ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 2, Spring 1994

A CHARACTERIZATION OF SOLUTIONS TO A PERTURBED LAPLACE EQUATION II

JOHN KELINGOS AND PETER MASSOPUST

1. Introduction. This report is a sequel to work done by P. Staples and the first author in [**12**]. Both papers concern the elliptic partial differential equation

$$\operatorname{div}(\sigma \operatorname{grad} u) = 0, \tag{1.1}$$

which is the model equation for a number of physical situations, e.g., steady state temperature distribution without heat sources where σ is the coefficient of heat conduction of the medium; magnetic potential with σ the magnetic permeability of the medium; the potential of the electric field of a steady current where σ is the conductivity of the medium [**2**, p. 387].

We are interested in finding a representation for the general solution to (1.1) in the case of variable σ, and where σ is *not* required to be *real analytic* in its variables. As in [**12**] we consider only the two dimensional case, where in polar coordinates (r,θ) (1.1) becomes

$$\sigma \Delta u + \sigma_r u_r + \frac{1}{r^2}\sigma_\theta u_\theta = 0. \tag{1.2}$$

Here Δ is the Laplace operator and subscripts denote partial derivatives. In [**12**] σ was assumed to depend on r only. The method of separation of variables was then used to find an eigen-function expansion for the general solution to (1.2) in the unit disk, where classical stability theory for ordinary differential equations was invoked to handle the ensuing r-equation.

In this paper we take the next obvious step and assume $\sigma(r,\theta) = \sigma_1(r)\sigma_2(\theta)$, in order to examine the θ-dependence in the method of separation of variables. An eigenfunction expansion (see (2.20) and (4.4)) for the general solution to (1.2) on the unit disk in this case

Received by the editors on April 6, 1992, and in revised form on September 22, 1992.

follows easily using standard periodic Sturm-Liouville theory for the ensuing θ-equation.

Analyzing the boundary behavior of these solutions in a distributional sense is more interesting however. In a series of papers culminating in [**10**] Lions and Magenes consider elliptic systems of partial differential equations with *analytic* coefficients, defined in domains of $\mathbf{R}^n$ with compact *analytic* boundary. They then establish a linear and topological isomorphism between the space of solutions to the equations, provided with the topology of uniform convergence on compact subsets of the domain, and a certain space of distributions defined on the boundary, the "boundary values" of the solutions. These boundary values are called *analytic functionals* or *hyperfunctions*, and have been studied in various settings by Gelfand and Silov [**6**], Sato [**11**], Köthe [**9**] and others.

The topology on the test function space for these hyperfunctions usually appears as the inductive limit of a sequence of normed spaces, although the choice of the normed spaces differs among the authors. In [**8**] Johnson considers the very special case of Laplace's equation ((1.2) with constant σ) in the unit disk of the plane, and presents an independent description of the topologies for the hyperfunctions on the boundary and their function test space using sequence spaces of Fourier coefficients. This approach uses no intermediate normed spaces. It relies on the fact that every (complex) solution to Laplace's equation in the unit disk of the plane has an (eigenfunction) expansion of the form

$$u(r,\theta) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}, \tag{1.3}$$

for a unique sequence of (complex) constants a_k satisfying

$$\limsup_{|k|\to\infty} |a_k|^{1/k} \le 1. \tag{1.4}$$

In this paper the eigenfunction expansion for the general solution to (1.2) in the unit disk with $\sigma = \sigma_1(r)\sigma_2(\theta)$ is also shown to have unique coefficients satisfying (1.4). This enables us to adapt Johnson's approach in defining the space of distributional boundary values for solutions to (1.2). These generalized hyperfunctions are different from, but algebraically and topologically equivalent to, the hyperfunctions of Lions and Magenes. This difference is to be expected since σ is not analytic and therefore the solutions to (1.2) are not analytic. On the other hand, in [**12**] $\sigma = \sigma(r)$ was also not analytic, but the solutions $u(r, \theta)$ *were* analytic in θ, and as a result the "boundary values" were exactly the analytic functionals of Lions and Magenes, an unexpected result, which comprises a form of stability in that case.

In the final section we discuss extensions of our results to higher dimensions.

