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REES RINGS AND DERIVATIONS

SILVIA MOLINELLI AND GRAZIA TAMONE

ABSTRACT. Let A be a ring, $\{I_n\}$ a filtration of ideals of A and $R = \oplus I_n T^n$ (contained in $A[T]$) the Rees ring associated with $\{I_n\}$. We study the derivations D of $A[T]$ such that $D(A) \subset A$ and $D(R) \subset R$.

Introduction. Let A be a noetherian ring and $\{I_n\}_{n\in\mathbf{Z}}$ a filtration of ideals of A. Let $R = \oplus_{n\geq 0} I_n T^n$ (respectively $R' = \oplus_{n\in\mathbf{Z}} I_n T^n$) be the Rees ring associated with $\{I_n\}$ for $n \geq 0$ (respectively for $n \in \mathbf{Z}$). One can remark that $R \subset A[T]$ and when $F = \{I^n\}$ (where I is an ideal of A) then R is the well-known "Rees algebra."

In this paper we first consider derivations D of the polynomial ring $A[T]$ such that $D(A) \subset A$, and we determine several conditions on $D(T)$ and $D(I_n)$ in order that $D(R) \subset R$ and $D(R') \subset R'$. In particular, we discuss five filtrations, namely $\{I^n\}$, $\{I^{(n)}\}$, $\{I^n : \langle J\rangle\}$, $\{(I^n)_\Delta\}$, $\{(I^n)_a\}$ (see definitions 1.4, 1.6, 1.8, 1.9).

In Section 2 we consider the Rees rings associated to the previous five filtrations. If $D \in \operatorname{Der}(A[T])$ is a derivation of one of these rings, we wonder on which of the others D is also a derivation. We give several implications and show some examples of implications which do not hold.

Further, if A is a noetherian domain containing a field of characteristic zero, for any filtration $\{I_n\}$ in A we show that each $D \in \operatorname{Der}(A[T])$ such that $D(R) \subset R$ is also a derivation of the Rees rings associated respectively to $\{\bar{I}_n\}$ and $\{(I_n)_a\}$ (where $\bar{I}_n$ (respectively, $(I_n)_a$) is the integral closure of I_n in $\bar{A}$ (respectively in A), see definition 1.9).

We recall that several properties of R have been studied in some cases. For example, when $F = \{I^{(n)}\}$ (I prime, i.e., R is the "symbolic Rees algebra"), many authors have studied when R is Noetherian, Gorenstein, Cohen-Macaulay (see [**1, 2, 3, 4**]). Further, in [**12**] there are some finiteness results related to certain filtrations.

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Finally, we remark that the problem studied in this paper can be seen as a particular aspect of the following general question: if A, B, C are rings such that $A \subset C \subset B$ and $D \in \mathrm{Der}\,(B)$ is such that $D(A) \subset A$, under what conditions does one have $D(C) \subset C$? The question has been studied by the authors in some cases (see, e.g., [**5, 6, 7**]).

1. Let A be a noetherian ring and $F = \{I_n\}$ be a filtration of ideals of A, i.e., a sequence $\{I_n\}$ of ideals such that $I_0 = A$, $I_n \subset I_{n-1}$ for all $n \geq 1$ and $I_m I_n \subset I_{m+n}$ for all $m, n \in \mathbf{N}$. Further, let $F' = \{I_n, n \in \mathbf{Z}\}$ where $I_n = A$ for $n < 0$ and I_n is as before for $n \geq 0$. From now on let $R = \oplus_{n \geq 0} I_n T^n$ (contained in $A[T]$) and $R' = \oplus_{n \in \mathbf{Z}} I_n T^n$ (contained in $A[T, T^{-1}]$) be the associated Rees rings with respect to F and F'.

Further, for each ring B, we let $\mathrm{Der}\,(B)$ denote the B-module of all the derivations of B.

