ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 1, Winter 1994

SUMMABILITY METHODS FOR OSCILLATION OF LINEAR SECOND-ORDER MATRIX DIFFERENTIAL EQUATIONS

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Dedicated to Paul Waltman on the occasion of his 60th birthday

ABSTRACT. Familiar oscillation criteria of Wintner [**11**] and Hartman [**7**] for the equation (1) $y'' + q(t)y = 0$ on $[0,\infty)$ using limits of the mean $(1/t)\int_0^t(\int_0^s q(u)\,du)\,ds$ have been extended by various authors in many ways. For equation (1) itself, these extensions have included the use of weighted means [**3, 10, 9**] or of iterated weighted means [**5**]. Other extensions of the scalar results have been made to the matrix equation (2) $Y'' + Q(t)Y = 0$ [**1, 2, 6**] and to the self-adjoint matrix equation (3) $(PY')' + Q(t)Y = 0$ [**4**]. Meanwhile, Hartman [**8**] gave conditions on very general means which allowed simplification of the proofs in [**5**] and which properly included their results and some others. In this paper, all these considerations are combined to derive improved results for (5).

1. Introduction. In 1949, Wintner [**11**] proved that a hypothesis sufficient for the oscillation of

$$y'' + q(t)y = 0 \tag{1}$$

is

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_0^s q(u)\,du\,ds = +\infty. \tag{2}$$

In 1952, Hartman [**7**] weakened this hypothesis to the following:

$$\begin{aligned}&\liminf_{t\to\infty} \frac{1}{t}\int_0^t \int_0^s q(u)\,du\,ds > -\infty \quad \text{and}\\ &\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_0^s q(u)\,du\,ds \quad \text{does not exist.}\end{aligned} \tag{3}$$

Received by the editors on March 9, 1993.

Crucial to Hartman's generalization is the following lemma.

Lemma A. *Assume that* (1) *is nonoscillatory. Then the following are equivalent*:

(i) *For some solution* $r(t)$ *of the related Riccati equation* $r' = q(t) + r^2$, $\int^\infty r^2(t)\,dt$ *exists*;

(ii) $\lim_{t\to\infty}(1/t)\int_0^t\int_0^s q(u)\,du\,ds$ *exists*;

(iii) $\liminf_{t\to\infty}(1/t)\int_0^t\int_0^s q(u)\,du\,ds > -\infty$.

A number of authors have since made use of the ideas and techniques of Hartman's paper to extend his criterion (3) and Lemma A using weighted means (Willett [**10**], Macki and Wong [**9**]), and to matrix equations (Butler, Erbe, and Mingarelli [**1**]). Others have extended the original criterion (2) of Wintner using weighted means (Coles [**3**]) and iterated weighted means (Coles and Willett [**5**]); these results have also been extended to matrix equations (Byers, Harris, and Kwong [**2**], Coles [**4**]).

The matrix equation

$$Y'' + Q(t)Y = 0 \tag{4}$$

was considered in [**1, 2, 6**] and the results in [**2**] were extended in [**4**] to the self-adjoint matrix equation

$$(P(t)Y')' + Q(t)Y = 0 \tag{5}$$

with hypotheses that were more general in other ways. Equation (5) must be considered separately from equation (4) because, unlike the scalar case, there is no oscillation-preserving transformation of the independent variable that allows passage between the two forms.

Meanwhile, for the scalar equation (1), Hartman [**8**] had defined a summability kernel more general than the iterated means used in [**5**] and thereby was able to simplify the proofs and to weaken the hypotheses in [**5**], finally obtaining a generalization of his own Lemma A and criterion (3).

In this paper we draw on Riccati methods, a generalized kernel like that in [**8**], and other ideas from the papers cited above to obtain an extension of Lemma A and Hartman's (3) (and the corresponding results of [**1**]) to the self-adjoint matrix equation (5), with proofs as relatively simple as those of Hartman [**8**]. We also apply the same method of proof to obtain an improved version of the theorems in [**4**] which generalize the original Wintner criterion (2).

2. The setup and the statements. Consider the $n \times n$ matrix differential equation (5) on $[0, \infty)$ where $P(t)$ and $Q(t)$ are real, continuous, and symmetric, and $P(t)$ is positive definite. A solution $Y(t)$ is *prepared* if $Y^*(PY') = (PY')^*Y$. Equation (5) is said to be *oscillatory* on $[0, \infty)$ if, for each $a \geq 0$, the determinant of each nontrivial prepared solution has a zero on $[a, \infty)$.