2. The kernel solution. In [**12**] the case $\sigma = \sigma(r)$ was considered, hence in this section we consider only $\sigma = \sigma(\theta) > 0$, and we assume σ is $C^2(\mathbf{R}^1)$ and 2π-periodic. In Section 4 we will combine these two cases. Our goal is to find all (real) solutions $u(r, \theta)$ to (1.2) in the punctured unit disk, $0 < r < 1$, which are bounded in a neighborhood of the origin. If we set $\varepsilon(\theta) = \sigma'(\theta)/\sigma(\theta)$, then

$$\varepsilon(\theta) \in C^1(\mathbf{R}^1); \qquad \int_0^{2\pi} \varepsilon(\theta)\, d\theta = 0, \tag{2.1}$$

and equation (1.2) becomes

$$\Delta u + \frac{\varepsilon(\theta)}{r^2} \frac{\partial u}{\partial \theta} = 0, \qquad 0 < r < 1. \tag{2.2}$$

Separating variables in (2.2), we arrive at the following two boundary value problems in r and θ, respectively:

$$r^2 R'' + rR' - \lambda R = 0; \qquad R(0) \text{ finite}, \; R(1) = 1, \tag{2.3}$$

$$\Theta'' + \varepsilon(\theta)\Theta' + \lambda\Theta = 0; \qquad \Theta(0) = \Theta(2\pi), \; \Theta'(0) = \Theta'(2\pi). \tag{2.4}$$

The θ-problem (2.4) is a periodic eigenvalue problem, and we will show below that all the eigenvalues λ_k are nonnegative. The r-problem (2.3) is a Cauchy-Euler equation, whose unique solution satisfying the boundary conditions is

$$R_k(r) = r^{\sqrt{\lambda_k}}, \qquad k \geq 0. \tag{2.5}$$

If $\lambda = 0$, the general solution to (2.4) is

$$\Theta = a + b\int_0^\theta \exp\left(\int_0^t \varepsilon(t)\,dt\right)d\theta,$$

and the periodicity of Θ implies $b = 0$, so up to a constant multiple

$$\Theta_0(\theta) \equiv 1.$$

We put (2.4) into self-adjoint form by multiplying by

$$\eta(\theta) = \exp\left(\int_0^\theta \varepsilon(t)\,dt\right) > 0. \tag{2.6}$$

Then (2.1) implies η is $C^2(\mathbf{R}^1)$ and 2π-periodic. Also $\eta(0) = \eta(2\pi) = 1$ and η is bounded away from 0. The eigenvalue problem (2.4) becomes

$$(\eta(\theta)\Theta')' + \lambda\eta(\theta)\Theta = 0, \qquad \Theta(0) = \Theta(2\pi),\ \Theta'(0) = \Theta'(2\pi). \tag{2.7}$$

Using known results on such boundary value problems (see, e.g., [**3**, p. 214, Theorem 3.1 or **5**, pp. 19–27]. See also [**4**, p. 293].) we list the facts we will need concerning the eigenvalues and eigenfunctions.

All eigenvalues $\lambda \geq 0$, and as we saw, $\lambda_0 = 0$ is an eigenvalue with eigenfucntion $\Theta_0 \equiv 1$. The remaining eigenvalues occur in *pairs*. Denote the eigenvalues by

$$\{\lambda\} = \{0 = \lambda_0 < \lambda_1' \leq \lambda_1'' < \lambda_2' \leq \lambda_2'' < \cdots \to \infty\}. \tag{2.8}$$

The pairs may or may not be double eigenvalues. Corresponding to each eigenvalue λ is a unique eigenfunction ψ_λ, which due to the pairing phenomenon we denote by

$$\{\psi_\lambda\} = \{C_0(\theta), C_1(\theta), S_1(\theta), C_2(\theta), S_2(\theta), \dots\}, \tag{2.9}$$

$C_0(\theta) \equiv$ constant. These functions can be chosen to be *real* and *orthonormal* with respect to the *weight function* $\eta(\theta)$,

$$\int_0^{2\pi} \psi_\lambda\psi_{\tilde{\lambda}}\eta\,d\theta = \begin{cases} 1 & \text{if } \lambda = \tilde{\lambda} \\ 0 & \text{if } \lambda \neq \tilde{\lambda}. \end{cases} \tag{2.10}$$

Here λ and $\tilde{\lambda}$ refer generically to any λ_k' or λ_k'' and ψ_λ refers to the corresponding C_k or S_k.

