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SURJECTIVITY RESULTS FOR COMPACT PERTURBATIONS OF STRONGLY MONOTONE OPERATORS IN BANACH SPACES

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ABSTRACT. The operator equation $Au + Lu + cFu = h$ is studied in a Banach space X and its dual space. The operators L, F are compact and A is strongly monotone. Degree arguments are used to show the existence of solutions of the equation and extension of the results in [**2, 3**] are established.

1. Introduction. In the following, the symbols R and R_+ denote the sets $(-\infty, \infty)$ and $[0, \infty)$, respectively. X stands for a real Banach space having a Schauder basis $\{x_i\}$. Without loss of generality, we will assume that X is normed so that both X and X^* are locally uniformly convex and $||x_i|| = 1$, $i = 1, 2, 3, \dots$. Referring to the book ([**4**, pages 25, 272]) there exists a constant $M \geq 1$, independent of n, such that

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq M \left\| \sum_{i=1}^{\infty} a_i x_i \right\|, \qquad n = 1, 2, 3, \dots \tag{1}$$

and

$$\sup\{|\langle \Phi, x \rangle| : ||x|| \leq 1, x \in E_k^\perp\} \to 0, \qquad k \to \infty \tag{2}$$

for each $\Phi \in X^*$. Here

$$E_k = \text{span}\,\{x_1, x_2, \dots, x_k\}$$

and

$$E_k^\perp = \text{span}\,\{x_{k+1}, x_{k+2}, \dots\}.$$

Lemma 1. *Let $L : X \to X^*$ be a compact mapping. Then*

$$\lim_{k\to\infty} \sup\{|\langle Lf, f \rangle| : f \in E_k^\perp, ||f|| \leq 1\} = 0.$$

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Proof. Since L is compact, the set $\{Lf : ||f|| \leq 1\}$ is totally bounded, i.e., for all $\varepsilon > 0$, there exists a finite ε-net $\{\Phi_1, \Phi_2, \dots, \Phi_n\} \subset X^*$. By (2), for $\varepsilon > 0$, there exists a number k_0 such that

$$\sup\{|\langle \Phi_i, f\rangle| : ||f|| \leq 1, f \in E_k^\perp\} < \varepsilon$$

for all $k \geq k_0$ and $i = 1, 2, \dots, n$. Now, for any $f \in E_k^\perp$, $||f|| \leq 1$, $k \geq k_0$, we have

$$\begin{aligned} |\langle Lf, f\rangle| &\leq |\langle Lf - \Phi_{i0}, f\rangle + \langle \Phi_{i0}, f\rangle| \\ &\leq ||Lf - \Phi_{i0}||\,||f|| + |\langle \Phi_{i0}, f\rangle| \\ &\leq \varepsilon + \varepsilon \leq 2\varepsilon \end{aligned}$$

so that

$$\sup\{|\langle Lf, f\rangle| : ||f|| \leq 1, f \in E_k^\perp\} < 2\varepsilon.$$

This completes the proof. □

An operator $A : D \subset X \to X^*$ is said to be *strongly monotone* if there exists a constant $c > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq c||x - y||^2 \tag{3}$$

for all $x, y \in D$, and it is *maximal monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0$$

and

$$R(A + \lambda J) = X^*$$

for every $\lambda > 0$. Here the *duality mapping* $J : X \to X^*$ is defined by

$$Jx = \{f^* \in X^* : ||f^*||^2 = ||x||^2 = \langle x, f^*\rangle\}. \tag{4}$$

An operator $F : D \subset X \to X^*$ is said to be *compact* if it is continuous and maps bounded subset of D into relatively compact subset of X^*. It is *weakly continuous* if $x_n \rightharpoonup x$ implies $Fx_n \rightharpoonup F_x$. It is *demicontinuous* if $x_n \to x$ implies $Fx_n \rightharpoonup fx$. It is *demiclosed* at 0 if $x_n \rightharpoonup x$ and $Fx_n \to 0$ implies $Fx = 0$. A weakly continuous mapping is demiclosed at 0. Here, $\rightharpoonup$ (respectively, $\to$) denotes weak (respectively, strong) convergence. We denote by $B_b(0)$ the open ball with center at zero and radius $b > 0$. $\partial B_b(0)$ and $\overline{B_b(0)}$ denote the boundary and the closure of B_b, respectively.

