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THE FILLING SCHEME IN ARCHIMEDEAN RIESZ SPACES

RADU ZAHAROPOL

ABSTRACT. In the paper we extend the filling scheme as formulated by Akcoglu and Chacon to a class of Archimedean Riesz spaces; we then apply the filling scheme (in our more general setting) in order to prove two ratio ergodic theorems.

The Archimedean Riesz spaces under consideration are not necessarily spaces of classes of equivalence of measurable functions; therefore, in our approach we have to avoid measure theoretical considerations; we do so by using notions and arguments which we introduced in previous papers.

1. Introduction. Our goal here is to extend the filling scheme as defined by Akcoglu and Chacon in their paper [**1**] to a class of Archimedean Riesz spaces, and to use the scheme in order to prove two ratio ergodic theorems similar to an extension of the Ornstein ratio ergodic theorem [**6**, Theorem 1.1] which we obtained in [**9**] (the Ornstein ratio ergodic theorem is an extension of the famous Chacon-Ornstein theorem [**3**]; for a description of the evolution of the topic, as well as for additional references see Krengel's book [**4**] and our paper [**9**]).

The terminology used in this paper can be found in the books of Aliprantis and Burkinshaw [**2**], Luxemburg and Zaanen [**5**], Schaefer [**7**], and in our papers [**8, 9**].

Besides the Introduction, the paper has three more sections. Section 2 (the next section) has a preliminary character; the section contains several results of a rather general nature which will be used later on. In Section 3 we extend the filling scheme to a class of Archimedean Riesz spaces. Finally, in the last section (Section 4) we apply the results obtained in the previous sections in order to prove two ratio ergodic theorems.

We will now describe the mathematical setting and the results of the paper.

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Let E be a Riesz space, let $T : E \to E$ be a positive linear operator (unless the contrary is explicitly stated, all the operators considered in this paper are assumed to be linear), and let H be a Riesz subspace of E. Let $u \in E$, $u \geq 0$, and let $\omega \in E$. We say that ω is an H-modification of u if $\omega \geq 0$ and if $\omega = u - v + Tv$ for some $v \in H$, $v \geq 0$.

Assume now that E is an Archimedean Riesz space, and let $\tilde{E}$ be the Dedekind completion of E. Thus, we may think of E as being a Riesz subspace of $\tilde{E}$ (we will be doing that throughout the paper without stating it explicitly every time).

Let $\tilde{E}'$ be the order dual of $\tilde{E}$, and assume that $\tilde{E}'$ separates the points of $\tilde{E}$. Let $\tilde{E}'_n$ be the order continuous dual of $\tilde{E}$ (that is, $\tilde{E}'_n$ is the projection band in $\tilde{E}'$ of all order continuous linear functionals on $\tilde{E}$).

Let $g \in E$, $g \geq 0$, and let B be the (projection) band in $\tilde{E}$ generated by the set $\{T^n g \mid n \in \mathbf{N} \cup \{0\}\}$. Set $\Gamma(B) = \{x \in \tilde{E}'_n \mid$ the carrier of x (in $\tilde{E}$) is included in $B\}$. We will prove in the next section (Section 2) that $\Gamma(B)$ is a projection band in $\tilde{E}'$.

Given $u \in E$, we will denote by H_u the ideal in E generated by the set $\{T^n u \mid n \in \mathbf{N} \cup \{0\}\}$.

Let $f \in E$, $f \geq 0$. For every $z \in \Gamma(B)$, $z \geq 0$, set $a_z = \sup\{\langle \bar{f}, z\rangle \mid \bar{f}$ is an H_{f+g}-modification of $f\}$, and $b_z = \sup\{\langle \bar{f}, z\rangle \mid \bar{f}$ is an H_f-modification of $f\}$.

Let Δ be a projection band in $\tilde{E}'$, $\Delta \subseteq \Gamma(B)$, and set $\Omega_\infty(f, T, \Delta) = \{x \in \Delta \mid a_y = 0$ or $+\infty$ for every $y \in \Delta$, $0 \leq y \leq |x|\}$; $\Sigma_\infty(f, T, \Delta) = \{x \in \Delta \mid b_y = 0$ or $+\infty$ for every $y \in \Delta$, $0 \leq y \leq |x|\}$. It will be shown in Section 2 that $\Omega_\infty(f, T, \Delta)$ and $\Sigma_\infty(f, T, \Delta)$ are projection bands in $\tilde{E}'$.

Now let $u, v \in E$, $u \geq 0$, $v \geq 0$, and let $n \in \mathbf{N} \cup \{0\}$. We will use the notation $u \overset{n}{\to} v$ as defined in [**1**]. Thus, $u \overset{0}{\to} v$ if and only if $u = v$; $u \overset{1}{\to} v$ whenever there exist r, $s \in E$, $r \geq 0$, $s \geq 0$ such that $u = r + s$, and $v = r + Ts$; if $n \geq 2$, then $u \overset{n}{\to} v$ whenever there exists $w \in E$, $w \geq 0$ such that $u \overset{n-1}{\to} w$ and $w \overset{1}{\to} v$.

Let $u \in E$, $u \geq 0$, and let $x \in \tilde{E}'$, $x \geq 0$. Set $\psi_x^n u = \sup_{u \overset{n}{\to} v} \langle v, x\rangle$ for every $n \in \mathbf{N} \cup \{0\}$. The sequence $(\psi_x^n u)_{n \in \mathbf{N} \cup \{0\}}$ is monotonic nondecreasing; therefore, $\lim_{n\to+\infty}\psi_x^n u$ exists. Set $\psi_x u = \lim_{n\to+\infty}\psi_x^n u$.

For every $w \in \tilde{E}$, let $B(w)$ be the (principal projection) band in $\tilde{E}$ generated by the singleton $\{w\}$.

Let $\bar{u}, \bar{v} \in E$, $\bar{u} \geq 0$, $\bar{v} \geq 0$, be such that there is no $v \in \tilde{E}$, $v \neq 0$, and no sequence $(\omega_k)_{k\in\mathbf{N}}$ of $H_{\bar{u}+\bar{v}}$-modifications of $\bar{v}$ such that $(\omega_k)_{k\in\mathbf{N}}$ diverges individually to ∞ on $B(v)$. Let $\rho \in \mathbf{R}$, $0 < \rho < 1$, let $u \in \tilde{E}$, $u \geq 0$, and set

$$u_\rho = \limsup_n \left(\left((1-\rho)\left(\sum_{i=0}^n T^i \bar{u}\right) - \left(\sum_{i=0}^n T^i \bar{v}\right)\right)^+ \wedge u \right).$$

Since we assume that $\tilde{E}'$ separates the points of $\tilde{E}$, it follows that we may think of the elements of $\tilde{E}$ as order continuous linear functionals on $\tilde{E}'$; hence, it makes sense to consider the carrier of u_ρ in $\tilde{E}'$. The main result of Section 3 is the following extension of Theorem 0.4 of the paper of Akcoglu and Chacon [**1**]: if x is an element of the carrier of u_ρ in $\tilde{E}'$, $x \in \tilde{E}'_n$, $x \geq 0$, then $\psi_x \bar{u} \geq \psi_x \bar{v}$.

Now let again $f, g \in E$ be such that $f \geq 0$, $g \geq 0$, and set $u_n = \sum_{i=0}^n T^i f$, $v_n = \sum_{i=0}^n T^i g$ for every $n \in \mathbf{N} \cup \{0\}$.

Let B and $\Gamma(B)$ be the projection bands we defined earlier, and let $w \in B$. Then we may think of w as an element of $\tilde{E}''$ ($\tilde{E}''$ being the second order dual of $\tilde{E}$). As an element of $\tilde{E}''$, w is an order continuous linear functional on $\tilde{E}'$. We will denote by $\Gamma(w, g, T)$ the intersection of the carrier (in $\tilde{E}'$) of w with $\Gamma(B)$.

Recall that in our paper [**8**] we proved that if the sequence $((u_n, v_n))_{n\in\mathbf{N}\cup\{0\}}$ does not converge individually on B, then there exists a nonzero (projection) band B_d in $\tilde{E}$, $B_d \subseteq B$ such that B_d has the following two properties:

(a) For any nonzero (projection) band C in $\tilde{E}$, $C \subseteq B_d$, it follows that $((u_n, v_n))_{n\in\mathbf{N}\cup\{0\}}$ does not ratio converge individually on C.

(b) B_d is the largest band in $\tilde{E}$ contained in B which has property (a).

If the sequence $((u_n, v_n))_{n\in\mathbf{N}\cup\{0\}}$ does ratio converge individually on B, then set (as in [**8**]) $B_d = 0$.

We say that g has property $\mathcal{S}$ if for every $u \in B$, $u \neq 0$, there exists $x \in \Gamma(B)$ such that $\langle u, x \rangle \neq 0$.

In Section 4 we will use the results obtained in the previous two sections in order to prove the following two ratio ergodic theorems (in both theorems we assume that g has property $\mathcal{S}$ and that the sequence $((u_n, v_n))_{n\in\mathbf{N}\cup\{0\}}$ does not ratio converge individually on B):

Theorem 1. *Let $u \in B_d$, $u \neq 0$. Then at least one of the following three assertions is true*:

(1) *There exists $v \in B(u)$, $v \geq 0$, $v \neq 0$ such that $\Sigma_\infty(f, T, \Gamma(v, g, T)) = \Gamma(v, g, T)$.*

(2) *There exist $v \in B(u)$, $v \geq 0$, $v \neq 0$ and a sequence $(\omega_n)_{n\in\mathbf{N}}$ of H_{f+g}-modifications of g such that $(\omega_n)_{n\in\mathbf{N}}$ diverges individually to ∞ on $B(v)$.*

(3) *There exist $v \in B(u)$, $v \geq 0$, $v \neq 0$ and a sequence $(\rho_n)_{n\in\mathbf{N}}$ of H_{f+g}-modifications of f such that $(\rho_n)_{n\in\mathbf{N}}$ diverges individually to ∞ on $B(v)$.*

Theorem 2. *Let $u \in B_d$, $u \neq 0$. Then at least one of the following two assertions is true*:

(i) *There exists $v \in B(u)$, $v \geq 0$, $v \neq 0$, such that $\Omega_\infty(f, T, \Gamma(v, g, T)) = \Gamma(v, g, T)$.*

(ii) *There exist $v \in B(u)$, $v \geq 0$, $v \neq 0$ and a sequence $(\omega_n)_{n\in\mathbf{N}}$ of H_{f+g}-modifications of g such that $(\omega_n)_{n\in\mathbf{N}}$ diverges individually to ∞ on $B(v)$.*

Both Theorem 1 and Theorem 2 can also be proved by suitably modifying the proof of Theorem 3 of [**9**]. However, the proofs presented here (which are applications of the filling scheme as discussed in Section 3) are much simpler (note that the proof of Theorem 3 of [**9**] is rather complicated and involves all the results of both papers [**8**] and [**9**]).

2. Preliminaries. In this section we gather several facts which belong to the general theory of Riesz spaces and which are needed in order to prove Theorem 1 and Theorem 2.

Lemma 3. *Let G be an order complete Riesz space, let G' be the* *order dual of G, and let G'_n be the order continuous dual of G. Let D be a (projection) band in G, and set $\Gamma(D) = \{x \in G'_n \mid$ the carrier of x (in G) is included in $D\}$. Then $\Gamma(D)$ is a (projection) band in G'.*

Proof. We first prove that $\Gamma(D)$ is a vector subspace of G'.