Furthermore, the $\{\psi_\lambda\}$ are *complete*, and hence the *generalized Fourier coefficients* of any periodic function $\phi \in L^1[0, 2\pi]$ with weight η

$$\hat{\phi}(\lambda) = \int_0^{2\pi} \eta(\theta)\phi(\theta)\psi_\lambda(\theta)\, d\theta, \tag{2.11}$$

are *uniquely* determined.

In addition each periodic $C^2(\mathbf{R}^1)$ function ϕ can be expanded

$$\phi(\theta) = \sum_{\lambda \geq 0} \hat{\phi}(\lambda)\psi_\lambda(\theta), \tag{2.12}$$

and the series converges *absolutely* and *uniformly*.

In order to obtain asymptotic formulae for the eigenvalues and eigenfunctions for large k, we apply the Liouville transformation

$$u = \Theta\eta^{1/2} = \Theta \exp\left(\frac{1}{2}\int_0^\theta \varepsilon(t)\, dt\right) \tag{2.13}$$

to change the dependent variable in (2.4) and obtain

$$u'' + (\lambda - \phi(\theta))u = 0; \qquad u(0) = u(2\pi),\ u'(0) = u'(2\pi), \tag{2.14}$$

where

$$\phi(\theta) = \frac{1}{4}\varepsilon(\theta)^2 + \frac{1}{2}\varepsilon'(\theta). \tag{2.15}$$

We see ϕ is continuous and 2π-periodic. We conclude that the eigenvalues to (2.4) and (2.14) are identical, the eigenfunctions are connected by (2.13), and the eigenfunctions $\{u_\lambda\}_{\lambda \geq 0}$ to (2.14) are *orthonormal* with *weight* $\equiv 1$. It follows from Theorem 4.2.3 of [**5**] that

$$\sqrt{\lambda_k} = k + o(1) \quad \text{as } k \to \infty, \tag{2.16}$$

where λ_k refers to λ_k' or λ_k''.

Next, using the periodic boundary conditions of (2.14), the methods of [**4**, pp. 334–339] can be adapted virtually without change to conclude that the *orthonormal* eigenfunctions u_λ to (2.14) satisfy the following estimates: $|u_\lambda| \le C$, $|u'_\lambda| \le C\sqrt{\lambda}$, $|u''_\lambda| \le C\lambda$, where C is some positive constant independent of λ and θ. Using (2.13) and the fact that $\eta(\theta)$ is $C^2(\mathbf{R}^1)$, bounded and bounded away from zero, we have

Lemma 2.1. *The orthonormal eigenfunctions* (2.9) *to the eigenvalue problem* (2.7) *satisfy*

$$|\psi_k(\theta)| \le C, \qquad |\psi'_k(\theta)| \le C_k; \qquad |\psi''_k(\theta)| \le Ck^2, \tag{2.17}$$

$k \ge 0$, *where C is a constant independent of k and θ, and ψ_k refers to either $C_k(\theta)$ or $S_k(\theta)$.*

Furthermore, for $k \ge 1$,

$$\begin{aligned} C_k(\theta) &= (\pi\eta(\theta))^{-1/2}\cos k\theta + \mathcal{O}(1/k), \\ S_k(\theta) &= (\pi\eta(\theta))^{-1/2}\sin k\theta + \mathcal{O}(1/k). \end{aligned} \tag{2.18}$$

Next we define the *generalized Poisson kernel* by

$$Q_r(\theta) = \sum_{\lambda \ge 0} r^{\sqrt{\lambda}}\psi_\lambda(\theta) = C_0 + \sum_{k>0}(r^{\sqrt{\lambda'_k}}C_k(\theta) + r^{\sqrt{\lambda''_k}}S_k(\theta)), \tag{2.19}$$

where the first series is a generic representation of the second series. Since the ψ_λ's are uniformly bounded and the λ_k's satisfy (2.16), this series converges absolutely and uniformly on compact subsets of the unit disk by the root test. Furthermore, the series can be twice differentiated with respect to r or θ because of (2.16), Lemma 2.1 and the root test again. Since each $r^{\sqrt{\lambda_k}}\psi_{\lambda_k}(\theta)$ is a solution to the differential equation (2.2) (separation of variables), so is $Q_r(\theta)$.

Lemma 2.2. *The generalized Poisson kernel defined by* (2.19) *is a solution to the differential equation* (2.2) *in the unit disk $r < 1$.*

We now come to the main result of the paper which characterizes *all* solutions to (2.2) in the unit disk in terms of their generalized Fourier series.

