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ON TWO EXTREMAL PROBLEMS RELATED TO UNIVALENT FUNCTIONS

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ABSTRACT. For an integrable $\Lambda : [0,1] \to \mathbf{R}$, nonnegative on $(0,1)$, and $f \in \mathcal{S}$, the class of normalized univalent functions in the unit disk $\mathbf{D}$, we are interested in the functional

$$L_\Lambda(f) := \inf_{z\in\mathbf{D}} \int_0^1 \Lambda(t)\left(\operatorname{Re}\frac{f(tz)}{tz} - \frac{1}{(1+t)^2}\right) dt,$$

and, in particular, in $L_\Lambda(\mathcal{S}) := \inf_{f\in\mathcal{S}} L_\Lambda(f)$. Note that $L_\Lambda(\mathcal{S}) \le 0$ for every Λ. We show that $L_\Lambda(f) \ge 0$ for f close-to-convex and a set of functions Λ containing $\Lambda_c(t) := (1-t^c)/c$, $c \in (-1,2]$. This result turns out to be instrumental for our solution of the following problem: find the best (least) bound β_c so that for each $g \in \mathcal{H}(\mathbf{D})$ with $g(0)=0$, $g'(0)=1$, $\operatorname{Re}[e^{i\alpha}(g'(z)-\beta)] > 0$ in $\mathbf{D}$ with $\beta \ge \beta_c$ the function

$$(c+1)\int_0^1 t^{c-1} g(tz)\, dt, \qquad z \in \mathbf{D},$$

is starlike univalent in $\mathbf{D}$. Weaker bounds for β_c have been obtained by a number of authors (cf. Ali [**1**], Nunokawa [**6**]). We are using the duality principle for Hadamard products to obtain our results.

1. Introduction and statement of the results. Let $\mathcal{S}$ denote the set of univalent functions f in the unit disk $\mathbf{D}$, normalized by $f(0)=0$, $f'(0)=1$. The Koebe distortion theorem then states that, for $f \in \mathcal{S}$,

$$\frac{1}{(1+|z|)^2} \le \left|\frac{f(z)}{z}\right| \le \frac{1}{(1-|z|)^2}, \qquad |z| < 1.$$

Generally, however, we do not have

$$\frac{1}{(1+|z|)^2} \le \operatorname{Re}\frac{f(z)}{z}, \tag{1}$$

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in $\mathcal{S}$, not even for the Koebe function. Grunsky's [**4**] evaluation of the range of the functional $f(z)/z$ over $\mathcal{S}$ for z fixed shows that (1) holds for $0 < |z| = r \leq (e-1)/(e+1) = .462\ldots$ and $f \in \mathcal{S}$ arbitrary, but for no larger r, and that

$$\min_{f\in\mathcal{S}} \operatorname{Re}\frac{f(z)}{z} = \frac{-1}{(1-|z|)^2}(1+o(1)), \qquad |z| \to 1.$$

For an integrable function $\Lambda : [0,1] \to \mathbf{R}$, which we assume to be positive in $(0,1)$, we define

$$L_\Lambda(f) := \inf_{z\in\mathbf{D}} \int_0^1 \Lambda(t)\left(\operatorname{Re}\frac{f(tz)}{tz} - \frac{1}{(1+t)^2}\right) dt, \qquad f \in \mathcal{S},$$

and

$$L_\Lambda(\mathcal{S}) := \inf_{f\in\mathcal{S}} L_\Lambda(f).$$

Since obviously $L_\Lambda(F) \leq 0$ for the Koebe function F, we have $L_\Lambda(\mathcal{S}) \leq 0$ for every admissible weight function Λ. It is therefore of interest to know whether there are such weight functions for which $L_\Lambda(\mathcal{S}) = 0$, and, possibly, to characterize them.

We cannot solve this problem for the whole of $\mathcal{S}$. For the important subclass of close-to-convex functions, however, we can find a set of weight functions with the desired property.

A function $f \in \mathcal{S}$ is called *starlike* ($f \in \mathcal{S}^*$) if $f(\mathbf{D})$ is a starlike domain with respect to the origin, and $f \in \mathcal{S}$ is called close-to-convex ($f \in \mathcal{C}$), if there exists $g \in \mathcal{S}^*$ and $\alpha \in \mathbf{R}$ such that

$$\operatorname{Re} e^{i\alpha}\frac{zf'(z)}{g(z)} > 0, \qquad z \in \mathbf{D}.$$

We refer to Duren [**3**] for some basic results concerning these function classes. Our main result is

Theorem 1. *Let Λ be integrable over $[0,1]$ and positive on $(0,1)$. If*

$$\frac{\Lambda(t)}{1-t^2} \quad \textit{is decreasing on } (0,1), \tag{2}$$

then $L_\Lambda(\mathcal{C}) = 0$.