If $Y(t)$ is a prepared, nonoscillatory solution of (5), then $R = -PY'Y^{-1}$ satisfies the matrix Riccati equation

$$R' = Q + RP^{-1}R \tag{6}$$

on $[a, \infty)$, where a exceeds the largest zero of $\det(Y(t))$. The integrated form of (6) is

$$R(t) = C(a) + Q_1(t) + \int_a^t RP^{-1}R\,ds \tag{7}$$

where $Q_1(t) := \int_0^t Q(s)\,ds$ and $C(a) := R(a) - Q_1(a)$. Our proofs will be based on (7).

For an $n \times n$ symmetric matrix A, $\lambda_1(A)$, $\lambda_n(A)$, and $\operatorname{tr}(A)$ shall denote, respectively, the largest and smallest eigenvalues and the trace.

We consider oscillation criteria for (5) in terms of general means of symmetric matrix-valued functions $F(t)$:

$$\int_0^t K(t,s)F(s)\,ds,$$

where $K(t,s) \geq 0$ is real-valued and continuous for $0 \leq s \leq t$, $0 < t < \infty$. The following are hypotheses on $K(t,s)$ (and $P(t)$) that will occasionally be assumed ($f(t)$ denotes an arbitrary continuous real-valued function).

(H1) $\int_0^t K(t,s)\lambda_1(P(s))f(s)\,ds \to 0$ as $t \to \infty$ if $\int^\infty f(s)\,ds$ exists;

(H2) $\int_0^t K(t,s)f(s)\,ds \to 0$ as $t \to \infty$ if $f(t) \to 0$ as $t \to \infty$;

(H3) $\int_0^t K(t,s)\,ds = O(1)$ as $t \to \infty$;

(H4) $\int_0^t K(t,s)f(s)\,ds \to \infty$ as $t \to \infty$ if $f(t) \to \infty$ as $t \to \infty$;

(H5) There exists a continuous function $k(t) > 0$ and a positive definite matrix $C(t,s)$, continuous for $0 \le s \le t$, $0 < t < \infty$, such that

(i) $g(t,s) := (\partial/\partial t)\int_s^t k(t)K(t,s)\,ds \ge 0$ exists and is continuous for $0 \le s \le t$, $0 < t < \infty$, and

(ii) $g(t,s)P^{-1}(s) \ge k^2(t)K^2(t,s)C^{-1}(t,s)$ for s sufficiently large and $s \le t < \infty$, and

(iii) for $H(t) := \int_0^t C(t,s)\,ds$, $\int^\infty \lambda_n(H^{-1}(t))\,dt = \infty$ or $\limsup_{t\to\infty} k(t)\int_t^\infty \lambda_n(H^{-1}(t))\,dt > 0$.

Remarks. (1) Our technical condition (H5) is simpler than the corresponding technical condition (d) of Hartman [**8**], and, furthermore, the latter condition implies (H5) even in the scalar case; we omit the proof.

(2) See Hartman [**8**] for examples of kernels satisfying (his formulations of) the hypotheses.

(3) It will be seen from the proof of Theorem 2.1 that (H1) can be replaced by the weaker hypothesis:

(H1′) $\int_0^t K(t,s)F^2(s)\,ds \to 0$ if $\int^\infty (FP^{-1}F)(s)\,ds$ exists.

This is a condition on $\int_0^t K(t,s)(\cdot)\,ds$ as a *quadratic* mapping on the space

$$L^2(P) = \{X(t) : \int^\infty (X^TP^{-1}X)(s)\,ds \text{ exists}\}.$$

We retain hypothesis (H1) because it is closer to the idea that $\int_0^t K(t,s)(\cdot)\,ds$ is a generalized summability method.

In our generalization of Lemma A (and Lemma 5.1 of [**1**]), we will use hypotheses (H1) through (H5) to obtain a chain of implications among the following collection of statements.

(S1) For every prepared nonoscillatory solution $Y(t)$ of (5), $\int^\infty (RP^{-1}R)(s)\,ds$ exists, where $R = -PY'Y^{-1}$.