Let $x, y \in \Gamma(D)$. Clearly, $x + y \in G'_n$ since $x \in G'_n$ and $y \in G'_n$.

Note that the carrier of $x+y$ is included in D. Indeed, if we assume that the carrier of $x + y$ is not included in D, then it follows that there exists $u \in G$, $u \geq 0$, $u \neq 0$ such that u is in the carrier of $x + y$ while the projection of u on the projection band D is zero. Since u is in the carrier of $x + y$, $u \geq 0$, $u \neq 0$, it follows that $0 < \langle u, |x+y|\rangle \leq \langle u, |x| + |y|\rangle = \langle u, |x|\rangle + \langle u, |y|\rangle$. We obtain that $\langle u, |x|\rangle > 0$ or $\langle u, |y|\rangle > 0$. Assume that $\langle u, |x|\rangle > 0$ (similar arguments hold in the case in which $\langle u, |y|\rangle > 0$). Taking into consideration that G is the order direct sum of the carrier and the null ideal of $|x|$ (since $|x| \in G'_n$), we obtain that the projection of u on the carrier of $|x|$ is a nonzero component $v \in G$ of u. Clearly, the projection of v on D is zero (since $0 \leq v \leq u$). Thus, $v \notin D$; therefore, we have obtained a contradiction since the carrier of $|x|$ is included in D.

It follows that $x + y \in \Gamma(D)$.

Obviously, $\alpha x \in \Gamma(D)$ whenever $\alpha \in \mathbf{R}$ and $x \in \Gamma(D)$. Thus, $\Gamma(D)$ is a vector subspace of G'.

Taking into consideration that $\Gamma(D)$ is clearly a solid set, it follows that $\Gamma(D)$ is an ideal in G'.

In order to complete the proof of the lemma we have to show that the ideal $\Gamma(D)$ is actually a band in G'.

To this end, let $A \subseteq \Gamma(D)$, $A \neq \varnothing$ be such that $\sup A$ exists in G'. We have to prove that $\sup A \in \Gamma(D)$.

Clearly, $\sup A \in G'_n$ since $A \subseteq G'_n$ and since G'_n is a band in G'. Thus we only have to prove that the carrier of $\sup A$ is included in D.

Assume that the carrier of $\sup A$ is not included in D. Then there exists $u \in G$, $u \geq 0$, $u \neq 0$ such that u is in the carrier of $\sup A$ and such that the projection of u on D is zero.

It follows that

$$0 < \langle u, |\sup A|\rangle = \langle u, (\sup A)^+\rangle + \langle u, (\sup A)^-\rangle.$$

We have to study two cases:

(i) $\langle u, (\sup A)^+\rangle > 0$;

(ii) $\langle u, (\sup A)^-\rangle > 0$.

(i) Let $\mathcal{F} = \{y \in G' \mid y = \sup_{x\in F} x^+$ for some finite set $F, F \subseteq A\}$.

Clearly, $\mathcal{F}$ is an upward directed set in G'. Taking into consideration that $(\sup A)^+ = \sup_{x\in A} x^+$, we obtain that $\sup \mathcal{F} = (\sup A)^+$. Accordingly, (using [**7**, Proposition 4.2, p. 72]) it follows that $0 < \langle u, (\sup A)^+\rangle = \langle u, \sup \mathcal{F}\rangle = \sup_{y\in\mathcal{F}}\langle u, y\rangle$. Hence, there exists $y \in \mathcal{F}$, $y = \sup_{x\in F} x^+$ for some finite set F, $F \subseteq A$ such that $\langle u, y\rangle \neq 0$.

Taking into consideration that $0 < \langle u, y\rangle \leq \Sigma_{x\in F}\langle u, x^+\rangle$, we obtain that there exists $x \in F \subseteq A$ such that $\langle u, x^+\rangle \neq 0$.

Let v be the projection of u on the carrier of x^+. Clearly, $v \neq 0$. It is also obvious (since $x^+ \leq |x|$) that v is in the carrier of x. Taking into consideration that $0 \leq v \leq u$, $v \neq 0$ and that the projection of u on D is zero, we obtain that $v \notin D$. We have obtained a contradiction since v is in the carrier of x and the carrier of x is included in D.

(ii) Assume now that $\langle u, (\sup A)^-\rangle > 0$ and let $x \in A$.

Taking into consideration that $(\sup A)^- = (-\sup A)\vee 0 = (\inf\ (-A))\vee 0 = \inf_{z\in A} z^-$, we obtain that $x^- \geq \inf_{z\in A} z^- = (\sup A)^-$; therefore $\langle u, x^-\rangle \neq 0$.

Let v be the projection of u on the carrier of x^-. Then v is a nonzero component of u. It follows that v is in the carrier of x (since v is in the carrier of x^- and since $x^- \leq |x|$). Taking into consideration that the projection of v on D is zero (since $0 \leq v \leq u$ and since the projection of u on D is zero), it follows that $v \notin D$; we have obtained a contradiction since the carrier of x is included in D.

We have therefore proved that in both cases (i) and (ii) we obtain a contradiction. It follows that the carrier of $\sup A$ is included in D. □

We will now discuss a general procedure for constructing projection bands in the order dual of a Riesz space, and its use in the study of the sets $\Omega_\infty(f, T, \Delta)$ and $\Sigma_\infty(f, T, \Delta)$ defined in Introduction.

Let G be a Riesz space, and let G' be the order dual of G. Let G_+ be the positive cone of G, that is, $G_+ = \{x \in G \mid x \geq 0\}$, and let $A \subseteq G_+$.

Set

$$\Lambda_\infty(A) = \{x \in G' \mid \sup_{u \in A} \langle u, y\rangle = 0 \text{ or } +\infty \text{ for every } y \in G', 0 \leq y \leq |x|\}$$

and

$$\Lambda_F(A) = \left\{ x \in G' \;\middle|\; \begin{array}{l} x = 0 \text{ or } x \neq 0 \text{ and for every} \\ y \in G', 0 \leq y \leq |x|, y \neq 0 \text{ there exists} \\ z \in G', 0 \leq z \leq y \text{ such that} \\ 0 < \sup_{u \in A} \langle u, z\rangle < +\infty \end{array} \right\}.$$

Proposition 4. *$\Lambda_\infty(A)$ and $\Lambda_F(A)$ are projection bands in G' and G' is the order direct sum of $\Lambda_\infty(A)$ and $\Lambda_F(A)$.*

Proof. We first prove that $\Lambda_\infty(A)$ is an ideal in G'.

Clearly, $\Lambda_\infty(A)$ is a solid set in G'.

Let $x_1, x_2 \in \Lambda_\infty(A)$. Let $y \in G'$ be such that $0 \leq y \leq |x_1 + x_2|$. Using the decomposition property [**7**, Proposition 1.6, p. 53], we obtain that there exist y_1, $y_2 \in G'$ such that $0 \leq y_1 \leq |x_1|$, $0 \leq y_2 \leq |x_2|$, and $y = y_1 + y_2$. Assume that $\sup_{u \in A}\langle u, y\rangle > 0$. Then $\langle u', y\rangle > 0$ for some $u' \in A$. Since $\langle u', y\rangle = \langle u', y_1\rangle + \langle u', y_2\rangle$, it follows that $\langle u', y_1\rangle > 0$ or $\langle u', y_2\rangle > 0$. Assume that $\langle u', y_1\rangle > 0$ (similar arguments hold if $\langle u', y_2\rangle > 0$). Then $\sup_{u \in A}\langle u, y_1\rangle = +\infty$ since $\Lambda_\infty(A)$ is a solid set, $x_1 \in \Lambda_\infty(A)$, and $0 \leq y_1 \leq |x_1|$. It follows that $\sup_{u \in A}\langle u, y\rangle = +\infty$. Accordingly, $x_1 + x_2 \in \Lambda_\infty(A)$.

Now let $\alpha \in \mathbf{R}$ and $x \in \Lambda_\infty(A)$. If $|\alpha| \leq 1$, then, clearly, $\alpha x \in \Lambda_\infty(A)$ since $\Lambda_\infty(A)$ is a solid set. Assume that $|\alpha| > 1$, and let $y \in G'$, $0 \leq y \leq |\alpha x|$; it follows that $\sup_{u \in A}\langle u, y\rangle = 0$ or $+\infty$ since $0 \leq y/|\alpha| \leq |x|$ and since $x \in \Lambda_\infty(A)$. Thus, $\alpha x \in \Lambda_\infty(A)$ whenever $\alpha \in \mathbf{R}$ and $x \in \Lambda_\infty(A)$.

We now prove that $\Lambda_F(A)$ is an ideal in G'.

Let $x_1, x_2 \in \Lambda_F(A)$ be such that $x_1 + x_2 \neq 0$, and let $y \in G'$ be such that $0 \leq y \leq |x_1 + x_2|$, $y \neq 0$. By the decomposition property it follows that there exist y_1, $y_2 \in G'$, $0 \leq y_1 \leq |x_1|$, $0 \leq y_2 \leq |x_2|$ such that $y = y_1 + y_2$. Since we assume that $y \neq 0$, it follows that $y_1 \neq 0$

or $y_2 \neq 0$. Assume that $y_1 \neq 0$ (the case $y_2 \neq 0$ is similar). Taking into consideration that $x_1 \in \Lambda_F(A)$ we obtain that there exists $z \in G'$, $0 \leq z \leq y_1 \leq y$, $z \neq 0$ such that $0 < \sup_{u \in A}\langle u, z\rangle + \infty$. We have therefore proved that $x_1 + x_2 \in \Lambda_F(A)$ whenever $x_1, x_2 \in \Lambda_F(A)$.

Now let $\alpha \in \mathbf{R}$ and $x \in \Lambda_F(A)$ be such that $\alpha \neq 0$ and $x \neq 0$. Let $y \in G'$ be such that $0 \leq y \leq |\alpha x|$, $y \neq 0$. It follows that $0 \leq y/|\alpha| \leq |x|$. Taking into consideration that $x \in \Lambda_F(A)$, we obtain that there exists $z \in G'$, $0 \leq z \leq y/|\alpha|$, $z \neq 0$ such that $0 < \sup_{u \in A}\langle u, z\rangle < +\infty$. Thus, we have shown that $\alpha x \in \Lambda_F(A)$ whenever $\alpha \in \mathbf{R}$ and $x \in \Lambda_F(A)$.

We have therefore proved that $\Lambda_F(A)$ is an ideal in G' since $\Lambda_F(A)$ is, obviously, a solid set.

Now note that, in view of [**7**, Proposition 2.7, p. 61] our proposition is completely proved if we show that G' is the order direct sum of $\Lambda_\infty(A)$ and $\Lambda_F(A)$.

Clearly, $\Lambda_\infty(A) \cap \Lambda_F(A) = 0$. Therefore, in order to prove that G' is the order direct sum of $\Lambda_\infty(A)$ and $\Lambda_F(A)$, it is enough to show that $G' = \Lambda_\infty(A) + \Lambda_F(A)$. It follows that we only have to prove that for every $x \in G'$, $x \geq 0$, there exist $x_\infty \in \Lambda_\infty(A)$ and $x_F \in \Lambda_F(A)$ such that $x = x_\infty + x_F$.