Our results in this paper are an improvement and extension of the results of Kesavan [**3**] and Kartsatos and Mabry [**2**]. Kesavan [**3**] showed that the operator equation

$$u - \lambda Lu + Fu = h$$

is solvable in a Hilbert space H for all $\lambda \in R$, $h \in H$. Here, L is linear, compact, self-adjoint, and strictly positive, while F is compact, positively homogeneous of degree $\rho > 1$, $\langle Fu, u\rangle \geq 0$, and $\langle Fu, u\rangle = 0$ implies $u = 0$. Kartsatos and Mabry [**2**] established some localized versions of the main results of Kesavan's.

In this paper, we prove the solvability of the equation

$$Au + Lu + \mu Fu = h \tag{5}$$

in a real Banach space which has Schauder basis without assuming L is self-adjoint and strictly positive.

2. Main results. In this section, we will always assume that X is a real Banach space and has Schauder basis $\{x_1, x_2, x_3, \dots\}$. The proofs of the Theorems follow from the important Lemma 2.

Let $a, b, d, B \geq 0$ and $c \in R$ be given. Set

$$W(a,b,c,d,B,E_k) \\ = \{v \in X : v = e + f, e \in E_k, f \in E_k^{\perp} \text{ such that (6) is satisfied}\}$$

where

$$a||f||^2 - b||f|| \leq c||e||^2 + d||e|| + B||e||\,||f||. \tag{6}$$

Lemma 2. (i) *Let $a > b > 0$. Then there exists a constant $\delta > 0$ depending on a, b, c, d, B and the space E_k such that for all $v \in W(a, b, c, d, B, E_k)$ with $||v|| \geq 1$, we have*

$$g(v/||v||) \geq \delta. \tag{7}$$

(ii) *Given a number* $\beta, 0 \leq \beta \leq 1$. *Let* $a\beta^2 > b\beta > 0$. *Then there exists a constant* $\delta > 0$ *depending on* a, b, c, d, B *and the space* E_k *such that for all* $v \in W(a, b, c, d, B, E_k)$ *with* $||v|| = \beta$, *we have*

$$g(v) \geq \delta.$$

Here the mapping $g : \overline{B_1(0)} \to R_+$ *is demiclosed at* 0 *and* $g(u) = 0$ *implies* $u = 0$.

Proof. (i) Suppose not. Then there exists a sequence u_n in $W(a, b, c, d, B, E_k)$, $||u_n|| = 1$, such that

$$g(u_n) \to 0, \qquad \text{as} \quad n \to \infty.$$

Let $u_n = e_n + f_n$, $e_n \in E_k$, $f_n \in E_k^\perp$. Then

$$||e_n|| \leq M||u_n|| = M$$

and

$$||f_n|| \leq M + 1.$$

Thus there exist subsequences $\{e_{n_j}\}$ and $\{f_{n_j}\}$ such that $e_{n_j} \to e \in E_k$ and $f_{n_j} \rightharpoonup f \in E_k^\perp$. Therefore, we have $u_{n_j} \rightharpoonup u = e + f$. Since g is demiclosed at 0, $g(u) = 0$. It follows that $u = 0$ and $e = f = 0$. Since $||u_n|| = 1$ and $u_{n_j} = f_{n_j} + e_{n_j}$. Thus

$$\lim_{n\to\infty} ||e_{n_j}|| = 0, \qquad \lim_{n\to\infty} ||f_{n_j}|| = 1.$$

But $\{f_{n_j}\}$ and $\{e_{n_j}\}$ satisfy (6) so that $0 < a - b \leq 0$, which is a contradiction. The proof of (ii) is exactly the same as (i). The proof is now complete. □