Note that $L_\Lambda(\mathcal{C}) = 0$ implies $L_\Lambda(\overline{\mathrm{co}}(\mathcal{C})) = 0$, where $\overline{\mathrm{co}}$ stands for the closed convex hull of a set. It is well known (see [**3**, Chapter 9]) that $\mathcal{S}_{\mathbf{R}}$, the set of functions in $\mathcal{S}$ with real Taylor coefficients about the origin, is contained in $\overline{\mathrm{co}}(\mathcal{C})$, so that Theorem 1 holds with $\mathcal{C}$ replaced by $\mathcal{S}_{\mathbf{R}}$ as well. There are reasons to believe that it in fact holds for the whole of $\mathcal{S}$.

The functions

$$\Lambda_c(t) := \begin{cases} (1-t^c)/c, & -1 < c \le 2,\ c \ne 0, \\ \log(1/t), & c = 0, \end{cases} \tag{3}$$

satisfy the condition (2). Numerical calculations show that for $\Lambda = \Lambda_c$ with c large (cf. $c > 7$) we have $L_\Lambda(\mathcal{C}) < 0$.

To describe the second extremal problem we are dealing with, we introduce

$$\begin{aligned} \mathcal{P}_\beta := \{f \in \mathcal{H}(\mathbf{D}) : f(0) = 0, f'(0) = 1,\ \exists \alpha \in \mathbf{R} \text{ s.t.} \\ \mathrm{Re}\,[e^{i\alpha}(f'(z) - \beta)] > 0 \text{ in } \mathbf{D}\}. \end{aligned} \tag{4}$$

Let $\lambda : [0,1] \to \mathbf{R}$ be nonnegative with $\int_0^1 \lambda(t)\,dt = 1$. On $\mathcal{A}_1 := \{f \in \mathcal{H}(\mathbf{D}) : f(0) = 0, f'(0) = 1\}$, we define the operator

$$V_\lambda(f) := \int_0^1 \lambda(t) \frac{f(tz)}{t}\,dt,$$

and the number $\beta(\lambda) < 1$ by

$$\frac{\beta(\lambda)}{1 - \beta(\lambda)} = -\int_0^1 \lambda(t) \frac{1-t}{1+t}\,dt.$$

Theorem 2. *For* $\beta = \beta(\lambda)$, *we have*

$$V_\lambda(\mathcal{P}_\beta) \subset \mathcal{S}. \tag{5}$$

If, in addition, $\Lambda(t) := \int_t^1 \lambda(s)\, ds/s$ *satisfies* $t\Lambda(t) \to 0$ *for* $t \to 0+$, *then*

$$V_\lambda(\mathcal{P}_\beta) \subset \mathcal{S}^* \iff L_\Lambda(\mathcal{C}) = 0. \tag{6}$$

For $\beta < \beta(\lambda)$ *there exists* $f \in \mathcal{P}_\beta$ *with* $V_\lambda(f)$ *not even locally univalent in* $\mathbf{D}$.

For $\lambda_c(t) := (c+1)t^c$, $c > -1$, the corresponding functions Λ are $(c+1)\Lambda_c$ (compare (3)). For easier reference we restate Theorem 2 for this special case.

Corollary 1. *Let* $c > -1$ *and* β_c *be defined by*

$$\frac{\beta_c}{1-\beta_c} = -(c+1)\int_0^1 t^c \frac{1-t}{1+t}\, dt, .$$

Then, for $f \in \mathcal{P}_{\beta_c}$, *and*

$$F_c(z) := (c+1)\int_0^1 t^{c-1} f(tz)\, dt,$$

we have $F_c \in \mathcal{S}$. *For* $-1 < c \leq 2$, *we even have* $F_c \in \mathcal{S}^*$. *Neither of the two conclusions can be drawn for any* $\beta < \beta_c$.