To this end, let $x \in G'$, $x \geq 0$, and let $\mathcal{U}_x$ be the set of all components of x, $\mathcal{U}_x = \{y \in G' \mid y \wedge (x - y) = 0\}$. Since G' is an order complete Riesz space, and since x is an upper bound for $\mathcal{U}_x \cap \Lambda_\infty(A)$, it follows that $\sup(\mathcal{U}_x \cap \Lambda_\infty(A))$ exists in G'. Let $x_\infty = \sup(\mathcal{U}_x \cap \Lambda_\infty(A))$. Using Theorem 3.15, [**2**, p. 37], it follows that $x_\infty \in \mathcal{U}_x$.

We now prove that $x_\infty \in \Lambda_\infty(A)$. To this end, let $y \in G'$, $0 \leq y \leq x_\infty$, and assume that $\sup_{u \in A}\langle u, y\rangle > 0$. Then there exists $u' \in A$ such that $\langle u', y\rangle > 0$.

The set $\mathcal{U}_x \cap \Lambda_\infty(A)$ is a directed one since $t_1 \vee t_2 \in \mathcal{U}_x \cap \Lambda_\infty(A)$ whenever $t_1, t_2 \in \mathcal{U}_x \cap \Lambda_\infty(A)$. Indeed, if $t_1, t_2 \in \mathcal{U}_x \cap \Lambda_\infty(A)$, then $t_1 \vee t_2 \in \mathcal{U}_x$ (since $\mathcal{U}_x$ is a Boolean algebra) and $t_1 \vee t_2 \in \Lambda_\infty(A)$ (since $0 \leq t_1 \vee t_2 \leq t_1 + t_2$, and since $\Lambda_\infty(A)$ is an ideal).

Accordingly, the set $\{y \wedge z \mid z \in \mathcal{U}_x \cap \Lambda_\infty(A)\}$ is also directed.

Taking into consideration that $y = y \wedge x_\infty = \sup_{z \in \mathcal{U}_x \cap \Lambda_\infty(A)}(y \wedge z)$, and using Proposition 4.2, [**7**, p. 72], we obtain that $0 < \langle u', y\rangle = \langle u', \sup_{z \in \mathcal{U}_x \cap \Lambda_\infty(A)}(y \wedge z)\rangle = \sup_{z \in \mathcal{U}_x \cap \Lambda_\infty(A)}\langle u', y \wedge z\rangle$. Hence, there exists $z' \in \mathcal{U}_x \cap \Lambda_\infty(A)$ such that $\langle u', y \wedge z'\rangle > 0$. Clearly, $y \wedge z' \in \Lambda_\infty(A)$

since $\Lambda_\infty(A)$ is an ideal.

It follows that $\sup_{u\in A}\langle u, y\wedge z'\rangle = +\infty$; therefore, $\sup_{u\in A}\langle u, y\rangle = +\infty$.

We have therefore proved that $x_\infty \in \Lambda_\infty(A)$.

Now set $x_F = x - x_\infty$ and assume that $x_F \notin \Lambda_F(A)$. Then there exists $y \in G'$, $y \neq 0$, $0 \leq y \leq x_F$ such that $\sup_{u\in A}\langle u, z\rangle = 0$ or $+\infty$ for every $z \in G'$, $0 \leq z \leq y$. Accordingly, $y \in \Lambda_\infty(A)$.

By the Freudenthal spectral theorem [**5**, Theorem 40.2, p. 257] there exist $\alpha \in \mathbf{R}$, $\alpha > 0$ and $t \in \mathcal{U}_x$, $t \neq 0$ such that $\alpha t \leq y$. It follows that $t \in \Lambda_\infty(A)$.

Since $t \wedge x_\infty = 0$ and since $t \neq 0$, it follows that $t \vee x_\infty \neq x_\infty$.

Taking into consideration that $t \vee x_\infty \in \mathcal{U}_x \cap \Lambda_\infty(A)$ we obtain a contradiction to the way we have defined x_∞. Hence, $x_F \in \Lambda_F(A)$. □

We will now discuss an application of Proposition 4.

To this end, let E be an Archimedean Riesz space, let $\tilde{E}$ be the Dedekind completion of E, and let $\tilde{E}'$ be the order dual of $\tilde{E}$. Let $f, g \in E$, $f \geq 0$, $g \geq 0$, and let B and $\Gamma(B)$ be the sets defined in Introduction (note that B is a projection band, by construction, and that $\Gamma(B)$ is also a projection band, by Lemma 3). Let Δ be a projection band in $\tilde{E}'$, $\Delta \subseteq \Gamma(B)$, and consider the sets $\Omega_\infty(f, T, \Delta)$ and $\Sigma_\infty(f, T, \Delta)$ defined in Introduction.

Set

$$\Omega_F(f,T,\Delta) = \left\{ x \in \Delta \quad \middle| \begin{array}{l} x = 0 \text{ or } x \neq 0 \text{ and for every } y \in \Delta, \\ 0 \leq y \leq |x|, y \neq 0 \text{ there exists } z \in \Delta, \\ 0 \leq z \leq y, z \neq 0 \text{ such that} \\ 0 < a_z < +\infty \end{array} \right\}$$

and

$$\Sigma_F(f,T,\Delta) = \left\{ x \in \Delta \quad \middle| \begin{array}{l} x = 0 \text{ or } x \neq 0 \text{ and for every} \\ y \in \Delta, 0 \leq y \leq |x|, y \neq 0 \text{ there exists} \\ z \in \Delta, 0 \leq z \leq y, z \neq 0 \text{ such that} \\ 0 < b_z < +\infty \end{array} \right\}.$$

Corollary 5. a) *The sets* $\Omega_\infty(f,T,\Delta)$ *and* $\Omega_F(f,T,\Delta)$ *are projection bands in* $\tilde{E}'$ *and* Δ *is their order direct sum.*

b) *The sets* $\Sigma_\infty(f,T,\Delta)$ *and* $\Sigma_F(f,T,\Delta)$ *are projection bands in* $\tilde{E}'$ *and* Δ *is their order direct sum.*

Proof. a) The proof consists in a straightforward application of Proposition 4. To this end, set $G = \tilde{E}$, $A = \{\bar{f} \in E / \bar{f}$ is an H_{f+g}-modification of $f\} \subseteq \tilde{E}$ and note that $\Omega_\infty(f,T,\Delta) = \Lambda_\infty(A) \cap \Delta$ and that $\Omega_F(f,T,\Delta) = \Lambda_F(A) \cap \Delta$.

b) Proceed as in a), the only difference being that in this case set $A = \{\bar{f} \in E \mid \bar{f}$ is an H_f-modification of $f\} \subseteq \tilde{E}$. □

3. The filling scheme in Archimedean Riesz spaces. In this section we will extend several results of Akcoglu and Chacon's paper [**1**] to the setting defined in Introduction.

Let E be an Archimedean Riesz space, and let $T : E \to E$ be a positive linear operator.

As in [**1**] we consider the nonlinear mapping $U : E \to E$, $U(u) = T(u^+) - u^-$ for every $u \in E$.

The next lemma is an extension of (and has the same proof as) Lemma 0.1 of [**1**].

Lemma 6. *Let* $u, v \in E$ *be such that* $u \geq 0$ *and* $v \geq 0$. *Let* $n \in \mathbf{N} \cup \{0\}$. *Then there exists* $w \in E$, $w \geq 0$, *such that* $u \overset{n}{\to} w$ *and* $U^n(u - v) = w - v$.

Now let $\tilde{E}$ be the Dedekind completion of E. Let $u \in \tilde{E}$, and let $B(u)$ be the projection band in $\tilde{E}$ generated by the singleton $\{u\}$; we will denote by P_u the band projection associated with the projection band $B(u)$.

The next proposition is an extension of Theorem 0.1 of [**1**].

Proposition 7. *Let* $u, v \in E$, $u \geq 0$, $v \geq 0$. *Set* $w = u - v$ *and set* $w_n = \sup_{0 \leq k \leq n} \sum_{l=0}^{k} T^l w$ *for every* $n \in \mathbf{N} \cup \{0\}$. *Then* $P_{w_n^+} U^n w \geq 0$ *for every* $n \in \mathbf{N} \cup \{0\}$.

Proof. The proof is an extension of the proof of Theorem 0.1 of [**1**] to our setting.

Set $\phi_k = \sum_{l=0}^{k} (U^l w)^+$ for every $k \in \mathbf{N} \cup \{0\}$.

We will first prove by induction that $\phi_k \geq \sum_{l=0}^{k} T^l w$ for every $k \in \mathbf{N} \cup \{0\}$. The statement is obviously true for $k = 0$ (since $\phi_0 = w^+ \geq w$).

Now assume that $\phi_k \geq \sum_{l=0}^{k} T^l w$. Taking into consideration that T is a positive linear operator, we obtain that

$$T\phi_k \geq \sum_{l=1}^{k+1} T^l w. \tag{7.1}$$

Since $U^{l+1}w = T((U^l w)^+) - (U^l w)^-$ for every $l \in \mathbf{N} \cup \{0\}$, it follows that $T\phi_k = \sum_{l=0}^{k} T((U^l w)^+) = \sum_{l=0}^{k} (U^{l+1}w + (U^l w)^-) = \sum_{l=0}^{k} ((U^{l+1}w)^+ - (U^{l+1}w)^- + (U^l w)^-) = \phi_{k+1} - w^+ + w^- - (U^{k+1}w)^- = \phi_{k+1} - w - (U^{k+1}w)^-$. Accordingly, (7.1) implies that $\phi_{k+1} - w - (U^{k+1}w)^- \geq \sum_{l=1}^{k+1} T^l w$. It follows that $\phi_{k+1} \geq \phi_{k+1} - (U^{k+1}w)^- \geq w + \sum_{l=1}^{k+1} T^l w = \sum_{l=0}^{k+1} T^l w$. We have therefore proved that $\phi_k \geq \sum_{l=0}^{k} T^l w$ for every $k \in \mathbf{N} \cup \{0\}$.

We obtain that $\phi_k \geq w_k^+$ for every $k \in \mathbf{N} \cup \{0\}$ (since $\phi_k \geq 0$ and $\phi_k \geq w_k$ for every $k \in \mathbf{N} \cup \{0\}$).

We will now prove that for every nonzero component s of w_n^+ (in $\tilde{E}$) there exists a nonzero component t of w_n^+ (in $\tilde{E}$), $t \leq s$, and there exists $k \in \{0, 1, 2, \dots, n\}$ such that $(U^k w)^- \wedge t = 0$.

To this end, let s be a nonzero component of w_n^+ in $\tilde{E}$.

Let P_s be the band projection associated with the projection band $B(s)$. Taking into consideration that $\phi_n \geq w_n^+$, and using the fact that s is a component of w_n^+, we obtain that $P_s\phi_n \geq P_s w_n^+ = s$. It follows that $(P_s\phi_n) \wedge s = s \neq 0$.

Using a well-known consequence of the decomposition property in Riesz spaces (see, for example, [**7**, Corollary, p. 53]), we obtain that $(P_s\phi_n) \wedge s = (\sum_{i=0}^{n} P_s((U^i w)^+)) \wedge s \leq \sum_{i=0}^{n} ((P_s((U^i w)^+)) \wedge s)$. Accordingly, there exists $k \in \{0, 1, 2, \dots, n\}$ such that $(P_s((U^k w)^+)) \wedge s \neq 0$.