(S2) There exist $a \geq 0$ and a symmetric matrix $C_1 = C_1(a)$ such that

$$\lim_{t\to\infty} \int_a^t K(t,s)(Q_1(s) - C_1)^2\, ds = 0.$$

(S3) There exist $a \geq 0$ and a symmetric matrix $C_1 = C_1(a)$ such that

$$\lim_{t\to\infty} \int_a^t K(t,s)(Q_1(s) - C_1)\, ds = 0,$$

(S4)
$$\liminf_{t\to\infty} \int_0^t K(t,s)\mathrm{tr}\,(Q_1(s))\, ds > -\infty.$$

The limits in (S2) and (S3) can be interpreted in the operator norm sense, but our proofs will use the equivalent entrywise sense.

Theorem 2.1. *Assume that* (5) *is nonoscillatory.*

(a) *If $K(t,s)$ satisfies* (H1) *and* (H2)*, and if* (S1) *holds, then* (S2) *holds*;

(b) *If $K(t,s)$ satisfies* (H3) *and* (S2)*, then* (S3) *holds*;

(c) *If $K(t,s)$ satisfies* (H4) *and* (S3)*, then* (S4) *holds*;

(d) *If $K(t,s)$ satisfies* (H3), (H4), (H5)*, and* (S4)*, then* (S1) *holds.*

Remarks. (1) If the additional hypothesis

(H6)
$$\int_0^T K(t,s)\, ds \to 0 \qquad \text{as } t \to \infty \text{ for } T \geq 0 \text{ fixed}$$

is satisfied, then (S2) and (S3) can be replaced, respectively, by

(S2′) there exists a constant symmetric matrix C_1 such that

$$\lim_{t\to\infty} \int_0^t K(t,s)(Q_1(s) - C_1)^2\, ds = 0,$$

and

(S3′) there exists a constant symmetric matrix C_1 such that

$$\lim_{t\to\infty} \int_0^t K(t,s)(Q_1(s) - C_1)\, ds = 0.$$

(2) If (H6) and

$$\liminf_{t\to\infty}\int_0^t K(t,s)\,ds \geq \delta > 0 \tag{H7}$$

are satisfied, then in (S3′), we have

$$C_1 = \lim_{t\to\infty}(1/\int_0^t K(t,s)ds)\int_0^t K(t,s)Q_1(s)\,ds.$$

We reap the rewards of Theorem 2.1 in the following oscillation criteria. Parts (a) and (b) extend Theorems 2.1(B) and 2.1(A) of [**1**], respectively.

Theorem 2.2. *Suppose* $K(t,s)$ *satisfies* (HI), $i = 1,\dots,5$, *and suppose* (S4) *holds.*

(a) *If*

$$\limsup_{t\to\infty}\int_0^t K(t,s)\lambda_1(Q_1(s))^2 = \infty$$

then (5) *is oscillatory.*

(b) *If*

$$\limsup_{t\to\infty}\int_0^t K(t,s)\lambda_1(Q_1(s))\,ds = \infty,$$

then (5) *is oscillatory.*

We now turn our attention to an extension of Theorems 1 and 2 in [**4**]. Before stating this result, we establish some notation. Set

$$\begin{aligned}
S_\mu &:= \left\{t \geq 0 : \lambda_1\left\{\int_0^t K(t,s)Q_1(s)\,ds\right\} \geq \mu\int_0^t K(t,s)\,ds\right\},\\
S_\mu(t) &:= S_\mu \cap [t,\infty),\\
j(t) &:= \int_0^t k(t)K(t,s)\,ds,\\
L(\mu) &:= \limsup_{t\to\infty} j(t)\int_{S_\mu(t)}\lambda_n(H^{-1}(s))\,ds.
\end{aligned}$$

Theorem 2.3. *Suppose parts* (i) *and* (ii) *of* (H5) *hold. Then each of the following is sufficient for oscillation of* (5).

(a)
$$\limsup_{\mu\to\infty} \mu \int_{S_\mu} \lambda_n(H^{-1}(s))\,ds > \frac{n}{j(\infty)};$$

(b)
$$\limsup_{\mu\to\infty} \mu L(\mu) > n.$$

Note that part (ii) of (H5) implies that $j(\infty) = \lim_{t\to\infty} j(t) \le \infty$ makes sense.