In particular, we have

$$\beta_0 = \frac{1-2\log(2)}{2-2\log(2)} = -.629\ldots, \qquad \beta_1 = \frac{3-4\log(2)}{2-4\log(2)} = -.294\ldots,$$
$$\beta_2 = \frac{4-6\log(2)}{5-6\log(2)} = -.188\ldots.$$

The problem discussed in Corollary 1 has been studied by many authors, compare [**1, 9, 10, 11, 7, 5, 6**], mainly, however, in the context of the smaller class

$$\mathcal{P}_{0,\beta} := \{f \in \mathcal{H}(\mathbf{D}) : f(0)=0,\ f'(0)=1,\ \mathrm{Re}\,(f'(z)-\beta)>0 \text{ in } \mathbf{D}\}$$

instead of $\mathcal{P}_\beta$. It can be shown that β_c, as defined above, is also the sharp bound for the corresponding problem in $\mathcal{P}_{0,\beta}$. In this class the best previous results have been $\beta_0 \le -.273$ by Ali [**1**] and $\beta_1 \le -.0175$ by Nunokawa [**6**]. In all these previous papers the method of 'differential-subordination' has been applied which does not give very sharp results. here. For problems of this kind, the duality principle for Hadamard products (compare [**8**]) usually produces sharp estimates and seems to be the more adequate tool.

The problem concerning the starlikeness of F_c remains open for $c > 2$ since the calculations become more complicated. For large c, starlikeness and univalence will not coincide anymore, at least as far as $\mathcal{P}_\beta$ is concerned. It seems to be worthwhile to observe that F_c satisfies the differential equation

$$zF_c'(z) + cF_c(z) = (1+c)f(z).$$

The interaction of f and F_c in terms of geometric function theory has been studied on several occasions. We refer to [**2**].

2. Proof of Theorem 1. It is well known (see [**8**, Chapter 1]) that the extreme points of the closed convex hull of $\mathcal{C}$ are among the functions $h_T(xz)/x$, $|x| = 1$, where

$$h_T(z) := \frac{iT\dfrac{z}{1-z} + \dfrac{z}{(1-z)^2}}{1+iT}, \qquad T \in \mathbf{R}.$$

It is therefore clear that we only need to prove

$$\int_0^1 \Lambda(t)\left\{\operatorname{Re}\frac{h_T(tz)}{tz} - \frac{1}{(1+t)^2}\right\} dt \ge 0, \qquad z \in \mathbf{D}, T \in \mathbf{R}. \tag{7}$$

We first show that the left hand side of (7) is bounded from below. We write

$$\begin{aligned}\operatorname{Re}\frac{h_T(tz)}{tz} &= \operatorname{Re}\left\{\frac{iT}{1+iT}\frac{1}{1-tz}\right\} \\ &\quad + \frac{1}{1+T^2}\operatorname{Re}\frac{1}{(1-tz)^2} + \frac{T}{1+T^2}\operatorname{Im}\frac{1}{(1-tz)^2} \\ &= H_1 + H_2 + H_3.\end{aligned}$$

Then

$$\begin{aligned}\left|\int_0^1 \Lambda(t)H_1\,dt\right| &\le \int_0^1 \frac{\Lambda(t)}{1-t}\,dt\\ &\le \int_0^{1/2} 2\Lambda(t)\,dt + \int_{1/2}^1 (1+t)\frac{\Lambda(t)}{1-t^2}\,dt\\ &\le 2\int_0^1 \Lambda(t)\,dt + 2\Lambda\left(\frac{1}{2}\right)\\ &< \infty.\end{aligned}$$

Next we note that Λ is decreasing, and that

$$\int_0^1 \frac{\Lambda(t)}{(1-tz)^2}\,dt = -\int_0^1 \frac{t}{1-tz}\,d\Lambda(t).$$

Hence

$$\int_0^1 \Lambda(t)H_2\,dt = \frac{-1}{1+T^2}\int_0^1 \operatorname{Re}\frac{t}{1-tz}\,d\Lambda(t) \ge 0.$$

Finally,

$$\begin{aligned}\left|\int_0^1 \Lambda(t)H_3\,dt\right| &\le \int_0^1 \frac{\Lambda(t)}{1-t^2}\left|\operatorname{Im}\frac{1-t^2}{(1-zt)^2}\right|dt\\ &\le M_1 + M_2\int_0^1 \left|\operatorname{Im}\frac{1-t^2}{(1-zt)^2}\right|dt\\ &\le M_1 + 4M_2\int_0^1 \left|\operatorname{Im}\frac{1}{1-tz}\right|dt\\ &\le M_1 + 2\pi M_2\\ &< \infty.\end{aligned}$$