By the Freudenthal spectral theorem (see, for example, [**5**, Theorem 40.2, p. 257]), there exists a nonzero component, t, of s in $\tilde{E}$ (therefore t is also a nonzero component of w_n^+ in $\tilde{E}$), and there exists $\alpha \in \mathbf{R}$, $\alpha > 0$ such that $\alpha t \leq (P_s((U^k w)^+)) \wedge s \leq P_s((U^k w)^+) \leq (U^k w)^+$. It follows that $(\alpha t) \wedge (U^k w)^- = 0$ (since $(\alpha t) \wedge (U^k w)^- \leq (U^k w)^+ \wedge (U^k w)^- = 0$).

We obtain that $t \wedge (U^k w)^- = 0$ (since if $0 < \alpha \leq 1$, then $0 = (\alpha t) \wedge (U^k w)^- \geq (\alpha t) \wedge (\alpha (U^k w)^-) = \alpha (t \wedge (U^k w)^-)$ and if $\alpha > 1$, then $0 = (\alpha t) \wedge (U^k w)^- \geq t \wedge (U^k w)^-$).

Set $\mathcal{U} = \{q \in \tilde{E} \mid q$ is a component of w_n^+ in $\tilde{E}$ and $(U^n w)^- \wedge q = 0\}$. Clearly, $\mathcal{U}$ is an order bounded set in the order complete Riesz space $\tilde{E}$; hence, $\sup \mathcal{U}$ exists in $\tilde{E}$. Taking into consideration that w_n^+ is an upper bound for $\mathcal{U}$, we obtain that $\sup \mathcal{U} \leq w_n^+$.

Our goal now is to prove that $\sup \mathcal{U} = w_n^+$. To this end, assume that $\sup \mathcal{U} \neq w_n^+$. By Theorem 3.15 [**2**, p. 37] $\sup \mathcal{U}$ is a component of w_n^+; therefore, $w_n^+ - \sup \mathcal{U}$ is a nonzero component of w_n^+. Using our previous remarks we obtain that there exists a nonzero component t of w_n^+ in $\tilde{E}$, $t \leq w_n^+ - \sup \mathcal{U}$, and there exists $k \in \{0, 1, 2, \dots, n\}$ such that $(U^k w)^- \wedge t = 0$.

We now note that $(U^n w)^- \leq (U^k w)^-$. Indeed, let $l \in \mathbf{N} \cup \{0\}$; using the definition of the mapping U we obtain that $U^{l+1} w = T((U^l w)^+) - (U^l w)^-$. Since T is a positive (linear) operator, it follows that $T((U^l w)^+) \geq 0$. We obtain that $(U^{l+1} w)^- \leq (U^l w)^-$ for every $l \in \mathbf{N} \cup \{0\}$; hence, $(U^n w)^- \leq (U^k w)^-$.

It follows that $(U^n w)^- \wedge t = 0$; hence, $t \in \mathcal{U}$. We obtain a contradiction since $0 \leq t \leq (w_n^+ - \sup \mathcal{U}) \wedge (\sup \mathcal{U}) = 0$ while $t \neq 0$. Accordingly, $\sup \mathcal{U} = w_n^+$.

We obtain that $(U^n w)^- \wedge w_n^+ = (U^n w)^- \wedge (\sup \mathcal{U}) = \sup_{q \in \mathcal{U}} ((U^n w)^- \wedge q) = 0$; therefore, $P_{w_n^+}((U^n w)^-) = 0$. It follows that $P_{w_n^+}(U^n w) = P_{w_n^+}((U^n w)^+) - P_{w_n^+}((U^n w)^-) = P_{w_n^+}((U^n w)^+) \geq 0$. □

Let $\tilde{E}'$ and $\tilde{E}''$ be the first and second order dual of $\tilde{E}$, respectively. We will assume from now on throughout the paper that $\tilde{E}'$ separates the points of $\tilde{E}$. Then the canonical embedding $i : \tilde{E} \to \tilde{E}''$ is one-to-one. Thus, we may think of $\tilde{E}$ as a vector sublattice of $\tilde{E}''$, and from now on we will do so without stating it explicitly every time. Moreover, we may and do think of the elements of $\tilde{E}$ as being order continuous linear functionals on $\tilde{E}'$.

The next proposition is an extension of Theorem 0.2 of [**1**].

Proposition 8. *Let* $u, v \in E$, $u \geq 0$, $v \geq 0$, *and set* $w = u - v$. *Set* $w_n = \sup_{0 \leq k \leq n} \sum_{l=0}^{k} T^l w$ *for every* $n \in \mathbf{N} \cup \{0\}$. *Let* Γ_n *be the carrier of* w_n^+ *in* $\tilde{E}'$ *for every* $n \in \mathbf{N} \cup \{0\}$, *and let* Γ *be the (projection) band in* $\tilde{E}'$ *generated by* $\cup_{n=0}^{\infty} \Gamma_n$. *Then*

(a) $\psi_x^n u \geq \langle v, x \rangle$ *for every* $n \in \mathbf{N} \cup \{0\}$ *and for every* $x \in \Gamma_n$, $x \geq 0$.

(b) $\psi_x u \geq \langle v, x \rangle$ *for every* $x \in \Gamma$, $x \geq 0$.

Proof. (a) Let $n \in \mathbf{N} \cup \{0\}$.

By Lemma 6 there exists $u_n \in E$, $u_n \geq 0$, such that $u \overset{n}{\to} u_n$ and $U^n(u - v) = u_n - v$.

Let $B(w_n^+)$ be the projection band in $\tilde{E}$ generated by the singleton $\{w_n^+\}$, and let $P_{w_n^+}$ be the band projection associated with the projection band $B(w_n^+)$. Taking into consideration that $P_{w_n^+} u_n$ is a component of u_n, we obtain that $u_n - P_{w_n^+} u_n$ and $P_{w_n^+} u_n$ (thought of as elements of $\tilde{E}''$) are disjoint order continuous linear functionals on $\tilde{E}'$.

Let $x \in \Gamma_n$, $x \geq 0$. Taking into consideration that $(u_n - P_{w_n^+} u_n) \wedge w_n^+ = 0$, and since x is in the carrier of w_n^+, we obtain, using Nakano's theorem (see [**2**, Theorem 5.2, pp. 56–57]) that $\langle u_n - P_{w_n^+} u_n, x \rangle = 0$. Accordingly,

$$(8.1) \qquad \langle u_n, x \rangle = \langle P_{w_n^+} u_n, x \rangle + \langle u_n - P_{w_n^+} u_n, x \rangle = \langle P_{w_n^+} u_n, x \rangle.$$

Similarly, $v - P_{w_n^+} v$ and w_n^+ are disjoint order continuous linear functionals on $\tilde{E}'$, therefore, using again Nakano's theorem we obtain that $\langle v - P_{w_n^+} v, x \rangle = 0$. Hence,

$$(8.2) \qquad \langle v, x \rangle = \langle P_{w_n^+} v, x \rangle + \langle v - P_{w_n^+} v, x \rangle = \langle P_{w_n^+} v, x \rangle.$$

Taking into consideration the way in which u_n was chosen and using Proposition 7 we obtain that $P_{w_n^+}(u_n - v) = P_{w_n^+} U^n w \geq 0$. Accordingly, $P_{w_n^+} u_n \geq P_{w_n^+} v$. In view of (8.1) and (8.2), it follows that $\langle u_n, x \rangle \geq \langle v, x \rangle$.

Since $u \overset{n}{\to} u_n$ and using the definition of $\psi_x^n u$, we obtain that $\psi_x^n u \geq \langle v, x \rangle$.

(b) Let $x \in \Gamma$, $x \geq 0$.

Let P_n be the band projection associated with the projection band Γ_n for every $n \in \mathbf{N} \cup \{0\}$, and set $x_n = P_n x$ for every $n \in \mathbf{N} \cup \{0\}$.

Clearly, $(x_n)_{n \in \mathbf{N} \cup \{0\}}$ is a sequence of components of x. Taking into consideration that $\tilde{E}'$ is an order complete Riesz space and using Theorem 3.15 [**2**, pp. 37–38], we obtain that $\sup_{n \in \mathbf{N} \cup \{0\}} x_n$ exists in $\tilde{E}'$ and that $\sup_{n \in \mathbf{N} \cup \{0\}} x_n$ is a component of x.

We will now note that $\sup_{n \in \mathbf{N} \cup \{0\}} x_n = x$. Indeed, assume that $\sup_{n \in \mathbf{N} \cup \{0\}} x_n \neq x$. Then $x - \sup_{n \in \mathbf{N} \cup \{0\}} x_n$ is a nonzero component of x. Taking into consideration that $x - \sup_{n \in \mathbf{N} \cup \{0\}} x_n \in \Gamma$ and that Γ is the projection band generated by $\cup_{n \in \mathbf{N} \cup \{0\}} \Gamma_n$, we obtain (using [**7**, Proposition 2.11, p. 63 and Corollary, p. 53]) that there exists $y \in \cup_{n \in \mathbf{N} \cup \{0\}} \Gamma_n$, $y \geq 0$, $y \neq 0$, such that $y \leq x - \sup_{n \in \mathbf{N} \cup \{0\}} x_n$. Clearly, $y \in \Gamma_m$ for some $m \in \mathbf{N} \cup \{0\}$. Taking into consideration that $0 \leq y \leq x$, and using Theorem 24.5, [**5**, pp. 133–134], we obtain that $0 \leq y \leq x_m$; hence, $0 \leq y \leq \sup_{n \in \mathbf{N} \cup \{0\}} x_n$. We have obtained a contradiction since, on one hand, $0 \leq y \wedge (x - \sup_{n \in \mathbf{N} \cup \{0\}} x_n) \leq (\sup_{n \in \mathbf{N} \cup \{0\}} x_n) \wedge (x - \sup_{n \in \mathbf{N} \cup \{0\}} x_n) = 0$, while, on the other, $y \wedge (x - \sup_{n \in \mathbf{N} \cup \{0\}} x_n) \neq 0$.

Taking into consideration that $x \geq x_n \geq 0$ for every $n \in \mathbf{N} \cup \{0\}$, and using (a), we obtain that $\psi_x u \geq \psi_{x_n} u \geq \psi_{x_n}^n u \geq \langle v, x_n \rangle$ for every $n \in \mathbf{N} \cup \{0\}$.

Since $(w_n^+)_{n \in \mathbf{N} \cup \{0\}}$ is a monotonic nondecreasing sequence, it follows that $\Gamma_n \subseteq \Gamma_{n+1}$ for every $n \in \mathbf{N} \cup \{0\}$; accordingly, the sequence $(x_n)_{n \in \mathbf{N} \cup \{0\}}$ is monotonic nondecreasing.

Taking into consideration that $x = \sup_{n \in \mathbf{N} \cup \{0\}} x_n$ and using Proposition 4.2 [**7**, p. 72], we obtain that $\langle v, x \rangle = \sup_n \langle v, x_n \rangle \leq \psi_x u$. □

For the next proposition, we need the following lemma:

Lemma 9. *Let $u, v \in E$, $u \geq 0$, $v \geq 0$. If $u \overset{n}{\to} v$ for some $n \in \mathbf{N} \cup \{0\}$, then $v \in H_u$.*

Proof. The proof is by induction over n.

For $n = 0$ the statement of the lemma is obviously true.

Now assume that the lemma is true for $n-1$, and let $v \in E$, $v \geq 0$, be such that $u \overset{n}{\to} v$. Then there exists $v_1 \in E$, $v_1 \geq 0$, such that $u \overset{n-1}{\to} v_1$ and $v_1 \overset{1}{\to} v$. It follows that there exist r, $s \in E$, $r \geq 0$, $s \geq 0$, such that $v_1 = r + s$ and $v = r + Ts$.