3. The proofs. As usual, for symmetric matrices A and B, we write $A \ge B$ to mean $A - B \ge 0$, i.e., $A - B$ is nonnegative. We will be using various properties of this ordering; in particular, for A, B, C, and D symmetric, we have:

$$\begin{aligned}
&A > B \Rightarrow CAC > CBC,\\
&A \ge B, C \ge D \Rightarrow A + C \ge B + D,\\
&A \not\le B, B \ge C \Rightarrow A \not\le C,\\
&A \ge 0, A \not\ge B, C > 0 \Rightarrow ACA \not\ge BCB,\\
&A \not\le B, B \ge 0 \Rightarrow \operatorname{tr}(A) \ge \lambda_n(B),\\
&C \ge 0 \Rightarrow (A+B)C(A+B) \le 2ACA + 2BCB.
\end{aligned}$$

(Proof of the last property: $(A+B)C(A+B) \le (A+B)C(A+B) + (A-B)C(A-B) = 2ACA + 2BCB$.)

We also need the following lemma. It is a matrix extension of the Schwarz inequality. We refer the reader to [**4**] for the proof. As noted in that paper, this lemma provides a way to prove theorems for (5) rather than (4).

Lemma 3.1. *If $X(t)$ and $Y(t)$ are integrable on $[a,b]$, and if $\int_a^b Y^*Y$ is nonsingular, then*

$$\left(\int_a^b X^*Y\right)\left(\int_a^b Y^*Y\right)^{-1}\left(\int_a^b Y^*X\right) \le \int_a^b X^*X.$$

Corollary 3.2. *If $A(t)$ and $B(t)$ are symmetric and if $A(t) > 0$, then*

$$\int_a^b BA^{-1}B \geq \Big(\int_a^b B\Big)\Big(\int_a^b A\Big)^{-1}\Big(\int_a^b B\Big).$$

Proof. Take $Y = A^{1/2}$ and $X = Y^{-1}B$ in Lemma 3.1. □

Proof of Theorem 2.1. (a) Suppose $\int_a^\infty (RP^{-1}R)(s)\,ds$ exists. Then we can rewrite the Riccati integral equation (7) as

$$R(t) + \int_t^\infty RP^{-1}R\,ds = Q_1(t) - C_1,$$

where $C_1 := -R(a) + Q_1(a) - \int_a^\infty RP^{-1}R\,ds$. Squaring, applying $K(t,s)$, integrating, and using the last listed property of symmetric matrices, we find

$$\begin{aligned}\int_a^b K(t,s)(Q_1(s) - C_1)^2\,ds &\leq 2\int_a^t K(t,s)R^2(s)\,ds \\ &\quad + 2\int_a^t K(t,s)\Big(\int_s^\infty RP^{-1}R\Big)^2\,ds \\ &\leq 2\int_a^t K(t,s)\lambda_1(P(s))(RP^{-1}R)(s))\,ds \\ &\quad + 2\int_a^t K(t,s)\Big(\int_s^\infty RP^{-1}R\Big)^2\,ds.\end{aligned}$$

We apply (H1) entrywise to the first term on the right hand side and apply (H2) entrywise to the second term to obtain

$$\int_a^b K(t,s)(Q_1(s) - C_1)^2\,ds \to 0 \qquad \text{as } t \to \infty.$$

This proves (a).

(b) By Corollary 3.2,

$$\begin{aligned}\int_a^b K(t,s)(Q_1(s)-C_1)^2\,ds \\ &\geq \frac{1}{\int_a^t K(t,s)\,ds}\left(\int_a^t K(t,s)(Q_1(s)-C_1)\,ds\right)^2 \\ &\geq \frac{1}{\int_0^t K(t,s)\,ds}\left(\int_a^t K(t,s)(Q_1(s)-C_1)\,ds\right)^2 \geq 0.\end{aligned}$$

Since the left side tends to zero, so does the right side, and (H3) implies that the denominator is bounded, so

$$\int_a^t K(t,s)(Q_1(s)-C_1)\,ds \to 0 \qquad \text{as } t\to\infty.$$

This proves (b).