Theorem 2.3. *Every real solution $u(r,\theta)$ of* (2.2) *in the punctured unit disk which is bounded in a neighborhood of the origin is continuous at the origin, and can be written*

$$u(r,\theta) = a_0 + \sum_{k>0}(a_k' r^{\sqrt{\lambda'}} C_k(\theta) + a_k'' k r^{\sqrt{\lambda_k''}} S_k(\theta)), \tag{2.20}$$

for some unique set of real constants a_0, a_k', a_k'', $k \geq 1$, satisfying

$$\limsup_{k\to\infty} |a_k|^{1/k} \leq 1, \tag{2.21}$$

where $a_k = a_k' + i a_k''$. Conversely each set of complex constants with property (2.21) *determines a solution through* (2.20).

The proof of the final statement is exactly as the proof of Lemma 2.2.

For the first part, suppose $u(r,\theta)$ is a solution of (2.2) in $0 < r < 1$, bounded near the origin. By results of partial differential equations [**1**, p. 136], since $\varepsilon(\theta)$ is $C^1(\mathbf{R}^1)$, $\varepsilon(\theta)$ is Hölder continuous of order α for each $0 < \alpha < 1$, so $u(r,\theta)$ has Hölder continuous second partial derivatives in the punctured disk. If we define the Fourier coefficients of $u(r,\theta) = u_r(\theta)$ as in (2.11),

$$\hat{u}_r(\lambda) = \int_0^{2\pi} \eta(\theta) u(r,\theta) \psi_\lambda(\theta)\, d\theta, \tag{2.22}$$

where as usual λ refers to any λ_k' or λ_k'' and $\psi_\lambda(\theta)$ refers to the corresponding $C_k(\theta)$ or $S_k(\theta)$, then by (2.12)

$$u(r,\theta) = \sum_{\lambda\geq 0} \hat{u}_r(\lambda)\psi_\lambda(\theta), \qquad 0 < r < 1, \tag{2.23}$$

and the series converges absolutely and uniformly in compact subsets of the punctured disk. Hence, if we set

$$Lu = \Delta u + \frac{\varepsilon(\theta)}{r^2} u_\theta \equiv 0,$$

we have

$$\sum_{\lambda\geq 0} L(\hat{u}_r(\lambda)\psi_\lambda(\theta)) = \sum_{\lambda\geq 0}\left[\hat{u}_r''\psi_\lambda + \frac{1}{r}\hat{u}_r'\psi_\lambda + \frac{1}{r^2}\hat{u}_r\psi_\lambda'' + \frac{\varepsilon}{r^2}\hat{u}_r\psi_\lambda'\right] \equiv 0.$$

But $\psi_\lambda'' + \varepsilon\psi_\lambda' = -\lambda\psi_\lambda$, so

$$Lu = \sum_{\lambda\geq 0}\left[\hat{u}_r'' + \frac{1}{r}\hat{u}_r - \frac{1}{r^2}\lambda\hat{u}_r\right]\psi_\lambda \equiv 0.$$

By the uniqueness of the Fourier coefficients, each bracketed quantity is 0, i.e., $\hat{u}_r(\lambda)$ is a solution of the Cauchy-Euler equation (2.3). But from (2.22), $\hat{u}_r(\lambda)$ is bounded near $r = 0$ since $u(r, \theta)$ is. Therefore, $\hat{u}_r(\lambda)$ is a *unique* constant multiple of $r^{\sqrt{\lambda}}$,

$$\hat{u}_r(\lambda) = a_\lambda r^{\sqrt{\lambda}}, \tag{2.24}$$

where $a_\lambda = a_k'$ or a_k'' depending on whether $\lambda = \lambda_k'$ or λ_k''. We have proved (2.20), that $u(r, \theta)$ is continuous at the origin, and we know the series in (2.20) converges absolutely for each $0 < r < 1$ and each θ. All that is left is to show the constants in (2.20) satisfy (2.21). Again, from the absolute convergence and the root test, we have

$$\limsup_{k\to\infty} |a_k' r^{\sqrt{\lambda_k'}} C_k(\theta) + a_k'' r^{\sqrt{\lambda_k''}} S_k(\theta)|^{1/k} \leq 1, \tag{2.25}$$

for each $0 < r < 1$, θ arbitrary. Using (2.18) we first set $\theta = 0$ and then set $\theta = \pi/2$. The result from (2.25) is then, on the one hand, $r(\limsup_{k\to\infty} |a_k'|^{1/k}) \leq 1$, for each $0 < r < 1$, and therefore $\limsup_{k\to\infty} |a_k'|^{1/k} \leq 1$, with a corresponding result on the other hand for a_k''. So if $a_k = a_k' + ia_k''$, $|a_k|^{1/k} \leq 2^{1/k}\max(|a_k'|^{1/k}, |a_k''|^{1/k})$, and (2.21) follows.