Taking into consideration that $0 \leq r \leq v_1$, $0 \leq s \leq v_1$, and since T is a positive operator, we obtain that $0 \leq v \leq r + Ts \leq v_1 + Tv_1$; it follows that $v_1 + Tv_1 \in H_u$ (since $v_1 \in H_u$ by our induction hypothesis and since H_u is T-invariant ($T(H_u) \subseteq H_u$)); therefore, $v \in H_u$ (since H_u is an ideal). □

The next proposition (Proposition 10) should be compared to Lemma 0.2 of [**1**].

Proposition 10. *Let $\bar{u}, \bar{v} \in E$ be such that $\bar{u} \geq 0$, $\bar{v} \geq 0$. Let $u \in \tilde{E}$, $u \geq 0$. Assume that there is no $v \in B(u)$, $v \geq 0$, $v \neq 0$, and there is no sequence $(\omega_k)_{k\in\mathbf{N}}$ of $H_{\bar{u}+\bar{v}}$-modifications of $\bar{v}$ such that $(\omega_k)_{k\in\mathbf{N}}$ diverges individually to ∞ on $B(v)$. Let $n \in \mathbf{N} \cup \{0\}$, and let $\eta \in E$, $\eta \geq 0$, be such that $\bar{v} \overset{n}{\to} \eta$. Set $\bar{u}_k = \sum_{i=0}^{k} T^i\bar{u}$, $\bar{v}_k = \sum_{i=0}^{k} T^i\bar{v}$, and $\bar{w}_k = \sum_{i=0}^{k} T^i\eta$ for every $k \in \mathbf{N} \cup \{0\}$. Let $\rho \in \mathbf{R}$, $0 < \rho < 1$. Then $\limsup_k(((1-\rho)\bar{u}_k - \bar{v}_k)^+ \wedge u) \leq \limsup_k((\bar{u}_k - \bar{w}_k)^+ \wedge u)$.*

Proof. Let $B_T(\bar{u})$ be the projection band in $\tilde{E}$ generated by the set $\{T^k\bar{u} \mid k \in \mathbf{N} \cup \{0\}\}$, and let $P_{T,\bar{u}}$ be the band projection associated with the projection band $B_T(\bar{u})$.

We start by noticing that we may assume that $u \in B_T(\bar{u})$. Indeed, let $u \in \tilde{E}$ be as in the proposition. Set $v = P_{T,\bar{u}}u$ and $w = u - v$. Since $\bar{u}_k \wedge w = 0$, it follows (using [**7**, Corollary, p. 53]) that $(\bar{u}_k - \bar{w}_k)^+ \wedge u = (\bar{u}_k - \bar{w}_k)^+ \wedge (v + w) \leq (\bar{u}_k - \bar{w}_k)^+ \wedge v + (\bar{u}_k - \bar{w}_k)^+ \wedge w \leq (\bar{u}_k - \bar{w}_k)^+ \wedge v + \bar{u}_k \wedge w = (\bar{u}_k - \bar{w}_k)^+ \wedge v$ for every $k \in \mathbf{N} \cup \{0\}$. Accordingly, $\limsup_k((\bar{u}_k - \bar{w}_k)^+ \wedge u) \leq \limsup_k((\bar{u}_k - \bar{w}_k)^+ \wedge w)$. Since, obviously, $\limsup_k((\bar{u}_k - \bar{w}_k)^+ \wedge w) \leq \limsup_k((\bar{u}_k - \bar{w}_k)^+ \wedge u)$, it follows that $\limsup_k((\bar{u}_k - \bar{w}_k)^+ \wedge w) = \limsup_k((\bar{u}_k - \bar{w}_k)^+ \wedge u)$. In a similar way, we obtain that $\limsup_k(((1-\rho)\bar{u}_k - v_k)^+ \wedge w) = \limsup_k(((1-\rho)\bar{u}_k - v_k)^+ \wedge u)$.

We will now note that, in order to prove the proposition, it is enough to prove that for every $n \in \mathbf{N} \cup \{0\}$, for every $\eta \in E$, $\eta \geq 0$, such that $\bar{v} \stackrel{n}{\to} \eta$, and for every $\rho, \rho' \in \mathbf{R}$, $0 < \rho' < \rho < 1$, it follows that

$$(10.1)\quad \limsup_k (((1-\rho)\bar{u}_k - \bar{v}_k)^+ \wedge u) \leq \limsup_k (((1-\rho')\bar{u}_k - \bar{w}_k)^+ \wedge u).$$

(The above remark is obvious in view of the fact that $\limsup_k (((1-\rho')\bar{u}_k - \bar{w}_k)^+ \wedge u) \leq \limsup_k ((\bar{u}_k - \bar{w}_k)^+ \wedge u)$).

We will prove (10.1) by induction on n. The inequality (10.1) is clearly true for $n = 0$ since, in this case, $\eta = \bar{v}$; therefore, $\bar{v}_k = \bar{w}_k$ for every $k \in \mathbf{N} \cup \{0\}$.

Now assume that (10.1) is true for $n-1$, and let $\eta \in E$, $\eta \geq 0$, be such that $\bar{v} \stackrel{n}{\to} \eta$. We want to prove that the inequality (10.1) is true for every $\rho, \rho' \in \mathbf{R}$, $0 < \rho' < \rho < 1$, provided that $\bar{w}_k = \sum_{i=0}^k T^i \eta$ for every $k \in \mathbf{N} \cup \{0\}$. Taking into consideration that $\bar{v} \stackrel{n}{\to} \eta$, we obtain that there exists $\eta' \in E$, $\eta' \geq 0$, such that $\bar{v} \stackrel{n-1}{\to} \eta'$ and $\eta' \stackrel{1}{\to} \eta$. Set $\bar{w}'_k = \sum_{i=0}^k T^i \eta'$ for every $k \in \mathbf{N} \cup \{0\}$.

Let $\rho'' \in \mathbf{R}$ be such that $\rho' < \rho'' < \rho$.

By our induction hypothesis, $\limsup_k (((1-\rho)\bar{u}_k - \bar{v}_k)^+ \wedge u) \leq \limsup_k (((1-\rho'')\bar{u}_k - \bar{w}'_k)^+ \wedge u)$. Thus, in order to prove that inequality (10.1) is true for η, ρ, and ρ', it is enough to prove that

$$(10.2)\quad \limsup_k (((1-\rho'')\bar{u}_k - \bar{w}'_k)^+ \wedge u) \leq \limsup_k (((1-\rho')\bar{u}_k - \bar{w}_k)^+ \wedge u).$$

Taking into consideration that $\eta' \stackrel{1}{\to} \eta$, we obtain that there exist $r, s \in E$, $r \geq 0$, $s \geq 0$, such that $\eta' = r + s$ and $\eta = r + Ts$.

We will now prove that

$$(10.3)\qquad \limsup_k ((T^{k+1}s - \varepsilon \bar{u}_k)^+ \wedge u) = 0$$

for every $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$.

To this end, assume that there exists $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$, such that $\limsup_k ((T^{k+1}s - \varepsilon \bar{u}_k)^+ \wedge u) \neq 0$.

Taking into consideration that $0 \leq \limsup_k ((T^{k+1}s - \varepsilon \bar{u}_k)^+ \wedge u) \leq u$ and, since we assume that $u \in B_T(\bar{u})$, we obtain that $\limsup_k ((T^{k+1}s -$

$\varepsilon\bar{u}_k)^+ \wedge u) \in B_T(\bar{u})$. Thus, using [**7**, Proposition 2.11, p. 63 and Corollary, p. 53], we obtain that there exists $l \in \mathbf{N} \cup \{0\}$ such that $(T^l\bar{u}) \wedge (\limsup_k((T^{k+1}s - \varepsilon\bar{u}_k)^+ \wedge u)) \neq 0$; hence, $\limsup_k((T^{k+1}s - \varepsilon\bar{u}_k)^+ \wedge u \wedge (T^l\bar{u})) \neq 0$.

Since $\limsup_k((T^{k+1}s-\varepsilon\bar{u}_k)^+ \wedge u \wedge (T^l\bar{u})) \leq \limsup_{k\geq l}((T^{k-l+1}(T^l s) - \varepsilon(\sum_{i=0}^{k-l} T^i(T^l\bar{u})))^+ \wedge u \wedge (T^l\bar{u}))$, it follows that

$$\limsup_{k\geq l}\left(\left(T^{k-l+1}(T^l s) - \varepsilon\left(\sum_{i=0}^{k-l} T^i(T^l\bar{u})\right)\right)^+ \wedge u \wedge (T^l\bar{u})\right) \neq 0. \tag{10.4}$$

We will now apply (and use the notations of) Proposition 9 of [**9**]. To this end, let $\tilde{w}$ be the projection of $T^l\bar{u}$ on the projection band $B_\varepsilon((T^{k+1}(T^l s), \sum_{i=0}^{k} T^i(T^l\bar{u}))_{k\in\mathbf{N}\cup\{0\}})$.

Taking into consideration that (10.4) can be rewritten in the form (10.5)

$$0 \leq \limsup_{k\in\mathbf{N}\cup\{0\}}\left(\left(T^{k+1}(T^l s) - \varepsilon\left(\sum_{i=0}^{k} T^i(T^l\bar{u})\right)\right)^+ \wedge u \wedge (T^l\bar{u})\right) \neq 0,$$

we obtain that $\tilde{w} \neq 0$.

By Proposition 9 of [**9**] there exists a sequence $(\delta_m)_{m\in\mathbf{N}}$ of $H_{T^l s+T^l\bar{u}}$-modifications of $T^l s$ which diverges individually to ∞ on $B(\tilde{w})$.

In view of (10.5), it follows that $u \wedge \tilde{w} \neq 0$; therefore, the sequence $(\delta_m)_{m\in\mathbf{N}}$ diverges individually to ∞ on $B(u \wedge \tilde{w})$ as well.

By Lemma 9, $\eta' \in H_{\bar{v}}$; since $0 \leq s \leq \eta'$ and since $H_{\bar{v}}$ is an ideal, it follows that $s \in H_{\bar{v}}$; taking into consideration that $H_{\bar{v}}$ is T-invariant ($T(H_{\bar{v}}) \subseteq H_{\bar{v}}$), we obtain that $T^l s \in H_{\bar{v}}$. Accordingly, $H_{T^l s+T^l\bar{u}} \subseteq H_{\bar{u}+\bar{v}}$; hence, $(\delta_m)_{m\in\mathbf{N}}$ is a sequence of $H_{\bar{u}+\bar{v}}$-modifications of $T^l s$. Thus, given $m \in \mathbf{N}$, it follows that $\delta_m = T^l s - \lambda_m + T\lambda_m$ for some $\lambda_m \in H_{\bar{u}+\bar{v}}$, $\lambda_m \geq 0$.

Since $T^l s \in H_{\bar{v}}$, it follows that $T^l s \leq M\sum_{i=0}^{r} T^i\bar{v}$ for some $M \in \mathbf{R}$, $M > 0$ and $r \in \mathbf{N} \cup \{0\}$. Thus, $\delta_m + (M(\sum_{i=0}^{r} T^i\bar{v}) - T^l s) = T^l s - \lambda_m + T\lambda_m + (M(\sum_{i=0}^{r} T^i\bar{v}) - T^l s) = M(\sum_{i=0}^{r} T^i\bar{v}) - \lambda_m + T\lambda_m$ for every $m \in \mathbf{N}$; accordingly, $(\delta_m + (M(\sum_{i=0}^{r} T^i\bar{v}) - T^l s))_{m\in\mathbf{N}}$ is a sequence of $H_{\bar{u}+\bar{v}}$-modifications of $M(\sum_{i=0}^{r} T^i\bar{v})$.