(c) Taking the trace of (S3), we have

$$\lim_{t\to\infty}\int_a^t K(t,s)(\mathrm{tr}\,(Q_1(s))-c_1)\,ds = 0,$$

where $c_1 = \mathrm{tr}\,(C_1)$. Now

$$\begin{aligned}\liminf_{t\to\infty}\int_0^t K(t,s)\mathrm{tr}\,(Q_1(s))\,ds &\geq \liminf_{t\to\infty}\int_0^a K(t,s)\mathrm{tr}\,(Q_1(s))\,ds \\ &\quad + \liminf_{t\to\infty}\int_a^t K(t,s)\mathrm{tr}\,(Q_1(s))\,ds.\end{aligned}$$

Estimating the first term, we find

$$\begin{aligned}\liminf_{t\to\infty}\int_0^a K(t,s)\mathrm{tr}\,(Q_1(s))\,ds &\geq -\max_{0\leq s\leq a}|\mathrm{tr}\,(Q_1(s))|\liminf_{t\to\infty}\int_0^a K(t,s)\,ds \\ &\geq -\max_{0\leq s\leq a}|\mathrm{tr}\,(Q_1(s))|\liminf_{t\to\infty}\int_0^t K(t,s)\,ds \\ &> -\infty,\end{aligned}$$

using (H3).

Next we compute

$$\begin{aligned}\liminf_{t\to\infty}\int_a^t K(t,s)\operatorname{tr}(Q_1(s))\,ds \\ &= \liminf_{t\to\infty}\int_a^t K(t,s)(\operatorname{tr}(Q_1(s)) - c_1 + c_1)\,ds \\ &\geq \liminf_{t\to\infty}\int_a^t K(t,s)(\operatorname{tr}(Q_1(s)) - c_1)\,ds \\ &\quad + \liminf_{t\to\infty} c_1\int_a^t K(t,s)\,ds \\ &\geq -|c_1|\limsup_{t\to\infty}\int_a^t K(t,s)\,ds \\ &\geq -|c_1|\limsup_{t\to\infty}\int_0^t K(t,s)\,ds \\ &> -\infty,\end{aligned}$$

using (S3) and (H3). This proves (c).

(d) Suppose $\int^\infty RP^{-1}R\,ds$ does not exist. Then $\operatorname{tr}(\int^\infty RP^{-1}R\,ds)$ does not exist. Set $S(t) := \int_a^t K(t,s)\int_a^s \operatorname{tr}(RP^{-1}R)(u)\,du\,ds$. We apply $K(t,s)$ to equation (7), integrate, and take the trace to find

$$\begin{aligned}\int_a^t K(t,s)\operatorname{tr}(R(s))\,ds - \frac{1}{2}S(t) &= c(a)\int_a^t K(t,s)\,ds \\ &\quad + \int_a^t K(t,s)\operatorname{tr}(Q_1(s))\,ds + \frac{1}{2}S(t),\end{aligned}\tag{8}$$

where $c(a) = \operatorname{tr}(C(a))$. Hypothesis (H4) implies that $S(t) \to \infty$ as $t \to \infty$; hypothesis (H3) implies that $c(a)\int_a^t K(t,s)\,ds \geq -|c(a)|\int_0^t K(t,s)\,ds$ is bounded below. These together with (S4) show that the right hand side of (8) is positive for sufficiently large $t \geq 0$. Thus,

$$\int_a^t K(t,s)\operatorname{tr}(R(s))\,ds \geq \frac{1}{2}S(t) \tag{9}$$

for sufficiently large t. Now

$$(10)\qquad \int_a^t k(t)K(t,s)\int_a^s \operatorname{tr}(RP^{-1}R)\,du\,ds = \int_a^t \left\{\int_s^t k(t)K(t,u)\,du\right\}\operatorname{tr}(RP^{-1}R)\,ds,$$