Given the operator equation (5), let $h = h_n + h_n^\perp \in X^*$ and let M^* be a constant such that

$$\left\| \sum_{i=1}^{n} a_i x_i^* \right\| \leq M^* \left\| \sum_{i=1}^{\infty} a_i x_i^* \right\|, \qquad n = 1, 2, 3, \dots ,$$

where $\{x_i^*\}$ is the Schauder basis of X^* and

$$h_n \in \text{span}\,\{x_1^*, \dots, x_n^*\}, \qquad h_n^{\perp} \in \text{span}\,\{x_{n+1}^*, \dots\}.$$

Choose $\varepsilon > 0$ such that $c - \varepsilon > \varepsilon/2$, where $c > 0$ is the coefficient of strong monotonicity of A. Since L is a compact operator, by Lemma 1, there exists a number k such that

$$||h_n^{\perp}|| < \varepsilon/4$$

and

$$\sup\{|\langle Lf, f\rangle| : f \in E_n^{\perp}, ||f|| \leq 1\} < \varepsilon/4$$

for all $n \geq k$. Define the set

$$\overline{W} = W(c - \varepsilon, \varepsilon/2, ||L|| - c, M^*||h||, 2(||L|| + c), E_k) \tag{8}$$

We are now ready to prove the following theorems.

Theorem 1. *Let $A, L : D \to X^*$ with $(0 \in Int(D))$ be such that A is continuous strongly monotone and L is linear and compact. Let $F : D \to X^*$ be a completely continuous mapping, $\langle Fu, u\rangle \geq 0$, and $\langle Fu, u\rangle = 0$ implies $u = 0$. Then there exists a constant $\mu_0 \geq 0$ such that the operator equation* (5) *has a solution $u \in D$ for every μ which $\mu \geq \mu_0$.*

Proof. Without loss of generality, we may assume $A0 = 0$. Since the strongly monotone operator A has the local boundedness property (cf. [**1**]), there exists a number β such that $\overline{B_\beta(0)} \subset D$ and $A(B_\beta(0))$ is bounded. Moreover, since A is one-to-one and is onto $A(\overline{B_\beta(0)})$, it is sufficient to find solutions for the equation

$$v + (L + \mu F)A^{-1}v - h = 0, \qquad v \in A(\overline{B_\beta(0)}). \tag{9}$$

Assume that equation (9) has no solutions $v \in \partial A(B_\beta(0))$. We observe that the mapping $v \to (L+\mu F)A^{-1}v - h$ is compact, so that the Leray-Schauder degree (cf. [**1**]) $d(I + (L + \mu F)A^{-1} - h, A(B_\beta(0))), 0)$ is well defined. Consider the homotopy

$$S_t = I + t[(L + \mu F)A^{-1} - h], \qquad t \in (0, 1]. \tag{10}$$

If $0 \in S_t(\partial A(B_\beta(0)))$ for some $t \in (0,1]$, then

$$v_t + t(LA^{-1}v_t + \mu FA^{-1}v_t - h) = 0 \tag{11}$$

for some $v_t \in \partial A(B_\beta(0))$. Put $u_t = A^{-1}v_t$. Since $A(B_\beta(0))$ is open (cf. [**1**]), $u_t \in \partial B_\beta(0)$ and

$$Au_t + t(Lu_t + \mu Fu_t - h) = 0. \tag{12}$$

Now we want to show that there exists a constant $\delta > 0$, independent of $t \in (0,1]$ and μ, such that

$$\langle Fu_t, u_t\rangle \geq \delta \tag{13}$$

for all u_t which satisfies (12). Choose $\varepsilon > 0$ such that $(c-\varepsilon)\beta^2 > (\varepsilon/2)\beta$ and set

$$\overline{W} = W(c-\varepsilon, \varepsilon/2, ||L|| - c, M^*||h||, 2(||L||+c), E_k).$$