Since $T^i\bar{v}$ is an $H_{\bar{u}+\bar{v}}$-modification of $\bar{v}$ for every $i = 0, 1, \dots, r$, it follows that

$$\left(\frac{\delta_m + (M(\sum_{i=0}^r T^i\bar{v}) - T^l s)}{M(r+1)}\right)_{m\in\mathbf{N}}$$

is a sequence of $H_{\bar{u}+\bar{v}}$-modifications of $\bar{v}$. Clearly, the sequence

$$\left(\frac{\delta_m + (M(\sum_{i=0}^r T^i\bar{v}) - T^l s)}{M(r+1)}\right)_{m\in\mathbf{N}}$$

diverges individually to ∞ on $B(u \wedge \tilde{w})$. Since we have obtained a contradiction, it follows that (10.3) holds for every $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$.

We will now use the fact that (10.3) is true for every $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$ in order to prove (10.2). To this end, note that

$$\begin{aligned}(1-\rho')\sum_{i=0}^k T^i\bar{u} - \sum_{i=0}^k T^i\eta &= (1-\rho')\sum_{i=0}^k T^i\bar{u} - \sum_{i=0}^k T^i(r+Ts)\\ &= (1-\rho')\sum_{i=0}^k T^i\bar{u} - \sum_{i=0}^k T^i(r+s)\\ &\quad + \sum_{i=0}^k T^i(s-Ts)\\ &= (1-\rho')\sum_{i=0}^k T^i\bar{u} - \sum_{i=0}^k T^i\eta' + s - T^{k+1}s\\ &\geq (1-\rho')\sum_{i=0}^k T^i\bar{u} - \sum_{i=0}^k T^i\eta' - T^{k+1}s\end{aligned}$$

for every $k \in \mathbf{N} \cup \{0\}$.

Accordingly, we obtain that $\limsup_k(((1-\rho')\bar{u}_k - \bar{w}_k)^+ \wedge u) \geq \limsup_k(((1-\rho')\bar{u}_k - \bar{w}'_k - T^{k+1}s)^+ \wedge u) = \limsup_k(((1-\rho'')\bar{u}_k - \bar{w}'_k - (T^{k+1}s - (\rho''-\rho')\bar{u}_k))^+ \wedge u) \geq \limsup_k(((1-\rho'')\bar{u}_k - \bar{w}'_k)^+ \wedge u) - \limsup_k((T^{k+1}s - (\rho''-\rho')\bar{u}_k)^+ \wedge u)$.

Using (10.3) for $\varepsilon = \rho'' - \rho' > 0$, we obtain that $\limsup_k((T^{k+1}s - (\rho''-\rho')\bar{u}_k)^+ \wedge u) = 0$. Accordingly, (10.2) is true. □

Lemma 11. *Let $u, v \in E$ be such that $u \geq 0$ and $v \geq 0$. Set $w_n = \sup_{0\leq k\leq n}\sum_{l=0}^k T^l(u-v)$ for every $n \in \mathbf{N}\cup\{0\}$. Let $\bar{\Gamma}_n$ be* *the intersection of $\tilde{E}'_n$ with the carrier of w_n^+ in $\tilde{E}'$ (note that here we think of $\tilde{E}$ as being included in $\tilde{E}''$) for every $n \in \mathbf{N} \cup \{0\}$. Let $\bar{\Gamma}$ be the projection band (in $\tilde{E}'$) generated by $\cup_{n=0}^{\infty} \bar{\Gamma}_n$. Let $\bar{u} \in E$, $\bar{u} \geq 0$, and set $\bar{v} = \limsup_n (w_n^+ \wedge \bar{u})$. Let $\bar{\Gamma}(\bar{v})$ be the intersection of $\tilde{E}'_n$ with the carrier of $\bar{v}$ in $\tilde{E}'$. Then $\bar{\Gamma}(\bar{v}) \subseteq \bar{\Gamma}$.*

Proof. We will start by proving that, for every $x \in \bar{\Gamma}(\bar{v})$, $x \geq 0$, $x \neq 0$, there exists $y \in \bar{\Gamma}$, $0 \leq y \leq x$, $y \neq 0$.

To this end, let $x \in \bar{\Gamma}(\bar{v})$, $x \geq 0$, $x \neq 0$. Since $0 \leq \bar{v} = \limsup_n (w_n^+ \wedge \bar{u}) \leq \bigvee_{n=0}^{\infty} (w_n^+ \wedge \bar{u})$, it follows that x is in the carrier of $\bigvee_{n=0}^{\infty} (w_n^+ \wedge \bar{u})$.

Taking into consideration that x is an order continuous linear functional on $\tilde{E}$, and since the sequence $(w_n^+ \wedge \bar{u})_{n \in \mathbf{N} \cup \{0\}}$ is a monotonic nondecreasing sequence of elements of $\tilde{E}$, we obtain that $\langle \bigvee_{n=0}^{\infty} (w_n^+ \wedge u), x \rangle = \bigvee_{n=0}^{\infty} \langle w_n^+ \wedge u, x \rangle$; accordingly, $0 < \langle w_{n_0}^+ \wedge u, x \rangle \leq \langle w_{n_0}^+, x \rangle$ for some $n_0 \in \mathbf{N} \cup \{0\}$.

Since $w_{n_0}^+$ (thought of as an element of $\tilde{E}''$) is an order continuous linear functional on $\tilde{E}'$ (therefore $\tilde{E}'$ is the order direct sum of the carrier and the null space of $w_{n_0}^+$), it follows that the projection of x on the carrier of $w_{n_0}^+$ is a nonzero component of x. Let y be this projection. Then $y \in \tilde{E}'_n$ (since $0 \leq y \leq x$, $x \in \tilde{E}'_n$, and since $\tilde{E}'_n$ is a (projection) band in $\tilde{E}'$). Accordingly, $y \in \bar{\Gamma}_{n_0} \subseteq \bar{\Gamma}$.

Now assume that $\bar{\Gamma}(\bar{v}) \not\subseteq \bar{\Gamma}$. Then there exists $x \in \bar{\Gamma}(\bar{v})$ such that $x \notin \bar{\Gamma}$. Since both $\bar{\Gamma}(\bar{v})$ and $\bar{\Gamma}$ are projection bands, it follows that we may assume that $x \geq 0$, $x \neq 0$.

Let y be the projection of x on the projection band $\bar{\Gamma}$, and set $z = x - y$. Then $z \in \bar{\Gamma}(\bar{v})$ (since $0 \leq z \leq x$, $x \in \bar{\Gamma}(\bar{v})$ and since $\bar{\Gamma}(\bar{v})$ is a projection band).

Taking into consideration that $z \neq 0$ (since we assume that $x \notin \bar{\Gamma}$) and using what we proved previously, we obtain that there exists $s \in \bar{\Gamma}$, $s \neq 0$, $0 \leq s \leq z$.

Now note that $y \wedge z = 0$ (since y is a component of x); therefore, $0 \leq y \wedge s \leq y \wedge z = 0$; hence, $y \wedge s = 0$. We have obtained a contradiction since $0 \leq s \leq x$, $s \neq 0$, $s \in \bar{\Gamma}$, and since $y = \sup\{t \in \bar{\Gamma}/0 \leq t \leq x\}$. $\square$

The next (and last) proposition of this section is an extension of Theorem 0.4 of [**1**].

Proposition 12. *Let $\bar{u}, \bar{v} \in E$ be such that $\bar{u} \geq 0$ and $\bar{v} \geq 0$. Let $u \in \tilde{E}$, $u \geq 0$. Assume that there is no $v \in \tilde{E}$, $v \neq 0$ and no sequence $(\omega_k)_{k\in\mathbf{N}}$ of $H_{\bar{u}+\bar{v}}$-modifications of $\bar{v}$ such that $(\omega_k)_{k\in\mathbf{N}}$ diverges individually to ∞ on $B(v)$. Let $\rho \in \mathbf{R}$, $0 < \rho < 1$, and set $u_\rho = \limsup_n(((1-\rho)(\sum_{i=0}^n T^i\bar{u}) - (\sum_{i=0}^n T^i\bar{v}))^+ \wedge u)$. Let x be in the intersection of $\tilde{E}_n'$ with the carrier of u_ρ in $\tilde{E}'$, and assume that $x \geq 0$. Then $\psi_x\bar{u} \geq \psi_x\bar{v}$.*

Proof. Let $n \in \mathbf{N} \cup \{0\}$, and let $\eta_n \in E$, $\eta_n \geq 0$ be such that $\bar{v} \xrightarrow{n} \eta_n$. By Proposition 10,

$$u_\rho \leq \limsup_k \left(\left(\left(\sum_{i=0}^k T^i\bar{u} \right) - \left(\sum_{i=0}^k T^i\eta_n \right) \right)^+ \wedge u \right)$$
$$\leq \limsup_k \left(\left(\sup_{0\leq l\leq k} \left(\left(\sum_{i=0}^l T^i\bar{u} \right) - \left(\sum_{i=0}^l T^i\eta_n \right) \right) \right)^+ \wedge u \right).$$

It follows that x is in the intersection of $\tilde{E}_n'$ with the carrier of

$$\limsup_k \left(\left(\sup_{0\leq l\leq k} \left(\left(\sum_{i=0}^l T^i\bar{u} \right) - \left(\sum_{i=0}^l T^i\eta_n \right) \right) \right)^+ \wedge u \right).$$

We now apply Lemma 11 (the role played there by u, v, and $\bar{u}$ is taken here by $\bar{u}$, η_n, and u, respectively). Thus, set $w_k^{(n)} = \sup_{0\leq l\leq k} \sum_{i=0}^l T^i(\bar{u} - \eta_n)$ for every $k \in \mathbf{N} \cup \{0\}$; let $\bar{\Gamma}_k^{(n)}$ be the intersection of $\tilde{E}_n'$ with the carrier of $(w_k^{(n)})^+$ in $\tilde{E}'$ for every $k \in \mathbf{N} \cup \{0\}$, and let $\bar{\Gamma}^{(n)}$ be the (projection) band generated by $\cup_{k=0}^\infty \bar{\Gamma}_k^{(n)}$ in $\tilde{E}'$. Taking into consideration that x is in the intersection of $\tilde{E}_n'$ with the carrier of $\limsup_k((w_k^{(n)})^+ \wedge u)$ and using Lemma 11, we obtain that $x \in \bar{\Gamma}^{(n)}$.

By Proposition 8, $\psi_x\bar{u} \geq \langle \eta_n, x\rangle$. We have therefore proved that $\psi_x\bar{u} \geq \langle \eta_n, x\rangle$ for every $n \in \mathbf{N} \cup \{0\}$ and for every $\eta_n \in E$, $\eta_n \geq 0$, $\bar{v} \xrightarrow{n} \eta_n$; it follows that $\psi_x\bar{u} \geq \psi_x\bar{v}$. □

4. Modifications, the filling scheme, and two ratio ergodic theorems. Our goal in this section is to use the results obtained so far in order to prove Theorem 1 and Theorem 2 of Introduction.