3. Distributional boundary values. In this section we examine the behavior of solutions $u(r, \theta)$ to (2.2) near the boundary of the unit disk, in a distributional sense. That is to say, if we view $u(r, \theta) = u_r(\theta)$, $0 < r < 1$, as a *family* of functions with Hölder continuous second derivatives on the boundary of the unit disk, we want to decide whether or not $u_r(\theta)$ converges in some distributional sense as $r \to 1$, and if so, to what distribution does it converge?

The answer lies in Theorem 2.3 which gives a one-to-one correspondence between solutions to (2.2) and their generalized Fourier coefficients, namely real sequences $\{a_0, a_k', a_k''\}$, $k > 0$, satisfying (2.21).

Because the coefficient sequences satisfy (2.21), we can utilize all the results of G. Johnson [**8**] where he identifies the *distributional boundary values* of solutions to Laplace's equation, variously called *analytic functionals* or *hyperfunctions*, as *sequences* satisfying (2.21).

We begin with the description of the test space $\mathcal{G}$ of functions defined on the unit circle. The space of distributions which comprise the boundary values of real solutions to (2.2) will then be the dual $\mathcal{G}'$, the real continuous linear functionals defined on $\mathcal{G}$.

In order to avoid notational confusion, let us denote the *generalized Fourier coefficients* of a function $\phi \in L^1[0, 2\pi]$ by

$$\begin{aligned} \hat{\phi}_C(k) &= \int_0^{2\pi} \eta(\theta)\phi(\theta)C_k(\theta)\, d\theta, \qquad k \geq 0, \\ \hat{\phi}_S(k) &= \int_0^{2\pi} \eta(\theta)\phi(\theta)S_k(\theta)\, d\theta, \qquad k \geq 1. \end{aligned} \tag{3.1}$$

Set $\hat{\phi}_S(0) = 0$ and write

$$\hat{\phi}(k) = \hat{\phi}_C(k) + i\hat{\phi}_S(k), \qquad k \geq 0. \tag{3.2}$$

Define $\mathcal{G}$ to be the linear space of all real functions ϕ

$$\phi(\theta) = a_0 + \sum_{k \geq 1} (a_k' C_k(\theta) + a_k'' S_k(\theta)), \tag{3.3}$$

where $a_k = a_k' + i a_k''$ satisfies

$$\limsup_{k \to \infty} |a_k|^{1/k} < 1. \tag{3.4}$$

The space $\mathcal{G}$ depends of course on the perturbation coefficient function $\varepsilon(\theta)$ in (2.2), and therefore on the weight function $\eta(\theta)$ given by (2.6). If $\varepsilon \equiv 0$, then $\mathcal{G}$ is just (the real subspace of) the space of test functions $\mathcal{H}$ described in [**8**], and the coefficients a_k' and a_k'' in (3.3) are the usual Fourier cosine and sine coefficients. Of course, the coefficients in (3.3) are the generalized Fourier coefficients of ϕ given in (3.1). Hence, we may identify $\mathcal{G}$ with the set Γ of all *complex* sequences $\{a_k = a_k' + i a_k''$, a_0 real$\}$, satisfying (3.4), the correspondence being given by (3.3).

The correspondence is one-to-one since the eigenfunctions (2.9) form a complete orthonormal system.