The existence of this lower bound and the minimum principle for harmonic functions now permits a reduction of the proof of (7) to the boundary cases $|z| = 1$, $z \neq 1$. A minimization with respect to T gives

$$\operatorname{Re}\frac{h_T(tz)}{tz} \ge \frac{1}{2}\operatorname{Re}\left\{\frac{2-tz}{(1-tz)^2}\right\} - \frac{1}{2}\frac{t}{|1-tz|^2}, \qquad 0\le t\le 1,\ |z|=1,\ z\neq 1.$$

Taking this into account, we need to prove that

$$\int_0^1 \Lambda(t)\left[\mathrm{Re}\left\{\frac{2-tz}{(1-tz)^2}\right\}-\frac{t}{|1-tz|^2}-\frac{2}{(1+t)^2}\right]\geq 0, \qquad |z|=1,\ z\neq 1.$$

Here we have equality in $z=-1$. Using this fact, an evaluation (using $y=\mathrm{Re}\,z$) yields the equivalent condition:

$$H_\Lambda(y):=\int_0^1 t\Lambda(t)\frac{3-4(1+y)t+2(2y-1)t^2+4(y-1)t^3-t^4}{(1+t^2-2yt)^2(1+t)^2}\,dt\geq 0,$$

for $-1\leq y<1$. $H_\Lambda(y)$ has an expansion

$$H_\Lambda(y)=\sum_{k=0}^{\infty}H_{k,\Lambda}(1+y)^k, \qquad |1+y|<2,$$

and it is our aim to prove that all coefficients in this series are nonnegative. A simple calculation shows that $H_{k,\Lambda}$ is a positive multiple of

$$\tilde{H}_{k,\Lambda}:=\int_0^1 \Lambda(t)s_k(t)\,dt,$$

where

$$s_k(t):=\frac{t^{k+1}}{(1+t)^{2k+4}}\left(1-2t+\frac{k-1}{k+3}t^2\right).$$

Note that $s_k(t)>0$ if $0\leq t<t_k$ and $s_k(t)<0$ if $t_k<t\leq 1$, where t_k is the unique zero of s_k in (0,1). Assume now that

$$\tilde{H}_k:=\int_0^1(1-t^2)s_k(t)\,dt>0$$

and define

$$\tilde{\Lambda}(t):=\Lambda(t)-\frac{\Lambda(t_k)}{1-t_k^2}(1-t^2).$$

Then, because of our condition (2), $\tilde{\Lambda}$ is of the same sign as s_k in (0,1). This shows that

$$0\leq\int_0^1\tilde{\Lambda}(t)s_k(t)\,dt=\tilde{H}_{k,\Lambda}-\frac{\Lambda(t_k)}{1-t_k^2}\tilde{H}_k,$$

which clearly implies $\tilde{H}_{k,\Lambda} \geq 0$. What remains to show is that $\tilde{H}_k > 0$. An immediate calculation gives

$$\begin{aligned}
\tilde{H}_0 &= \frac{25}{6} - 6\log(2) = 0.0077\ldots, \\
\tilde{H}_1 &= \frac{-133}{96} + 2\log(2) = 0.00087\ldots, \\
\tilde{H}_2 &= \frac{111}{800} - \frac{1}{5}\log(2) = 0.00012\ldots, \\
\tilde{H}_3 &= \frac{1}{53760} = 0.000018\ldots,
\end{aligned}$$

and, for $k \geq 3$, we find the explicit representation

$$\tilde{H}_k = \frac{18 + 13k + k^2 - (9 + 12k + 3k^2)B(1/2, k+1)}{2^{2k}(k+3)(k+1)k(k-1)(k-2)},$$

where $B(z, w)$ denotes the Beta function. Thus, for each $k \geq 3$ the relation $\tilde{H}_k > 0$ is equivalent to

$$q_k := \frac{9 + 12k + 3k^2}{18 + 13k + k^2} B\left(\frac{1}{2}, k+1\right) < 1.$$