Let $u_t = e + f$, $e \in E_k$, $f \in E_k^\perp$. Then

$$\begin{aligned}\langle Lu_t, u_t\rangle &= \langle Le, e\rangle + \langle Le, f\rangle + \langle Lf, e\rangle + \langle Lf, f\rangle \\ \langle h, u_t\rangle &= \langle h_k, e\rangle + \langle h_k^\perp, f\rangle\end{aligned}$$

and

$$\begin{aligned}c||e||^2 + c||f||^2 - 2c||e||\,||f|| \leq c||u_t||^2 &\leq (1/t)\langle Au_t, u_t\rangle \\ &= -\langle Lu_t, u_t\rangle - \mu\langle Fu_t, u_t\rangle + \langle h, u_t\rangle \\ &\leq ||L||\,||e||^2 + \varepsilon||f||^2 + 2||L||\,||e||\,||f|| \\ &\quad + M^*||h||\,||e|| + (\varepsilon/2)||f||.\end{aligned}$$

Therefore,

$$(c-\varepsilon)||f||^2 - (\varepsilon/2)||f|| \leq (||L||-c)||e||^2 + M^*||h||\,||e|| + 2(||L||+c)||e||\,||f||$$

i.e., $u_t \in \overline{W}$. Using (ii) of Lemma 2, there exists a constant $\delta > 0$ such that $\langle Fu_t, u_t\rangle \geq \delta > 0$.

Now, set

$$t_0 = \inf\{t \in (0,1]\colon \text{ there exists } u_t \in \partial B_\beta(0) \text{ such that (12) is satisfied}\}.$$

Then $t_0 > 0$. Suppose not; then there exist sequences $t_n \in (0,1]$, $t_n \to 0$ and μ_n such that u_{t_n} satisfies

$$Au_{t_n} + t_n(Lu_{t_n} + \mu_n Fu_{t_n} - h) = 0.$$

Since the mapping A is strongly monotone and $\langle Fu, u\rangle \geq 0$, we have

$$\begin{aligned} cb^2 &= c||u_{t_n}||^2 \leq \langle Au_{t_n}, u_{t_n}\rangle \\ &= -t_n\langle Lu_{t_n}, u_{t_n}\rangle - t_n\mu_n\langle Fu_{t_n}, u_{t_n}\rangle + t_n\langle h, u_{t_n}\rangle \\ &\leq t_n||L||b^2 + t_n||h||b \\ &\to 0, \qquad \text{as } n \to \infty, \end{aligned}$$

which is a contradiction.

Now, from (12) we have the estimate

$$\begin{aligned} 0 &= |\langle (1/t)Au_t + Lu_t + \mu Fu_t - h, u_t\rangle| \\ &\geq \mu\langle Fu_t, u_t\rangle - (1/t_0)\sup\{||Au_t||\,||u_t||\} \\ &\quad - \sup\{||Lu_t||\,||u_t||\} - ||h||\,||u_t||. \end{aligned}$$

We can choose $\mu_0 > 0$ sufficiently large such that for all μ with $\mu > \mu_0$ we have

$$\mu\delta - (1/t_0)\sup\{||Au_t||\}b - \sup\{||Lu_t||\}b - ||h||b > 0.$$

This contradiction shows that $0 \bar{\in} S_t(\partial A(B_\beta(0)))$ for all $t \in (0,1]$. By the properties of the Leray-Schauder degree, equation (5) has solutions in $\overline{B_\beta(0)} \subset D$. This completes the proof. □

Theorem 2. *Let $A : \overline{D(A)} \subset X \to X^*$ be maximal monotone and strongly monotone with constant c. Let $f : \overline{D(A)} \subset X \to X^*$ and $L : X \to X^*$ be compact and linear, compact, respectively. Assume that there exists a mapping $g : \overline{B_1(0)} \to R_+$ which is demiclosed at 0, $g(u) = 0$ implies $u = 0$, and that*

$$\langle Fu, u\rangle \geq g(u/||u||)||u||^{\rho+1} \tag{14}$$

holds for a fixed number $\rho > 1$. Then the operator equation (5) is solvable for all $h \in X^*$ and $\mu > 0$.