so using (H5) and Corollary 3.2, we compute, for a sufficiently large,

$$\begin{aligned}
(kS)' &= \operatorname{tr}\left(\int_a^t g(t,s)(RP^{-1}R)(s)\,ds\right)\\
&\geq \operatorname{tr}\left(\int_a^t k^2(t)K(t,s)R(s)C^{-1}(t,s)R(s)\,ds\right)\\
&\geq \operatorname{tr}\left\{\left(\int_a^t k(t)K(t,s)R(s)\,ds\right)\left(\int_a^t C(t,s)\,ds\right)^{-1}\right.\\
&\qquad\qquad \left.\cdot\left(\int_a^t k(t)K(t,s)R(s)\,ds\right)\right\}\\
&\geq \operatorname{tr}\left\{\left(\int_a^t k(t)K(t,s)R(s)\,ds\right)H^{-1}(t)\left(\int_a^t k(t)K(t,s)R(s)\,ds\right)\right\}\\
&\geq k^2(t)\lambda_n(H^{-1}(t))\operatorname{tr}\left\{\left(\int_a^t K(t,s)R(s)\,ds\right)^2\right\}\\
&\geq \frac{k^2(t)}{n}\lambda_n(H^{-1}(t))\left(\int_a^t K(t,s)\operatorname{tr}(R(s))\,ds\right)^2.
\end{aligned}$$

Thus, using (9) we have

$$(kS)'(t) \geq \frac{1}{4n}\lambda_n(H^{-1}(t))(kS)^2(t),$$

for t sufficiently large. We divide by $(kS)^2$ and integrate the resulting differential inequality. For $t \geq b \geq a$, this gives

$$\frac{1}{k(b)S(b)} \geq \frac{1}{k(b)S(b)} - \frac{1}{k(t)S(t)} \geq \frac{1}{4n}\int_b^t \lambda_n(H^{-1}(s))\,ds.$$

This contradicts the first alternative of (H5)(iii). Thus,

$$\frac{1}{S(b)} \geq \frac{1}{4n}k(b)\int_b^\infty \lambda_n(H^{-1}(s))\,ds.$$

Letting $b \to \infty$, we contradict the other alternative of (H5)(iii). This completes the proof of (d) and the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. (a) We compute

$$\begin{aligned}\int_0^t K(t,s)\lambda_1(Q_1(s))^2\,ds &\le \int_0^t K(t,s)\lambda_1(Q_1^2(s))\,ds\\ &= \int_0^t K(t,s)\lambda_1((Q_1(s)-C_1+C_1)^2)\,ds\\ &\le 2\int_0^t K(t,s)\lambda_1((Q_1(s)-C_1)^2)\,ds\\ &\quad + 2\lambda_1 C_1^2\int_0^t K(t,s)\,ds\\ &\le 2\mathrm{tr}\left\{\int_0^t K(t,s)(Q_1(s)-C_1)^2\,ds\right\}\\ &\quad + 2\lambda_1 C_1^2\int_0^t K(t,s)\,ds.\end{aligned}$$

We estimate the first term on the right hand side. Setting $u(a) := \max_{0\le s\le a}\mathrm{tr}\,\{(Q_1(s)-C_1)^2\}$, we find

$$\begin{aligned}&\mathrm{tr}\left\{\int_0^t K(t,s)(Q_1(s)-C_1)^2\,ds\right\}\\ &= \mathrm{tr}\left\{\int_0^a K(t,s)(Q_1(s)-C_1)^2\,ds + tr\int_a^t K(t,s)(Q_1(s)-C_1)^2\,ds\right\}\\ &\le u(a)\int_0^a K(t,s)\,ds + \mathrm{tr}\left\{\int_a^t K(t,s)(Q_1(s)-C_1)^2\,ds\right\}\\ &\le u(a)\int_0^t K(t,s)\,ds + \mathrm{tr}\left\{\int_a^t K(t,s)(Q_1(s)-C_1)^2\,ds\right\}.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^t K(t,s)\lambda_1(Q_1(s))^2\,ds &\le (2\lambda_1(C_1^2)+2u(a))\int_0^t K(t,s)\,ds\\ &\quad + \mathrm{tr}\left\{\int_a^t K(t,s)(Q_1(s)-C_1)^2\,ds\right\}.\end{aligned}$$

Since (S4) and (HI), $i = 1, \dots, 5$, all hold, (S2) holds, and thus the second term on the right side approaches zero. By (H3), the first term is bounded above. Thus, the entire right hand side is bounded. This contradiction proves the result.