Now, let $D \in \mathrm{Der}\,(A[T])$ be such that $D(A) \subset A$. First, we find necessary and sufficient conditions in order that $D(R) \subset R$.

Proposition 1.1. *Let A, $F = \{I_n\}$, R as before, $D \in \mathrm{Der}\,(A[T])$ such that $D(A) \subset A$ and $D(T) = \Sigma_j q_j T^j$ (for $j = 0, \dots, r$). Then $D(R) \subset R$ if and only if the following conditions hold*:

I) $D(I_n) \subset I_n$ *for each* $n \geq 1$;

II) $(k - j + 1) q_j \in I_k : I_{k-j+1}$ *for each* (k, j) *such that* $1 \leq j \leq k$.

Proof. According to the assumptions on D, *for each* $f(T) = \Sigma_i p_i T^i \in R$ one has:

$$D(f(T)) = \Sigma_k [D(p_k) + \Sigma_{i+j=k} (i+1) p_{i+1} q_j] T^k.$$

Then $D(R) \subset R$ if and only if the following condition holds:

$$D(p_k) + p_1 q_k + 2 p_2 q_{k-1} + \cdots + (k+1) p_{k+1} q_0 \in I_k \tag{1}$$

for each $k \geq 1$ and each $p_h \in I_h$.

Now we show that condition (1) is equivalent to I) and II). In fact, if (1) holds, one has, in particular, (taking $p_1 = p_2 = \cdots = p_{k-1} = 0$): $D(I_k) \subset I_k$ for each $k \geq 1$. Further, for each (j, k) such that $1 \leq j \leq k$ one has (putting $p_h = 0$ for $h \neq k - j + 1$): $(k - j + 1) p_{k-j+1} q_j \in I_k$, i.e., $(k-j+1)q_j \in I_k : I_{k-j+1}$. On the other hand, conditions I) and II) obviously imply (1). □

Corollary 1.2. *Under the same notation as in Proposition* 1.1*, one has*:

(i) *if A contains a field of characteristic zero, then condition* II) *of Proposition* 1.1 *is equivalent to*:

$$\text{II}')\quad q_j \in \bigcap_{k\geq j}(I_k : I_{k-j+1}) \qquad \textit{for all } j \geq 1.$$

(ii) *if the filtration $F = \{I_n\}$ is such that*:

$$(*)\qquad I_j : I_1 \subset I_{j+1} : I_{1+i} \qquad \textit{for all } i \geq 0$$

then the condition: $\text{II}'')$ $q_j \in I_j : I_1$ *for all $j \geq 1$, is equivalent to* II) *of Proposition* 1.1.

Proof. (i) holds because $k-j+1$ is a unit for $j \leq k$.

(ii). If II) holds, for $k = j$ one has: $q_j \in I_j : I_1$ for all $j \geq 1$, i.e., $\text{II}'')$. On the other hand, if $\text{II}'')$ and $(*)$ hold, one has immediately II). □

Remark 1.3. In general, a filtration $F = \{I_n\}$ does not satisfy $(*)$. For example, let $A = k[X,Y]$ (k field), $F = \{I_n\}$ where $I_0 = A$, $I_n = (Y^n) \cap (XY^2)$ for each $n \geq 1$. One has: $I_1 = (XY^2) = I_2$, so that $I_2 : I_1 = A$. On the other hand, $I_3 : I_2 \neq A$, since $XY^2 \notin I_3$.

Nevertheless, we can show that some filtrations $\{I_n\}$ satisfy the property $(*)$ defined in Corollary 1.2. We need some definitions.

Definition 1.4. Let A be a noetherian ring, I an ideal of A, $M = A\backslash(\cup\wp)$, where $\wp \in \{\text{minimal prime ideals in } \mathrm{Ass}\,(A/I)\}$. The ideal $I^{(n)} = I^n A_M \cap A$ is called "*the n-th symbolic power of I*."