Proof. We consider the approximating equations

$$Au + Lu + \mu Fu + (1/n)Ju = h. \tag{15}$$

Since A is maximal monotone, $R(A + (1/n)J) = X^*$ and so (15) is equivalent to

$$v + (L + \mu F)J_n v - h = 0, \tag{16}$$

where $J_n = (A + (1/n)J)^{-1}$. First, we will show that (16) has solutions for any n. Since the mapping $v \to (L + \mu F)J_n v - h$ is compact, the Leray-Schauder degree $d(I + (L + \mu F)J_n - h, B_b^*(0), 0)$ is well defined. It suffices to show that there exists a number $b > 0$ such that

$$0 \bar{\in} (I + t[(L + \mu F)J_n - h])\partial B_b^*(0)$$

for all $t \in (0, 1]$. To this end, assume the contrary and let $t_m \in (0, 1]$, $v_m \equiv v_{t_m}$ be such that

$$(A + (1/n)J)J_n v_m + t_m[(L + \mu F)J_n v_m - h] = 0$$

and $||v_m|| \to \infty$ as $m \to \infty$. Set $x_m = J_n v_m$, we obtain

$$Ax_m + (1/n)Jx_m + t_m[(L + \mu F)x_m - h] = 0. \tag{17}$$

Since $L + \mu F$ is compact, by equation (17) and $v_m = Ax_m + (1/n)Jx_m$, there exists a subsequence of $\{x_m\}$ converging to ∞ (again denote by $\{x_m\}$).

Let $x_m = e + f$, $e \in E_k$, $f \in E_k^\perp$. Then

$$\langle Lx_m, x_m\rangle = \langle Le, e\rangle + \langle Le, f\rangle + \langle Lf, e\rangle + \langle Lf, f\rangle$$
$$\langle h, x_m\rangle = \langle h_k, e\rangle + \langle h_k^\perp, f\rangle$$

and

$$\begin{aligned} c||e||^2 + c||f||^2 - 2c||e||\,||f|| \le c||x_m||^2 &\le (1/t_m)\langle Ax_m, x_m\rangle \\ &= -\langle Lx_m, x_m\rangle - \mu\langle Fx_m, x_m\rangle + \langle h, x_m\rangle \\ &\quad - (1/nt_m)\langle Jx_m, x_m\rangle \\ &\le ||L||\,||e||^2 + \varepsilon||f||^2 + 2||L||\,||e||\,||f|| \\ &\quad + M^*||h||\,||e|| + (\varepsilon/2)||f||. \end{aligned}$$

So,

$$(c-\varepsilon)||f||^2-(\varepsilon/2)||f|| \le (||L||-c)||e||^2+M^*||h||\,||e||+2(||L||+c)||e||\,||f||,$$

i.e., $x_m \in \overline{W}$. Applying Lemma 2, there exists a number $\delta > 0$ such that $g(x_m/||x_m||) \ge \delta > 0$. Thus,

$$\begin{aligned}0 &= \langle (1/t_m)Ax_m + (1/nt_m)Jx_m + (L+\mu F)x_m - h, x_m\rangle \\ &\ge \mu\delta ||x_m||^{\rho+1} - ||L||\,||x_m||^2 - ||h||\,||x_m|| \\ &> 0, \qquad \text{as} \quad m \to \infty,\end{aligned}$$