Calculation yields

$$\frac{q_{k+1}}{q_k} = 1 - \frac{k(k^2 - 1)}{(k+3)(2k+3)(32 + 15k + k^2)} < 1$$

and therefore

$$q_{k+1} < q_k < \cdots < q_3.$$

But $q_3 < 1$ since $\tilde{H}_3 > 0$, and we are done. □

3. Proof of Theorem 2. Let $f \in \mathcal{P}_\beta$ and set $F = V_\lambda(f)$. We then have

$$\begin{aligned}
F'(z) &= f' * \int_0^1 \frac{\lambda(t)}{1 - tz}\, dt \\
&= g * \left(\beta + (1 - \beta)\int_0^1 \frac{\lambda(t)}{1 - tz}\, dt\right),
\end{aligned} \tag{8}$$

where $g = (f' - \beta)/(1 - \beta)$. Note that $f \in \mathcal{P}_\beta$ if and only if there exists

$$\alpha = \alpha(g) \in \mathbf{R} : \operatorname{Re} e^{i\alpha} g(z) > 0, \qquad z \in \mathbf{D}.$$

It now follows from the duality principle [**8**, p. 23] that $F'(z) \neq 0$ for all admissible F and $z \in \mathbf{D}$ if and only if

$$\operatorname{Re}\left(\beta + (1-\beta)\int_0^1 \frac{\lambda(t)}{1-tz}\,dt\right) > \frac{1}{2}, \qquad z \in \mathbf{D}. \tag{9}$$

It is easily seen that, due to our assumptions on λ, this latter relation holds if and only if $\beta \geq \beta(\lambda)$. On the other hand, it now follows from (8) and (9) that for these β the set $F'(\mathbf{D})$ is contained in a half plane not containing the origin, which implies the univalence of F. For $\beta < \beta(\lambda)$ the condition (9) is not satisfied, and then one can find an $f \in \mathcal{P}_\beta$ such that F is not even locally univalent in $\mathbf{D}$. This completes the proof of the first part of Theorem 2.

As far as starlikeness is concerned, the duality principle [**8**, Corollary 1.1, Theorem 1.6], applied to the functional $F(z)/(zF'(z))$, shows that we only have to establish the starlikeness of $F(z) = V_\lambda(f)$ where

$$f'(z) = (1-\beta(\lambda))\frac{1-xz}{1-yz} + \beta(\lambda), \qquad |x| \leq 1, |y| \leq 1.$$

Furthermore, it is well known (and easily verified) that $G \in \mathcal{A}_1$ is in $\mathcal{S}^*$ if and only if

$$\frac{1}{z}(G * h_T)(z) \neq 0, \qquad T \in \mathbf{R}, z \in \mathbf{D},$$

where h_T is as above. We have

$$\begin{aligned} 0 \neq \frac{1}{z}(F * h_T)(z) &= \int_0^1 \lambda(t)\frac{1}{1-tz}\,dt \\ &\quad * \left((1-\beta(\lambda))\frac{1}{z}\int_0^z \frac{1-xw}{1-yw}\,dw + \beta(\lambda)\right) * \frac{h_T(z)}{z} \\ &= (1-\beta(\lambda))\int_0^1 \lambda(t)\left(\frac{h_T(tz)}{tz} + \frac{\beta(\lambda)}{1-\beta(\lambda)}\right)dt \\ &\quad * \frac{1}{z}\int_0^z \frac{1-xw}{1-yw}\,dw \\ &= (1-\beta(\lambda))\int_0^1 \lambda(t)\frac{1}{z}\int_0^z \left(\frac{h_T(tw)}{tw} - \frac{1-t}{1+t}\right)dw\,dt \\ &\quad * \frac{1-xz}{1-yz}. \end{aligned}$$

The same conclusion as above, using [**8**, p. 23], shows that this condition holds true if and only if

$$\mathrm{Re}\,(1-\beta(\lambda))\int_0^1 \lambda(t)\frac{1}{z}\int_0^z \left(\frac{h_T(tw)}{tw}-\frac{1-t}{1+t}\right) dw\,dt > \frac{1}{2}, \qquad z \in \mathbf{D},$$

or, equivalently,

$$\mathrm{Re}\int_0^1 \frac{\lambda(t)}{t}\frac{1}{z}\int_0^z \left(\frac{h_T(tw)}{w}-\frac{t}{1+t}\right) dw\,dt > 0, \qquad z \in \mathbf{D}.$$

An integration by parts yields another equivalent formulation, namely (7), and this holds if and only if $L_\Lambda(\mathcal{C}) = 0$ as deducted in the proof of Theorem 1. □

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