that there is no real eigenvalue of $A(\delta, \varepsilon, \beta)$ for $\delta \neq 0$. Moreover, two eigenvalues of $A(\delta, \varepsilon, \beta)$ are the roots of

$$\lambda^2 + a_1(\delta, \varepsilon, \beta)\lambda + a_2(\delta, \varepsilon, \beta) = 0, \tag{24}$$

where

$$a_1(\delta, \varepsilon, \beta) = 4(1+\varepsilon)\delta - (1 + i\beta),$$
$$a_2(\delta, \varepsilon, \beta) = -4\varepsilon(1 + i\beta)\delta.$$

The analysis of (24) leads to the results obtained in [**8**] for the eigenvalues of $A(\delta, \varepsilon, \beta)$. We summarize these results in Proposition 5.

Proposition 5. *Suppose* $(\varepsilon, \beta) \in \Omega$.

(a) *For all negative* δ, $A(\delta, \varepsilon, \beta)$ *has four eigenvalues with positive real parts.*

(b) *If* $(\varepsilon, \beta) \in \Omega_1 \cup \Omega_3$, *then for all positive* δ, $A(\delta, \varepsilon, \beta)$ *has two eigenvalues with negative real parts and two with positive real parts.*

(c) *If* $(\varepsilon, \beta) \in L_1 \cup L_2$, *then for each* $\delta \in (0, \delta^*) \cup (\delta^*, \infty)$, $A(\delta, \varepsilon, \beta)$ *has two eigenvalues with negative real parts and two with positive real parts.*

(d) *If* $(\varepsilon, \beta) \in \Omega_2 \cup \Omega_4$, *then* $A(\delta, \varepsilon, \beta)$ *has two eigenvalues with negative ral parts and two with positive real parts for all* $\delta \in (0, \delta_3) \cup (\delta_4, 0)$, *and four eigenvalues with negative real parts for all* $\delta \in (\delta_3, \delta_4)$.

Next we analyze the multipliers of (11) to determine the stability of (16) as a solution of the full sytem (1). That is, we study the stability of the synchronized periodic solution (9) with $\bar{\phi}$ and ϕ_0 defined by (16). It is shown that the eigenvalue of the 2×2-matrix $Q(\Theta, \varepsilon, \beta)$ defined by

$$Q(\Theta, \varepsilon, \beta) = \begin{bmatrix} \frac{\beta \cos^2 \Theta \cot \Theta}{\cos^2 \Theta + \varepsilon} - 2h(\Theta, \varepsilon, \beta) & \frac{\beta \cos^2 \Theta}{\cos^2 \Theta + \varepsilon} \\ -\frac{\beta \cos^2 \Theta}{\cos^2 \Theta + \varepsilon} & \frac{\beta \cos^2 \Theta \cot \Theta}{\cos^2 \Theta + \varepsilon} \end{bmatrix}$$

are characteristic exponents of (11). The characteristic equation for $Q(\Theta, \varepsilon, \beta)$ is

$$\lambda^2 + q_1(\Theta, \varepsilon, \beta)\lambda + q_2(\Theta, \varepsilon, \beta) = 0,$$

where

$$\begin{aligned} q_1(\Theta, \varepsilon, \beta) &= 2\left[h(\Theta, \varepsilon, \beta) - \frac{\beta \cos^2 \Theta \cot \Theta}{\cos^2 \Theta + \varepsilon}\right], \\ q_2(\Theta, \varepsilon, \beta) &= \frac{\beta \cos^2 \Theta \cot \Theta}{\cos^2 \Theta + \varepsilon}\left[\frac{\beta \cos^2 \Theta \cot \theta}{\cos^2 \Theta + \varepsilon} - 2h(\Theta, \varepsilon, \beta)\right] \\ &\quad + \left(\frac{\beta \cos^2 \Theta}{\cos^2 \Theta + \varepsilon}\right)^2. \end{aligned}$$

One shows that, for all $\Theta \in (-\pi/2, 0)$ such that $h(\Theta, \varepsilon, \beta) > 0$, the eigenvalues of $Q(\Theta, \varepsilon, \beta)$ has negative real parts. Furthermore, it is shown that there is a $\tau_{11} = \tau_{11}(\varepsilon, \beta) > 0$ such that for each $\Theta \in (\pi/2 - \tau_{11}, \pi/2)$, $Q(\Theta, \varepsilon, \beta)$ has one eigenvalue with a negative real part and the other has a positive real part. It is also shown that there is a $\tau_{12} = \tau_{12}(\varepsilon, \beta) > 0$ such that, for each $\Theta \in (0, \tau_{12})$, $Q(\Theta, \varepsilon, \beta)$ has two eigenvalues with positive real parts. We summarize the results obtained for eigenvalues of $Q(\Theta, \varepsilon, \beta)$ in Proposition 6. The statement in Proposition 6(b) concerns the case $\delta \in (-\infty, 0)$ and $|\delta|$ is either sufficiently small or sufficiently large for which at least one periodic solution (16) exists for a given $(\varepsilon, \beta) \in \Omega$.

Proposition 6. (a) *For every* $(\varepsilon, \beta) \in \Omega$, *the* $2(N-1)$ *multipliers associated with* (9) *on the complement, which are determined from* (11), *have modulus less than* 1 *if* $\delta > 0$.

(b) *For every* $(\varepsilon, \beta) \in \Omega$, *there are* $d_{11} = d_{11}(\varepsilon, \beta) > 0$ *and* $d_{12} = d_{12}(\varepsilon, \beta) > 0$ *such that for* $\delta \in (-d_{11}, 0)$, *the* $N-1$ *multipliers associated with* (9) *on the complement which are determined from* (11), *have modulus less than* 1, *and the remaining* $N-1$ *have modulus greater than* 1, *and for* $\delta \in (-\infty, -d_{12})$, *the* $2(N-1)$ *multipliers have modulus greater than* 1.

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