Remark 1.5. Let I be as in Definition 1.4, let $n \geq 1$, and $I^n = Q_1 \cap \cdots \cap Q_k \cap Q_{k+1} \cap \cdots \cap Q_s$ be a reduced primary decomposition of I^n, where $Q_1, \dots, Q_k$ are the isolated primary components of I^n. Then $I^{(n)} = Q_1 \cap \dots \cap Q_k$. In particular, if $I = \wp$ is a prime ideal, one has $\wp^{(n)} = \wp^n A_\wp \cap A$ for each $n \geq 1$.

Definition 1.6. Let A be a noetherian ring, I, J ideals of A. For each $n \geq 1$, we put $I^n : \langle J \rangle = \{x \in A \mid xJ^k \subset I^n \text{ for some } k \geq 1\}$.

Remark 1.7. The filtration considered in Definition 1.6 has been studied in [**9, 10, 12**]. If $I^n = Q_1 \cap \dots \cap Q_r \cap Q'_1 \cap \dots \cap Q'_s$ is a reduced primary decomposition of I^n, where $\sqrt{Q_j} \not\supset J$ for $j = 1, \dots, r$ and $\sqrt{Q'_i} \supset J$ for $i = 1, \dots, s$, putting $S = \cap_j (A \backslash \sqrt{Q_j})$ (for $j = 1, \dots, r$), one has $I^n : \langle J \rangle = I^n A_S \cap A = Q_1 \cap \dots \cap Q_r$, as one can easily see.

Definition 1.8. Let A be a ring, I an ideal of A, Δ a multiplicatively closed set of nonzero ideals of A. The ideal $I_\Delta = \cup_{K \in \Delta} \{IK : K\}$ is called the "Δ-*closure of* I" (see [**11**, Theorem 2.1]).

If I, Δ are as in Definition 1.8, one can see that $\{(I^n)_\Delta\}$, $n \geq 0$, is a filtration [**11**, Theorems (2.4), (2.4.2), (2.4.4)].

Definition 1.9. Let A, A' be rings such that $A \subset A'$, and let I be an ideal of A. The set $\{x \in A' \mid x^k + \alpha_1 x^{k-1} + \dots + \alpha_i x^{k-i} + \dots + \alpha_k = 0$ for some $k \geq 1$, $\alpha_i \in I^i (i = 1, \dots, k)\}$ is called the *integral closure of* I *in* A'; we let I_a denote it when $A' = A$ and $\bar{I}$ denote it if A' is the integral closure $\overline{A}$ of A (see, e.g., [**8**, example 3 p. 34]).

Remark 1.10. 1) If Δ is the set of all the ideals of A that are not contained in any minimal prime ideal of A, then $I_\Delta = I_a$ for each ideal $I \in \Delta$ (see [**11**, Theorem (3.2.3)]).

2) It is well known that the integral closure of I in A' is an ideal of the integral closure of A in A'; in particular, one can see that $I_a \subset \sqrt{I}$ when $A = A'$.

From now on, we shall put (for an ideal I of A): $F_p = \{I^n\}$, $F_s = \{I^{(n)}\}$, $\langle F \rangle = \{I^n : \langle J \rangle\}$ (which depends on the fixed ideal J of A), $F_\Delta = \{(I^n)_\Delta\}$ (for each Δ as in Definition 1.8), $F_a = \{(I^n)_a\}$

for $n \geq 0$.

Lemma 1.11. *Let A be a noetherian ring, I and J ideals of A, Δ as in Definition* 1.8. *The filtrations $F_p, F_s, \langle F \rangle$ and* (*when $I \in \Delta$*) *F_Δ satisfy condition* $(*)$ *of Corollary* 1.2. *In particular, F_a satisfies condition* $(*)$ *when* $\operatorname{ht}(I) > 0$.

Proof. 1) If $F = F_p$, the proof is trivial.