Following Johnson [**8**] we put a topology on $\mathcal{G}$ (or equivalently Γ) as follows. Denote by $\mathcal{A}$ all real sequences $\alpha = \{\alpha_k\}$, $k \geq 0$, which satisfy

$$\begin{aligned} &\alpha_k \geq \alpha_{k+1} > 0, \qquad k \geq 0, \\ &\alpha_{k+1}/\alpha_k \to 1, \qquad \text{as } k \to \infty. \end{aligned} \tag{3.5}$$

The collection of sets

$$V(\alpha) = \{\theta \in \mathcal{G} : |\hat{\phi}(k)| \leq \alpha_k,\ k \geq 0\}$$

for all $\alpha \in \mathcal{A}$ is a base for the neighborhood system of the origin for the desired topology on $\mathcal{G}$. Johnson [**8**] then proves that Γ, and therefore $\mathcal{G}$, is a locally convex topological vector space which is a nonmetrizable, complete Montel space. He further shows on page 379 that the Fourier series (3.3) for each $\phi \in \mathcal{G}$ converges to ϕ in $\mathcal{G}$. In addition, [**8**, Proposition 5], the collection of sets

$$E(a, \rho) = \{\phi \in \mathcal{G} : |\hat{\phi}(k)| \leq a\rho^k,\ k \geq 0\}$$

for all $a > 0$ and $\rho < 1$ is a fundamental system of bounded sets for $\mathcal{G}$. Each $E(a, \rho)$ is compact.

Denote by $\mathcal{G}'$ the dual space of real continuous functionals on $\mathcal{G}$, and define the (generalized) Fourier coefficients of $f \in \mathcal{G}'$ by

$$\hat{f}_C(k) = \langle f, C_k \rangle, \qquad \hat{f}_S(k) = \langle f, S_k \rangle, \qquad k \geq 0, \tag{3.6}$$

where we set $\hat{f}_S(0) = 0$, and write $\hat{f} = \hat{f}_C + i\hat{f}_S$. We identify an element of $\mathcal{G}'$ with the sequence of its Fourier coefficients as follows:

Theorem 3.1. *If* $f \in \mathcal{G}'$, *then*

$$\limsup_{k \to \infty} |\hat{f}(k)|^{1/k} \leq 1, \tag{3.7}$$

and

$$\langle f, \phi \rangle = \sum_{k \geq 0} (\hat{\phi}_C(k)\hat{f}_C(k) + \hat{\phi}_S(k)\hat{f}_S(k)) = \sum_{k \geq 0} \mathrm{Re}\,(\overline{\hat{\phi}(k)}\hat{f}(k)). \tag{3.8}$$

The series is absolutely convergent. Conversely, any complex sequence $\{a_k = a'_k + ia''_k,\ a_0 \text{ real}\}$ *satisfying* $\limsup |a_k|^{1/k} \leq 1$ *determines a unique* $f \in \mathcal{G}'$ *with* $\hat{f}(k) = a_k$.

This is Theorem 2, page 379 of [**8**], and the proof there goes through without change. The completeness of the eigenfunctions (2.9) is used again to show the uniqueness in the above theorem.

As a corollary we see from (3.4) that $\mathcal{G} \subseteq \mathcal{G}'$. In particular, if $g \in \mathcal{G}$, define $\Lambda_g : \mathcal{G} \to R^1$ by $\langle \Lambda_g, \phi \rangle = \int_0^{2\pi} g\phi\eta\, d\theta$. Then $\hat{\Lambda}_g(k) = (\hat{\Lambda}_g)_C(k) + i(\hat{\Lambda}_g)_S(k) = \langle \Lambda_g, C_k \rangle + i\langle \Lambda_g, S_k \rangle = \int_0^{2\pi} gC_k\eta\, d\theta + i\int_0^{2\pi} gS_k\eta\, d\theta = \hat{g}(k)$. Hence $\hat{g}(k)$ and therefore $\hat{\Lambda}_g(k)$ satisfies (3.4) and therefore (3.7). By Theorem 3.1, $\Lambda_g \in \mathcal{G}'$.

We endow $\mathcal{G}'$ with the strong topology, which is the topology of uniform convergence on the bounded subsets of $\mathcal{G}$. Define a class $\mathcal{B}$ of sequences $\beta = \{\beta_k\}$, $k \geq 0$, which satisfy

$$\begin{aligned} &0 < \beta_k \leq \beta_{k+1}, \qquad k \geq 0 \\ &\beta_{k+1}/\beta_k \to 1, \qquad \text{as } k \to \infty. \end{aligned} \tag{3.9}$$

Lemma 3.2. *The sets*

$$V'(a, \rho) = \{f \in \mathcal{G}' : |\hat{f}(k)| \leq a\rho^k,\ k \geq 0\}$$

for all $\varepsilon > 0$ *and* $\rho > 1$ *form a base for the neighborhood system at the origin for the strong topology on* $\mathcal{G}'$. *The sets*

$$F'(\beta) = \{f \in \mathcal{G}' : |\hat{f}(k)| \leq \beta_k,\ k \geq 0\}$$

for all $\beta \in \mathcal{B}$ *is a fundamental system of bounded sets for* $\mathcal{G}'$. *Finally,* $\mathcal{G}'$ *is a Montel space whose strong dual is* $\mathcal{G}$.