We start with two lemmas. The setting in which the lemmas are stated is the one described in the previous section. Thus, we assume given an Archimedean Riesz space E, a positive linear operator $T : E \to E$; we denote by $\tilde{E}$ the Dedekind completion of E, and we denote by $\tilde{E}'$ the order dual of $\tilde{E}$.

Lemma 13. *Let $u \in E$, $u \geq 0$, and let $n \in \mathbf{N} \cup \{0\}$. Let $v \in E$, $v \geq 0$, be such that $u \overset{n}{\to} v$. Then v is an H_u-modification of u.*

Proof. We will prove the lemma by induction on n.

The lemma is clearly true for $n = 0$.

Now assume that the lemma is true for $n-1$, and let $v \in E$, $v \geq 0$, be such that $u \overset{n}{\to} v$. Then there exists $w \in E$, $w \geq 0$, such that $u \overset{n-1}{\to} w$ and $w \overset{1}{\to} v$. By our induction hypothesis, w is an H_u-modification of u. Thus, $w = u - h + Th$ for some $h \in H_u$, $h \geq 0$. Since H_u is T-invariant, it follows that $Th \in H_u$; therefore, $w \in H_u$.

Taking into consideration that $w \overset{1}{\to} v$, we obtain that there exist $r, s \in E$, $r \geq 0$, $s \geq 0$, such that $w = r + s$ and $v = r + Ts$; hence, $v = w - s + Ts$.

Clearly, $s \in H_u$ (since $0 \leq s \leq w$, $w \in H_u$ and since H_u is an ideal). Thus, it follows that v is an H_u-modification of w. Since w is an H_u-modification of u, we obtain that v is an H_u-modification of u. □

Lemma 14. *Let $u \in E$, $u \geq 0$, and let $x \in \tilde{E}'$, $x \geq 0$. If $\sup\{\langle \bar{u}, x\rangle \mid \bar{u}$ is an H_u-modification of $u\} < +\infty$, then $\psi_x u < +\infty$.*

Proof. Set $\alpha = \sup\{\langle \bar{u}, x\rangle \mid \bar{u}$ is an H_u-modification of $u\}$ and assume that $\psi_x u = +\infty$.

It follows that there exists $n \in \mathbf{N}$ such that $\psi_x^n u > \alpha + 1$ (actually, there exists $m \in \mathbf{N}$ such that $\psi_x^n u > \alpha + 1$ for every $n \in \mathbf{N}$,

$n \geq m$). Accordingly, there exists $v \in E$, $v \geq 0$, such that $u \overset{n}{\to} v$ and $\langle v, x \rangle > \alpha + 1$. By Lemma 13, v is an H_u-modification of u; therefore, $\langle v, x \rangle \leq \alpha$. We have obtained a contradiction which stems from the assumption that $\psi_x u = +\infty$; hence $\psi_x u < +\infty$. □

Observation. Note that the conclusion of Lemma 14 can be strengthened considerably, namely, one an prove that the following inequality is true: $\psi_x u \leq \sup\{\langle \bar{u}, x \rangle \mid \bar{u} \text{ is an } H_u\text{-modification of } u\}$. The above inequality can be proved by using arguments similar to the ones used in the proof of Lemma 14 or by noticing that, in view of Lemma 13, it follows that $\{v \in E \mid v \geq 0, u \overset{n}{\to} v \text{ for some } n \in \mathbf{N} \cup \{0\}\} \subseteq \{v \in E \mid v \text{ is an } H_u\text{-modification of } u\}$. However, for our purposes, the assertion of Lemma 14 is good enough.

From now on in the remainder of this section we will assume given the complete setting described in Introduction in order to state Theorem 1 and Theorem 2, and we will use the notations established there.

Everything done so far enables us to prove Theorem 1.

Proof of Theorem 1. Let $u \in B_d$, $u \neq 0$. Clearly, it is enough to prove the theorem under the assumption that $u \geq 0$.

Let B_∞ and B_{0S} be the projection bands in $\tilde{E}$ defined in [**8**]. Then it ws shown in [**8**] that B_d is the projection band in $\tilde{E}$ generated by $B_\infty \cup B_{0S}$. Accordingly, it follows (as in the proof of Theorem 3 of [**9**]) that given $u \in B_d$, $u \geq 0$, $u \neq 0$, there exists $u' \in B_\infty \cup B_{0S}$, $0 \leq u' \leq u$, $u' \neq 0$. Thus, it is enough to prove the theorem in the following two cases:

(α) $u \in B_\infty$;

(β) $u \in B_{0S}$.

(α) Our goal, in this case, is to prove that if (1) does not hold, then (2) has to be true.

To this end, let $u \in B_\infty$, $u \geq 0$, $u \neq 0$, and assume that (1) does not hold, that is, assume that $\Sigma_F(f, T, \Gamma(v, g, T)) \neq 0$ for every $v \in B(u)$, $v \geq 0$, $v \neq 0$. In particular, $\Sigma_F(f, T, \Gamma(u, g, T)) \neq 0$.

Taking into consideration that $\Sigma_F(f, T, \Gamma(u, g, T))$ is a (projection) band in $\tilde{E}'$ (by Corollary 5), we obtain that there exists $x' \in$

$\Sigma_F(f,T,\Gamma(u,g,T))$, $x' \geq 0$, $x' \neq 0$.

It follows (in view of the definition of $\Sigma_F(f,T,\Gamma(u,g,T)))$ that there exists $x \in \tilde{E}'$, $0 \leq x \leq x'$, $x \neq 0$, such that $0 < \sup\{\langle \bar{f}, x\rangle \mid \bar{f}$ is an H_f-modification of $f\} < +\infty$. Using Lemma 14 we obtain that $\psi_x f < +\infty$.

We will now prove that $\psi_x g > 0$. To this end, note that (since we assume that $u \in B_\infty \subseteq B$, $u \geq 0$) it follows that $u = \sup A$ provided that we set $A = \{u \wedge (l(\sum_{i=0}^k T^i g)) \mid l \in \mathbf{N}, k \in \mathbf{N} \cup \{0\}\}$. Clearly, $x \in \Sigma_F(f,T,\Gamma(u,g,T))$ since $0 \leq x \leq x'$, $x' \in \Sigma_F(f,T,\Gamma(u,g,T))$, and since $\Sigma_F(f,T,\Gamma(u,g,T))$ is a band. Hence, x is an order continuous linear functional on $\tilde{E}$. Taking into consideration that the set A is (increasingly) directed, we obtain that $0 < \langle u, x\rangle = \langle \sup A, x\rangle = \sup_{v\in A}\langle v, x\rangle$; accordingly, there exist $l \in \mathbf{N}$, $k \in \mathbf{N} \cup \{0\}$ such that $0 < l\sum_{i=0}^k \langle T^i g, x\rangle$; hence, $\langle T^j g, x\rangle > 0$ for some $j \in \{0,1,2,\dots,k\}$. Since $g \overset{j}{\to} T^j g$, it follows that $0 < \langle T^j g, x\rangle \leq \psi_x^j g \leq \psi_x g$.

Let $\alpha \in \mathbf{R}$, $\alpha > 0$, and let $\rho \in \mathbf{R}$, $0 < \rho < 1$. Since $u \in B_\infty$, it follows that the sequence $((u_n, v_n))_{n\in\mathbf{N}\cup\{0\}}$ is ratio unbounded on the (projection) band $B(u)$ in $\tilde{E}$ (recall that $u_n = \sum_{k=0}^n T^k f$, $v_n = \sum_{k=0}^n T^k g$ for every $n \in \mathbf{N} \cup \{0\}$). Thus, using the definition of ratio unboundedness on a band (see [**8**]) we obtain, in particular, that the sequence

$$\left(\left(u_n - \frac{\alpha}{1-\rho}v_n\right)^+\right)_{n\in\mathbf{N}\cup\{0\}}$$

is unbounded on $B(u)$. It follows that the sequence $(((1-\rho)u_n - \alpha v_n)^+)_{n\in\mathbf{N}\cup\{0\}}$ is unbounded on $B(u)$, as well (since

$$\begin{aligned}\sup_n(((1-\rho)u_n - \alpha v_n)^+ \wedge v) &= \sup_n\left(\left((1-\rho)\left(u_n - \frac{\alpha}{1-\rho}v_n\right)^+\right)\wedge v\right)\\ &= (1-\rho)\sup_n\left(\left(u_n - \frac{\alpha}{1-\rho}v_n\right)^+ \wedge\left(\frac{1}{1-\rho}v\right)\right)\\ &= (1-\rho)\cdot\frac{1}{1-\rho}v = v\end{aligned}$$

for every $v \in B(u)$, $v \geq 0$). Using Lemma 4 of [**8**] we obtain that the sequence $(((1-\rho)u_n - \alpha v_n)^+)_{n\geq k}$ is unbounded on $B(u)$ for every $k \in \mathbf{N} \cup \{0\}$; thus, $\sup_{n\geq k}((((1-\rho)u_n - \alpha v_n)^+) \wedge u) = u$ for every $k \in \mathbf{N} \cup \{0\}$; hence $\limsup_n((((1-\rho)u_n - \alpha v_n)^+) \wedge u) =$

$\inf_{k\in\mathbf{N}\cup\{0\}}\sup_{n\geq k}((((1-\rho)u_n-\alpha v_n)^+)\wedge u)=u$. Accordingly, we conclude that x is in the carrier of $\limsup_n((((1-\rho)u_n-\alpha v_n)^+)\wedge u)$.

Now assume that condition (2) of our theorem is not satisfied. Since $(\omega_k)_{k\in\mathbf{N}}$ is a sequence of H_{f+g}-modifications of g if and only if $(\alpha\omega_k)_{k\in\mathbf{N}}$ is a sequence of H_{f+g}-modifications of αg, and since the sequences $(\omega_k)_{k\in\mathbf{N}}$ and $(\alpha\omega_k)_{k\in\mathbf{N}}$ diverge individually to ∞ on the same projection bands in $\tilde{E}$, it follows that we may apply Proposition 12 (the role played there by $\bar{u}$ and $\bar{v}$ is taken here by f and αg, respectively) in order to conclude that $\psi_x f\geq\psi_x(\alpha g)$.

Taking into consideration that $g\xrightarrow{n}h$ if and only if $\alpha g\xrightarrow{n}\alpha h$ for every $n\in\mathbf{N}\cup\{0\}$, we obtain that $\psi_x(\alpha g)=\alpha\psi_x g$.

We have therefore proved that, if we assume that condition (2) is not satisfied, then it follows that $\psi_x f\geq\alpha\psi_x g$ for every $\alpha\in\mathbf{R}$, $\alpha>0$. Since $\psi_x f<+\infty$, it follows that $\psi_x g=0$. Thus, we have obtained a contradiction. Accordingly, the theorem is true for every $u\in B_\infty$.

(β) Let $u\in B_{0S}$, $u\geq 0$, $u\neq 0$, and assume that condition (1) is not satisfied. Our goal is to prove that (2) or (3) holds.

Since we assume that (1) is not satisfied, it follows that $\Sigma_F(f,T,\Gamma(u,g,T))\neq 0$. As in case ($\alpha$), it follows that there exists $x\in\Sigma_F(f,T,\Gamma(u,g,T))$, $x\geq 0$, $x\neq 0$, such that $0<\sup\{\langle\bar{f},x\rangle\mid\bar{f}$ is an H_f-modification of $f\}<+\infty$. Using Lemma 14 (as in (α)) we obtain that $\psi_x f<+\infty$.