(b) By the Schwarz inequality,

$$\int_0^t K(t,s)\lambda_1(Q_1(s))^2\,ds \geq \frac{1}{\int_0^t K(t,s)\,ds}\left(\int_0^t K(t,s)\lambda_1(Q_1(s))\,ds\right)^2.$$

By part (a), the left hand side approaches zero and so the right hand side must approach zero. Hypothesis (H3) shows that the denominator is bounded, so $\int_0^t K(t,s)\lambda_1(Q_1(s))^2\,ds \to 0$ as $t \to \infty$. This proves (b). □

Proof of Theorem 2.3. Assume that $Y(t)$ is a prepared, nonoscillatory solution of (5), and let $R = -PY'Y^{-1}$ be the corresponding solution to the Riccati equation (6) on $[a, \infty)$. For $t \geq b \geq a$, integrate this equation to get

$$R(t) = C(b) + Q_1(t) + \int_b^t RP^{-1}R\,ds,$$

where $C(b) = R(b) - Q_1(b)$. We apply our kernel and integrate:

$$\tag{11} \begin{aligned} \int_b^t K(t,s)R(s)\,ds = \left\{\int_b^t K(t,s)\,ds\right\}C(b) + \int_b^t K(t,s)Q_1(s)\,ds \\ + \int_b^t K(t,s)\int_b^s RP^{-1}R\,du\,ds. \end{aligned}$$

We now estimate from below the terms on the right hand side of (11).

First,

$$C(b) \geq \lambda_n(C(b)) \cdot I \geq -|\lambda_n(C(b))| \cdot I,$$

so

$$\tag{12} \begin{aligned} \left\{\int_b^t K(t,s)\,ds\right\}C(b) &\geq -\left\{\int_b^t K(t,s)\,ds\right\}|\lambda_n(C(b))| \cdot I \\ &\geq -\left\{\int_0^t K(t,s)\,ds\right\}|\lambda_n(C(b))| \cdot I. \end{aligned}$$

Next,

$$\int_b^t K(t,s)Q_1(s)\,ds = \int_0^t K(t,s)Q_1(s)\,ds + \int_0^b K(t,s)Q_1(s)\,ds,$$

so by the convexity of $\lambda_1(\cdot)$,

$$\begin{aligned}\lambda_1\bigg\{\int_b^t K(t,s)Q_1(s)\,ds\bigg\} &\geq \lambda_1\bigg\{\int_0^t K(t,s)Q_1(s)\,ds\bigg\} \\ &\quad - \lambda_1\bigg\{\int_0^b K(t,s)Q_1(s)\,ds\bigg\} \\ &\geq \lambda_1\bigg\{\int_0^t K(t,s)Q_1(s)\,ds\bigg\} \\ &\quad - |\lambda_1\bigg\{\int_0^b K(t,s)Q_1(s)\,ds\bigg\}|.\end{aligned}$$

Now

$$\begin{aligned}\bigg|\lambda_1\bigg\{\int_0^b K(t,s)Q_1(s)\,ds\bigg\}\bigg| &\leq \int_0^b K(t,s)|\lambda_1(Q_1(s))|\,ds \\ &\leq \max_{0\leq s\leq b} |\lambda_1(Q_1(s))| \cdot \int_0^b K(t,s)\,ds \\ &\leq \max_{0\leq s\leq b} |\lambda_1(Q_1(s))| \cdot \int_0^t K(t,s), ds \\ &=: \gamma(b)\int_0^t K(t,s)\,ds.\end{aligned}$$

Thus, for $t \in S_\mu(b)$,

$$\lambda_1\bigg\{\int_b^t K(t,s)Q_1(s)\,ds\bigg\} \geq (\mu - \gamma(b))\int_0^t K(t,s)\,ds.$$

In terms of matrix inequalities, this means

$$\int_b^t K(t,s)Q_1(s)\,ds \not\leq (\mu - \gamma(b))\bigg\{\int_0^t K(t,s)\,ds\bigg\}\cdot I. \tag{13}$$

Combining (11), (12) and (13), we have

$$\begin{aligned}\int_b^t K(t,s)R(s)\,ds &\not\leq (\mu - f(b))\Big\{\int_0^t K(t,s)\,ds\Big\}\cdot I \\ &\quad + \int_b^t K(t,s)\int_b^s RP^{-1}R\,du\,ds\end{aligned} \tag{14}$$

for $t \in S_\mu(b)$, where $f(b) := \gamma(b) + |\lambda_n(C(b))|$. For $t \geq b$, let $B(t)$ denote the right hand side of (14), and set $A(t) := k(t)B(t)$. As in equation (10), we have

$$\begin{aligned}&\int_b^t k(t)K(t,s)\int_b^s RP^{-1}R\,du\,ds \\ &\qquad = \int_b^t \Big\{\int_s^t k(t)K(t,u)\,du\Big\}RP^{-1}R\,ds.\end{aligned} \tag{15}$$