This is Propositions 9 and 10 and Theorem 3 of [**8**]. In addition, since the eigenfunctions (2.9) are complete, they are *total* in $\mathcal{G}$. Hence, [**9**] boundedness in $\mathcal{G}'$ and convergence of Fourier coefficients imply strong convergence. In particular, the Fourier series of each $f \in \mathcal{G}'$ converges to f in $\mathcal{G}'$. Finally, weak convergence in $\mathcal{G}$ or $\mathcal{G}'$ implies strong convergence to the same limit [**9**, p. 370].

Next we define a *convolution* operation in $\mathcal{G}$ and $\mathcal{G}'$ in the usual way by pointwise multiplication of Fourier coefficients:

$$(3.10)\qquad \begin{aligned}(f*g)(\theta) &= \sum_{k\geq 0}(\hat{f}_C(k)\hat{g}_C(k)C_k(\theta)+\hat{f}_S(k)\hat{g}_S(k)S_k(\theta))\\ &= \sum_{\lambda\geq 0}\hat{f}(\lambda)\hat{g}(\lambda)\psi_\lambda(\theta),\end{aligned}$$

the last being a generic representation of the first. The mapping $f \to f*g$ is continuous for each $g \in \mathcal{G}'$. Note that $\mathcal{G}'$ is now a *convolution algebra* with (unique) *identity*

$$(3.11)\qquad \Delta(\theta) = C_0(\theta) + \sum_{k\geq 1}(C_k(\theta)+S_k(\theta)) = \sum_{\lambda\geq 0}\psi_\lambda(\theta).$$

We now recast Theorem 2.3 in the setting of distributions as developed in this section and obtain a generalized Poisson integral representation for all solutions to (2.2).

Theorem 3.3. *A function $u(r,\theta)$ in the unit disk is a solution to (2.2) if and only if there is a generalized function f in $\mathcal{G}'$ such that*

$$(3.12)\qquad u(r,\theta) = u_r(\theta) = (Q_r * f)(\theta)$$

for each $0 < r < 1$. Furthermore, $u_r \to f$ in $\mathcal{G}'$ as $r \to 1$ and consequently f is uniquely determined.

Proof. If $u(r,\theta)$ is a solution to (2.2), then by Theorem 2.3, u can be represented by (2.20) with unique real constants $\{a_0, a_k', a_k''\}$, $k > 0$, satisfying (2.21). Hence, if we define

$$f(\theta) = a_0 + \sum_{k>0}(a_k' C_k(\theta) + a_k'' S_k(\theta)),$$

Theorem 3.1 implies $f \in \mathcal{G}'$, and (2.20) is equivalent to (3.12). Taking $\beta = \{1, 1, \dots\} \in \mathcal{B}$, Lemma 3.2 implies the family of functions $Q_r \in \mathcal{G} \subset \mathcal{G}'$ is *bounded* in $\mathcal{G}'$, since $0 \leq \hat{Q}_r(\lambda_k) = r^{\sqrt{\lambda_k}} \leq 1$. Hence, $Q_r \to \Delta$ in $\mathcal{G}'$ as $r \to 1$. Therefore, $u_r = Q_r * f \to \Delta * f = f$ in $\mathcal{G}'$, by the continuity of convolution. This completes the proof. □

To show the consistency with the classical solution of the Dirichlet problem for (2.2), suppose f is continuous on the boundary of the unit disk and $v(r,\theta) = v_r(\theta)$ is the unique solution to (2.2) in the open disk, continuous in the closed disk and equal to f on the boundary [**7**, p. 176]. Then $v_r \to f$ uniformly as $r \to 1$ and therefore $v_r \to f$ in $\mathcal{G}'$. By the uniqueness part of Theorem 3.3, $v_r \equiv u_r = Q_r * f$.