which gives us the desired contradiction. It follows that (15) is solvable for every $n = 1, 2, 3, \dots$. Let u_n denote a solution of (15). A similar argument to the one above shows that $\{u_n\}$ is uniformly bounded. Thus, there exists a sequence $\{(L+\mu F)u_{n_j}\}$ which converges strongly. Therefore,

$$\begin{aligned}c||u_{n_j} - u_{n_k}||^2 &\le \langle Au_{n_j} - Au_{n_k}, u_{n_j} - u_{n_k}\rangle \\ &= -\langle (L+\mu F)u_{n_j} - (L+\mu F)u_{n_k}, u_{n_j} - u_{n_k}\rangle \\ &\quad - \langle (1/n_j)Ju_{n_j} - (1/n_k)Ju_{n_k}, u_{n_j} - u_{n_k}\rangle \\ &\to 0, \qquad j,k \to \infty.\end{aligned} \tag{18}$$

Thus, $u_{n_j} \to u_0$, $j \to \infty$. Since A is demicontinuous and demiclosed, $Au_0 + Lu_0 + \mu F u_0 = h$. The proof is complete. □

Theorem 3. *Let $A, L : X \to X^*$ be such that A is strongly monotone and demicontinuous and L is linear and compact. Let $F : X \to X^*$ be compact, and there exists a weakly continuous function $g : \overline{B_1(0)} \to R_+$ such that $g(u) = 0$ implies $u = 0$ and*

$$\langle Fu, u\rangle \ge g(u/||u||)||u||^{\rho+1}$$

for all $u \in X$ and a fixed number $\rho > 1$. Then equation (5) has solutions $u \in X$ for every $\mu > 0$.

Proof. Without loss of generality, we assume $A0 = 0$. First, we consider the following finite dimensional problem: For a fixed number m, find $u_m \in E_m$ such that for all $u \in E_m$

$$\langle Au_m + Lu_m + \mu F u_m, u\rangle = \langle h, u\rangle. \tag{19}$$

Let $\xi = (\xi_i)_1^m \in R^m$ and $u = \sum_1^m \xi_i x_i \in E_m$. Define the mapping $T_m : R^m \to R^m$ by

$$(T_m(\xi))_i = \langle Au + Lu + \mu Fu - h, x_i \rangle, \qquad i = 1, 2, 3, \dots, m.$$

To find a solution u_m of (19) is equivalent to find a zero for map T_m. Due to the finite dimensionality of E_m and that g is weakly continuous, there exists a constant $\delta_m > 0$ such that

$$g(u/||u||) \geq \delta_m$$

for all $u \in E_m \backslash \{0\}$. Thus,

$$\begin{aligned}\langle T_m(\xi), \xi \rangle &= \langle Au + Lu + \mu Fu - h, u \rangle \\ &\geq \mu\delta_m ||u||^{\rho+1} - ||L|| \, ||u||^2 - ||h|| \, ||u|| > 0\end{aligned}$$

on $||\xi|| = r_m$. Hence equation (19) has a solution u_m. Next, we will sho that $\{u_m\}$ is uniformly bounded. As in the proof of Theorem 2, we have $u_m \in \overline{W}$. Using Lemma 2, there exists $\delta > 0$, independent of E_m, such that

$$\langle Fu_m, u_m \rangle \geq \delta ||u_m||^{\rho+1}.$$

By (19), we obtain

$$\mu\delta ||u_m||^{\rho+1} \leq ||L|| \, ||u_m||^2 + ||h|| \, ||u_m||.$$

Therefore, $\{u_m\}$ must be uniformly bounded. Referring to (18) in the proof of Theorem 2, there exists a subsequence $\{u_{n_j}\}$ which converges strongly to u_0. Since A is demicontinuous,

$$\langle Au_0 + Lu_0 + \mu Fu_0 - h, u \rangle = 0$$

for all $u \in E_m$, $m = 1, 2, \dots$. Furthermore, it is true for all $u \in X$. Thus,

$$Au_0 + Lu_0 + \mu Fu_0 = h$$

and the proof is complete. □

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