Thus,

$$A'(t) = (\mu - f(b))g(t,0)\cdot I + \int_b^t g(t,s)RP^{-1}R\,ds,$$

using part (i) of (H5). Now choose $b \geq a$ large enough that, for $t \geq s \geq b$, part (ii) of (H5) is applicable. Then choose μ sufficiently large so that $\mu \geq f(b)$. Then, using Corollary 3.3, we compute

$$\begin{aligned}A'(t) &\geq k^2(t)\int_b^t K^2(t,s)R(s)C^{-1}(t,s)R(s)\,ds \\ &\geq k^2(t)\Big(\int_b^t K(t,s)R(s)\,ds\Big)\Big(\int_b^t C(t,s)\,ds\Big)^{-1} \\ &\quad\cdot\Big(\int_b^t K(t,s)R(s)\,ds\Big) \\ &\geq k^2(t)\Big(\int_b^t K(t,s)R(s)\,ds\Big)H^{-1}(t)\Big(\int_b^t K(t,s)R(s)\,ds\Big).\end{aligned}$$

Thus, for $t \in S_\mu(b)$ we have

$$\begin{aligned}A'(t) &\not\leq k^2(t)B(t)H^{-1}(t)B(t) \\ &= A(t)H^{-1}(t)A(t)\end{aligned}$$

using (14).

Thus

$$A^{-1}(t)A'(t)A^{-1}(t) \not\leq H^{-1}(t), \qquad t \in S_\mu(b),$$

so

$$(-\mathrm{tr}\,(A^{-1}(t)))' \geq \lambda_n(H^{-1}(t)), \qquad t \in S_\mu(b).$$

Integrating the left hand side from $T \geq b$ to t, we have

$$\mathrm{tr}\,(A^{-1}(T)) \geq \mathrm{tr}\,(A^{-1}(T)) - \mathrm{tr}\,(A^{-1}(t)) = \int_T^t (-\mathrm{tr}\,(A^{-1}(s)))'\,ds;$$

thus

$$\mathrm{tr}\,(A^{-1}(T)) \geq \int_T^\infty (-\mathrm{tr}\,(A^{-1}(s)))'\,ds \geq \int_{S_\mu(T)} \lambda_n(H^{-1}(s))\,ds. \tag{16}$$

We estimate the left hand side of (16) by using the definition of $A(t)$:

$$\begin{aligned}\mathrm{tr}\,(A^{-1}(T)) = \mathrm{tr}\,\Big\{&(\mu - f(b))j(T)\cdot I \\ &+ \int_b^T k(T)K(T,s)\,ds \int_b^s RP^{-1}R\,du\,ds\Big\}^{-1} \\ \leq \mathrm{tr}\,&\{(\mu - f(b))j(T)\cdot I\}^{-1} \\ = &\frac{n}{(\mu - f(b))j(T)}.\end{aligned} \tag{17}$$

From (16) and (17), it follows that

$$\int_{S_\mu(T)} \lambda_n(H^{-1}(s))\,ds \leq \frac{n}{(\mu - f(b))j(T)}. \tag{18}$$

For T fixed, choose μ so large that $S_\mu \subseteq [T, \infty)$. Then from (18) we have

$$\limsup_{\mu\to\infty} \mu \int_{S_\mu} \lambda_n(H^{-1}(s))\,ds \leq \frac{n}{j(T)},$$

so

$$\limsup_{\mu\to\infty} \mu \int_{S_\mu} \lambda_n(H^{-1}(s))\,ds \leq \frac{n}{j(\infty)}.$$

This contradicts (a).

Next note that (18) implies that

$$j(T)\int_{S_\mu(T)}\lambda_n(H^{-1}(s))\,ds \le \frac{n}{\mu - f(b)},$$

and so

$$L(\mu) \le \frac{n}{\mu - f(b)}.$$

Therefore,

$$\limsup_{\mu\to\infty} \mu L(\mu) \le n.$$

This contradicts (b) and completes the proof of the theorem. □

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