4. Generalizations. The results of this paper can, of course, be combined with the results of [**12**] to completely solve equation (1.2) in the case

$$\sigma(r,\theta) = \sigma_1(r)\sigma_2(\theta). \tag{4.1}$$

The variables will still separate in this case.

If one defines $\varepsilon_1(r) = r\sigma_1'(r)/\sigma_1(r)$ as in [**12**], and $\varepsilon_2(\theta) = \sigma_2'(\theta)/\sigma_2(\theta)$ as in Section 2, and imposes the conditions on $\varepsilon_1(r)$ given in (2.3a) and (2.3b) of [**12**], and assumes $\varepsilon_2(\theta)$ satisfies (2.1), then the two boundary value problems that result after separating variables in (1.2) are (2.4) for the θ-problem, whereas the r-problem becomes

$$\begin{gathered} R'' + \frac{1+\varepsilon_1(r)}{r}R' - \frac{\lambda_k}{r^2}R = 0, \\ R_k(0) \text{ finite}, \ R_k(1) = 1, \ k \geq 0, \end{gathered} \tag{4.2}$$

where λ_k refers to λ_k' or λ_k''. Because of (2.16), the results of [**12**] carry through, namely, for each $k > 0$ there exists a unique pair of solutions $R_{\lambda_k'}(r)$ and $R_{\lambda_k''}(r)$ to (4.2) satisfying the estimates (2.4)–(2.6) of [**12**].

In summary, every solution to

$$\Delta u + \frac{\varepsilon_1(r)}{r}\frac{\partial u}{\partial r} + \frac{\varepsilon_2(r)}{r^2}\frac{\partial u}{\partial \theta} = 0 \tag{4.3}$$

in the punctured unit disk which is bounded near the origin, is continuous at the origin and can be represented by

$$u(r,\theta) = a_0 + \sum_{k>0}(a_k' R_{\lambda_k'}(r)C_k(\theta) + a_k'' R_{\lambda_k''}(r)S_k(\theta)). \tag{4.4}$$

The constants are unique and satisfy (2.21) as before.

The distributional boundary values remain $\mathcal{G}'$ in this case, and Theorem 3.3 is still valid where, of course, the new generalized Poisson kernel $Q_r(\theta)$ is given by (2.19) with $r^{\sqrt{\lambda_k}}$ replaced by $R_{\lambda_k}(r)$.

The results of this paper, as well as [**12**], can be extended to n-dimensions. In order to accomplish this one first needs to extend Johnson's results [**8**] on harmonic functions to higher dimensions. Much of the necessary background for this task is already in print from 1966. What appears to be missing is a definitive representation of harmonic functions in the n-ball using vector coefficients which satisfy some form of (2.21). In a forthcoming paper the present authors will present these details, in addition to solving the perturbed cases, both in the radial and angular components.

REFERENCES

1. L. Bers, F. John and M. Schechter, *Partial differential equations*, Lectures in Applied Mathematics, Vol. III, Interscience, New York, 1964.

2. B.M. Budak, A.A. Samarskii and A.N. Tikhonov, *A collection of problems on mathematical physics*, Macmillan, New York, 1964.

3. E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw Hill, NY, 1955.

4. R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. II, Interscience, New York, 1966.

5. M.S.P. Eastham, *The spectral theory of periodic differential equations*, Hafner Press, New York, 1973.

6. I.M. Gelfand and G.E. Silov, *Quelques applications de la théorie des fonctions généralisées*, J. Math. Pures Appl. **35** (1956), 383–413.

7. G. Hellwig, *Partial differential equations*, Blaisdell, New York, 1960.

8. G. Johnson, *Harmonic functions on the unit disk* I, Illinois J. Math. **12** (1968), 143–154.

9. G. Köthe, *Topological vector spaces* I, Springer-Verlag, New York, 1969.

10. J.L. Lions and E. Magenes, *Problèmes aux limites non homogénes* (VII), Ann. Mat. Pura Appl. **63** (1963), 201–224.

11. M. Sato, *Theory of hyperfunctions* I *and* II, J. Fac. Sci. Univ. Tokyo **8** (1959–60), 139–193, 387–437.

12. P.O. Staples and J.A. Kelingos, *A characterization of solutions to a perturbed Laplace equation*, Illinois J. Math. **22** (1978), 208–216.

1326 SC., Vanderbilt University, Nashville, TN 37240