In case (α) we proved that $\psi_x g>0$; here we need a slightly stronger statement which is valid in case (α), as well, namely, we will prove that $\psi_y g>0$ for every $y\in\tilde{E}'$, $0\leq y\leq x$, $y\neq 0$. Thus, let $y\in\tilde{E}'$, $0\leq y\leq x$, $y\neq 0$. As in case (α), set $A=\{u\wedge(l(\sum_{i=0}^k T^i g))\mid l\in\mathbf{N}, k\in\mathbf{N}\cup\{0\}\}$, note that the set A is (increasingly) directed and note that $\sup A=u$. Taking into consideration that y is an order continuous linear functional on $\tilde{E}$, we obtain, as in case (α), that there exists $j\in\mathbf{N}\cup\{0\}$ such that $\langle T^j g,y\rangle>0$. Since $g\xrightarrow{j}T^j g$, it follows that $0<\langle T^j g,y\rangle\leq\psi_y^j g\leq\psi_y g$.

Clearly, $\langle u,x\rangle\neq 0$ (since $x\in\Sigma_F(f,T,\Gamma(u,g,T))\subseteq\Gamma(u,g,T)$, $x\geq 0$, $x\neq 0$). Taking into consideration that x is a nonzero order continuous linear functional on $\tilde{E}$ (therefore, the carrier of x is a nonzero projection band in $\tilde{E}$, and $\tilde{E}$ is the order direct sum of the null space and the carrier of x), we obtain that the projection of u on the carrier of x is a nonzero component u' of u.

Since $u \in B_{0S}$, $0 \leq u' \leq u$, $u' \neq 0$, it follows (in view of the definition of the band B_{0S} (see [**8**])) that there exist $\alpha, \beta \in \mathbf{R}$, $0 < \beta < \alpha$ such that $(\limsup_n(((u_n - \beta v_n)^-) \wedge u')) \wedge (\limsup_n(((u_n - \alpha v_n)^+) \wedge u')) \neq 0$.

Let $\rho \in \mathbf{R}$, $0 < \rho < 1$, be small enough such that $\beta < (1-\rho)^2\alpha$.

Set $w_1 = \limsup_n((((1-\rho)u_n - (1-\rho)\beta v_n)^-) \wedge ((1-\rho)u'))$, $w_2 = \limsup_n((((1-\rho)u_n - (1-\rho)\alpha v_n)^+) \wedge ((1-\rho)u'))$, and $w = w_1 \wedge w_2$.

It follows that $0 \leq w \leq w_i \leq u' \leq u$, $i = 1, 2$. It also follows that $w \neq 0$ since $0 \neq (1-\rho)((\limsup_n(((u_n - \beta v_n)^-) \wedge u')) \wedge (\limsup_n(((u_n - \alpha v_n)^+) \wedge u'))) = w$.

Clearly, we may think of w as an element of $\tilde{E}''$ and, as such, w is an order continuous linear functional on $\tilde{E}'$. Let y be the projection of x on the carrier of w in $\tilde{E}'$. Since $0 \leq w \leq u'$, $w \neq 0$, and since u' is in the carrier (in $\tilde{E}$) of x, it follows that $\langle w, x \rangle \neq 0$; therefore, $y \neq 0$. If we think of w_1 and w_2 as elements of $\tilde{E}''$, and if we take into consideration that $0 \leq w \leq w_i$, $i = 1, 2$, then it follows that y is in both the carrier of w_1 and of w_2.

Now assume that neither condition (2) nor condition (3) are true.

Note that $w_1 = \limsup_n((((1-\rho)\beta v_n - (1-\rho)u_n)^+) \wedge ((1-\rho)u'))$. Note also that $H_{\beta g + (1-\rho)f} = H_{f+g}$; therefore, $(1-\rho)\bar{f}$ is an $H_{\beta g+(1-\rho)f}$-modification of $(1-\rho)f$, if and only if $\bar{f}$ is an H_{f+g}-modification of f. The above remarks allow us to conclude (since we assume that condition (3) is not satisfied) that we may apply Proposition 12, the role played by $\bar{u}, \bar{v}, u$, and x there being taken here by βg, $(1-\rho)f$, $(1-\rho)u'$, and y, respectively. Accordingly, we obtain that $\psi_y(\beta g) \geq \psi_y((1-\rho)f)$, that is,

$$\beta\psi_y g \geq (1-\rho)\psi_y f. \tag{1.1}$$

Now note that $H_{f+(1-\rho)\alpha g} = H_{f+g}$; consequently, $(1-\rho)\alpha\bar{g}$ is an $H_{f+(1-\rho)\alpha g}$-modification of $(1-\rho)\alpha g$ if and only if $\bar{g}$ is an H_{f+g}-modification of g. Thus, it follows (since we assume that condition (2) is not satisfied) that we may apply Proposition 12 once again, the role played by $\bar{u}, \bar{v}, u$, and x there being taken here by f, $(1-\rho)\alpha g$, $(1-\rho)u'$, and y, respectively. Accordingly, it follows that $\psi_y f \geq \psi_y((1-\rho)\alpha g)$, that is,

$$\psi_y f \geq (1-\rho)\alpha\psi_y g. \tag{1.2}$$

Using (1.1) and (1.2) we obtain that $\beta\psi_y g \geq (1-\rho)\psi_y f \geq (1-\rho)^2\alpha\psi_y g$.

Since $\psi_y g > 0$, it follows that $\beta \geq (1-\rho)^2\alpha$. Thus, we have obtained a contradiction. Accordingly, the theorem is true whenever $u \in B_{0S}$, $u \geq 0$, $u \neq 0$. □

Our intention now is to prove Theorem 2. For the proof we need the following proposition:

Proposition 15. *Let $v \in B$, $v \neq 0$. If there exists a sequence $(\rho_n)_{n\in\mathbf{N}}$ of H_{f+g}-modifications of f such that $(\rho_n)_{n\in\mathbf{N}}$ diverges individually to ∞ on $B(v)$, then*

$$\Omega_\infty(f, T, \Gamma(v, g, T)) = \Gamma(v, g, T).$$

Proof. Clearly, we may assume that $v \geq 0$.

Obviously, the proof of the proposition is completed if we prove that $a_x = +\infty$ for every $x \in \Gamma(v, g, T)$, $x \geq 0$, $x \neq 0$.

To this end, let $x \in \Gamma(v, g, T)$, $x \geq 0$, $x \neq 0$. Let $\alpha \in \mathbf{R}$, $\alpha > 0$, and set

$$\mathcal{V}_\alpha = \{w \in \tilde{E} \mid w \text{ is a component of } v \text{ and there exists } m \in \mathbf{N} \\ \text{such that } \alpha w \leq \rho_n \text{ for every } n \in \mathbf{N}, n \geq m\}.$$

Note that $\mathcal{V}_\alpha$ (endowed with the order induced by the order of $\tilde{E}$) is a directed set. Indeed, let $w_1, w_2 \in \mathcal{V}_\alpha$. It follows that there exist $m_1, m_2 \in \mathbf{N}$ such that $\alpha w_i \leq \rho_n$ for every $n \in \mathbf{N}$, $n \geq m_i$, $i = 1, 2$. Therefore, $\alpha(w_1 \vee w_2) = (\alpha w_1) \vee (\alpha w_2) \leq \rho_n$ for every $n \in \mathbf{N}$, $n \geq \max\{m_1, m_2\}$. Thus, $w_1 \vee w_2 \in \mathcal{V}_\alpha$.

Clearly, $\sup \mathcal{V}_\alpha$ exists and is a component of v in $\tilde{E}$ (since $\tilde{E}$ is order complete).

We will now prove that $v = \sup \mathcal{V}_\alpha$. To this end, assume that $v \neq \sup \mathcal{V}_\alpha$. Then $v - \sup \mathcal{V}_\alpha$ is a nonzero component of v. Since $(\rho_n)_{n\in\mathbf{N}}$ diverges individually to ∞ on $B(v)$, it follows that there exist a nonzero component w of $v - \sup \mathcal{V}_\alpha$ and $m \in \mathbf{N}$ such that $\alpha w \leq \rho_n$ for every $n \geq m$. Thus, $w \in \mathcal{V}_\alpha$. We have obtained a contradiction since, on one hand, $w \neq 0$, while, on the other hand, $0 \leq w = w \wedge \sup \mathcal{V}_\alpha \leq (v - \sup \mathcal{V}_\alpha) \wedge \sup \mathcal{V}_\alpha = 0$.

Taking into consideration that x is an order continuous linear functional on $\tilde{E}$, we obtain that $\sup\{\langle \bar{f}, x\rangle \mid \bar{f}$ is an H_{f+g}-modification of $f\} \geq \sup_{n\in\mathbf{N}} \langle \rho_n, x\rangle \geq \sup_{w\in\mathcal{V}_\alpha} \langle \alpha w, x\rangle = \alpha \sup_{w\in\mathcal{V}_\alpha} \langle w, x\rangle = \alpha\langle v, x\rangle$.

We have therefore proved that $a_x \geq \alpha\langle v, x\rangle$ for every $\alpha \in \mathbf{R}$, $\alpha > 0$. Since $\langle v, x\rangle > 0$, it follows that $a_x = +\infty$. □

In view of Proposition 15 we may now proceed to the proof of Theorem 2.

Proof of Theorem 2. As in the proof of Theorem 1, it follows that we may assume that $u \in B_d$, $u \geq 0$, $u \neq 0$, and that it is enough to prove the theorem in the following two cases:

(a) $u \in B_\infty$;

(b) $u \in B_{0S}$.

(a) Let $u \in B_\infty$, $u \geq 0$, $u \neq 0$, and assume that (i) is not true. It follows that $\Omega_F(f, T, \Gamma(u, g, T)) \neq 0$; hence, there exists $x \in \Omega_F(f, T, \Gamma(u, g, T))$, $x \geq 0$, $x \neq 0$ such that $0 < \sup\{\langle \bar{f}, x\rangle \mid \bar{f}$ is an H_{f+g}-modification of $f\} < +\infty$.

We obtain that $\sup\{\langle \bar{f}, x\rangle \mid \bar{f}$ is an H_f-modification of $f\} < +\infty$. Accordingly, using Lemma 14 it follows that $\psi_x f < +\infty$.

As in the proof of Theorem 1, it also follows that $\psi_x g > 0$.

Now, if we assume that assertion (ii) is not true, it follows, as in case (α) of Theorem 1, that we obtain a contradiction (note that assertions (ii) of Theorem 2 and (2) of Theorem 1 coincide).

(b) Let $u \in B_{0S}$, $u \geq 0$, $u \neq 0$, and assume that assertion (i) is not true. Then, as in case (a) it follows that there exists $x \in \Omega_F(f, T, \Gamma(u, g, T))$, $x \geq 0$, $x \neq 0$, such that $0 < \sup\{\langle \bar{f}, x\rangle \mid \bar{f}$ is an H_{f+g}-modification of $f\} < +\infty$.

We obtain as in case (a) using Lemma 14 that $\psi_x f < +\infty$.

Now assume that both assertions (i) and (ii) are not true. It follows that (2) and (3) of Theorem 1 are not true, either (indeed, assertions (ii) of Theorem 2 and (2) of Theorem 1 coincide; by Proposition 15, the fact that (i) of Theorem 2 is not true implies that (3) of Theorem 1 is not true, either). Using arguments completely similar to the ones offered in the proof of case (β) of Theorem 1 we obtain a contradiction. □

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Department of Mathematical Sciences, State University of New York at Binghamton, Binghamton, New York 13902-6000