2) As regards F_s, the result $I^{(j)} : I^{(1)} \subset I^{(j+i)} : I^{(1+i)}$ for each $i \geq 0$ and $j \geq 1$ follows from 1), after noting that $I^{(a)} : I^{(b)} = (I^a A_M : I^b A_M) \cap A = ((IA_M)^a : (IA_M)^b) \cap A$ for each $a, b \geq 0$.

3) For the filtration $\langle F \rangle$, according to Remark 1.7, the proof is similar to the one of 2), if A_M (respectively, $I^{(h)}$) is replaced by A_S (respectively by $I^h : \langle J \rangle$) for each h.

4) As concerns the filtration F_Δ, first we show: $(I^j)_\Delta : I_\Delta \subset (I^{j-1})_\Delta$, for each $j \geq 1$, where $I \in \Delta$. In fact, let $x \in (I^j)_\Delta : I_\Delta$; then there is a $K \in \Delta$ such that $xI_\Delta K \subset I^j K$, so in particular $x(IK) \subset I^j K$ (since $I \subset I_\Delta$, see [**11**, Theorems (2.4), (2.4.1)]) $= I^{j-1}(IK)$, where $IK \in \Delta$ according to the assumptions on I and Δ, so that $x \in (I^{j-1})_\Delta$. It follows that $(I^j)_\Delta : (I)_\Delta \subset (I^{j-1})_\Delta \subset$ (according to [**11**, Theorem (2.4.4)]) $\subset (I^{j+i})_\Delta : (I^{i+1})_\Delta$. □

In particular, if Δ is as in Remark 1.10 1) and ht $(I) > 0$, then $F_a = F_\Delta$ satisfies condition $(*)$.

Now our aim is to characterize the condition $D(R) \subset R$ when R is the associated Rees ring with respect to the filtrations considered in Lemma 1.11.

Lemma 1.12. *Let A be a noetherian ring, I, J ideals of A, $F = \{I_n\}$ a filtration of ideals of A, D a derivation of A. If $F = F_p$* (*or F_s, or $\langle F \rangle$*), *then condition* I) *in Proposition* 1.1 *is equivalent to*:

I′) $D(I_1) \subset I_1$.

Proof. 1) If $F = F_p$, the proof is trivial.

2) Let $F = F_s$. We suppose $D(I^{(1)}) \subset I^{(1)}$; then, if M is as in

Definition 1.4, one has: $D(IA_M) \subset IA_M$, so $D(I^n A_M) \subset I^n A_M$, which implies that $D[(I^n A_M) \cap A] \subset I^n A_M \cap A$.

3) Let $F = \langle F \rangle$. Since $I^n : \langle J \rangle = I^n A_S \cap A$ (see Remark 1.7), we can proceed as in the proof of 2). $\square$

Remark 1.13. In general condition I$'$) does not imply condition I). Let us consider the filtration F_a; we exhibit a ring A, a derivation $D \in \mathrm{Der}\,(A)$ and an ideal I of A such that $D(I_a) \subset I_a$ but $D((I^n)_a) \not\subset (I^n)_a$ for some $n > 1$. Let $A = k[X, Y]/(Y^p - X^{p \cdot p}(1 + X)) = k[x, y]$ where k is a field and $\mathrm{ch}\,(k) = p$, $I = (x)$, D the derivation of A induced by $\widetilde{D} = X\partial/\partial Y \in \mathrm{Der}\,(k[X, Y])$. We can see that $I_a = (x, y)$ and $y \in (I^p)_a$, since $y^p = (x^p)^p(1 + x)$. One has obviously: $D(I_a) \subset I_a$. On the other hand, $D((I^p)_a) \not\subset (I^p)_a$, since $D(y) = x \notin (I^p)_a$, otherwise in $k[x, y] = A$ one has:

$$x^r + \alpha_1 x^{r-1} + \cdots + \alpha_i x^{r-i} + \cdots + \alpha_r = 0$$

for some $r \geq 1$ (with $\alpha_i = \beta_i x^{pi}$, $\beta_i \in A$), then $x^r(1 + \beta_1 x^{p-1} + \cdots + \beta_r x^{pr-r}) = 0$. It follows that $x^r = 0$ in $k[x, y]_{(x,y)}$, a contradiction, since x is a parameter in $k[x, y]_{(x,y)}$.

From Proposition 1.1, Corollary 1.2, Lemma 1.11 and Lemma 1.12 it follows:

Corollary 1.14. *Let A, F, R, D be as in Proposition* 1.1 *and let I, J be ideals of A.*

a) *If $F = F_p$ (or F_s, or $\langle F \rangle$), then $D(R) \subset R$ if and only if $D(I_1) \subset I_1$ and $q_j \in I_j : I_1$ for all $j \geq 1$.*

b) *If $F = F_\Delta$ with $I \in \Delta$ (in particular, if $F = F_a$ with* $\mathrm{ht}\,(I) > 0$*), then $D(R) \subset R$ if and only if $D(I_n) \subset I_n$ for all $n \geq 1$, and $q_j \in I_j : I_1$ for $j \geq 1$.*

Now we wonder when one has $D(R') \subset R'$ (where $R' = \oplus_{n \in \mathbf{Z}} I^n T^n$ and $D \in \mathrm{Der}\,(A[T])$ is such that $D(A) \subset A$, $D(T) = \Sigma_j q_j T^j$ for $j = 0, \ldots, r$). First we note the following facts:

Lemma 1.15. *Let R, R', D be as above. If $D(R') \subset R'$, then $D(R) \subset R$.*

Remark 1.16. In general, the converse of Lemma 1.15 is not true. We show the following examples.

1) Let $A = k[x,y]$ (k field) where $x^3 = xy$, and let $D \in \mathrm{Der}\,(A[T])$ be such that $D(x) = x$, $D(y) = 2y$, $D(T) = yT^3$. Further, let $F = \{I^n\}_{n\in\mathbf{Z}}$ where $I = (x)$. It is easy to see that $D(A) \subset A$, $D(I) \subset I$, and condition II$''$) of Corollary 1.2 is satisfied. Then $D(R) \subset R$ (Corollary 1.14 a)). On the other hand, $D(R') \not\subset R'$ since $D(T^{-1}) = -D(T)T^{-2} = -yT \notin R'$ because $y \notin I$.

2) Let $A = k[t^5, t^{11}, t^{24}, t^{28}] \subset k[t]$ where k is a field of characteristic zero (here A is an integral domain), $F = \{I^n\}_{n\in\mathbf{Z}}$ where $I = (t^5, t^{11})$, $D = t\partial/\partial t \in \mathrm{Der}\,(A)$ such that $D(T) = t^{28}T^3$. It is easy to verify that $D(R) \subset R$, according to Corollary 1.14 a). On the other hand, one has: $D(T^{-1}) = -t^{28}T \notin R'$ since $t^{28} \notin I$; then $D(R') \not\subset R'$.

In general we can prove the following result.

Proposition 1.17. *Under the same assumptions as in Lemma* 1.15*, the following conditions are equivalent*:

1) $D(R') \subset R'$;

2) $D(R) \subset R$ *and* $q_j \in I_{j-2}$ *for all* $j \geq 3$.

Proof. For each $g(T) = \sum_{k\geq 0} p_k T^k + \Sigma_h a_h T^h$ ($h = -n, \dots, -1$) belonging to R' ($a_h \in A$ for $h < 0$, $p_k \in I_k$ for $k \geq 0$), one has:

$$D(g(T)) = D(\Sigma_k p_k T^k) + \Sigma_h D(a_h)T^h + \Sigma_h h a_h T^{h-1} D(T),\quad h = -n, \dots, -1,$$

where $D(T) = \Sigma_j q_j T^j$, $j = 0, \dots, r$. So 1) is equivalent to: $D(R) \subset R$ and

$$(2)\qquad \Sigma_h h a_h T^{h-1} D(T) \in R', \quad \text{for each } a_h \in A, \quad -n \leq h \leq -1.$$

Now (2) can be written as:

$$\Sigma_{k<0}[\Sigma_{h+j=k}(h+1)a_{h+1}q_j]T^k + \Sigma_{k\geq 0}[\Sigma_{h+j=k}(h+1)a_{h+j}q_j]T^k \in R',$$

which is equivalent to:

$$(3)\quad (-n+1)a_{-n+1}q_{n+k} + (-n+2)a_{-n+2}q_{n+k-1} + \cdots + a_{-1}q_{k+2} \in I_k$$

for each k, $1 \leq k \leq r-2$, and each $a_h \in A$, $-n \leq h \leq -1$. By putting in (3) $a_{-n+1} = \cdots = a_{-2} = 0$ and $a_{-1} = 1$, in particular we obtain $q_{k+2} \in I_k$. On the other hand, it is obvious that the condition "$q_j \in I_{j-2}$" for all $j \geq 3$ implies (3). $\square$

2. From now on, we let R_p (respectively, R_s, $\langle R \rangle$, R_Δ, R_a) denote the associated Rees rings R with respect to $F = F_p$ (respectively $F_s, \langle F \rangle, F_\Delta, F_a$) defined in Section 1 (see definition following Remark 1.10).

Let $D \in \mathrm{Der}\,(A[T])$ be such that $D(A) \subset A$. Let us consider the following conditions:

1) $D(R_p) \subset R_p$;

2) $D(R_s) \subset R_s$;

3) $D(\langle R \rangle) \subset \langle R \rangle$;

4) $D(R_\Delta) \subset R_\Delta$;

5) $D(R_a) \subset R_a$.

We wonder whether there is some connection between condition 1) and each of the other ones. One has:

Proposition 2.1. *Let A be a noetherian ring, $D \in \mathrm{Der}\,(A[T])$ such that $D(A) \subset A$. If condition* 1) *holds, then also conditions* 2) *and* 3) *hold.*

Proof. 1) $\Rightarrow$ 2). According to the assumption 1) and Corollary 1.14 a), we have: $D(I) \subset I$. Then $D(IA_M \cap A) \subset IA_M \cap A$ (where M is as in Definition 1.4), i.e., $D(I^{(1)}) \subset I^{(1)}$ (see Definition 1.4). Besides, let $D(T) = \Sigma_j q_j T_j$, $j = 0, \ldots, r$. According to 1) and Corollary 1.14 a), one has $q_j \in I^j : I$ for $j \geq 1$. Further, $I^j : I \subset I^{(j)} : I^{(1)}$; in fact, if $xI \subset I^j$, then $xI^{(1)} = x(IA_M \cap A) \subset (xI)A_M \cap A \subset I^{(j)}$ (see Definition 1.4). So $q_j \in I^{(j)} : I^{(1)}$ for $j \geq 1$. Then 2) follows from Corollary 1.14 a).

1) $\Rightarrow$ 3). One can proceed as in "1) $\Rightarrow$ 2)," by recalling that $I^j : \langle J \rangle = I^j A_S \cap A$ for $j \geq 1$, where S is as in Remark 1.7. $\square$

Remark 2.2. In general, neither condition 2) nor condition 3) implies condition 1) in Proposition 2.1. We show two examples.

2) $\not\Rightarrow$ 1). Let $A = k[X,Y]$, k field, $I = (X^2, XY) = (X) \cap (X^2, Y)$, $F_p = \{I^n, n \geq 0\}$, $F_s = \{I^{(n)}, n \geq 0\}$. Further, let $D = X\partial/\partial X + \partial/\partial Y \in \mathrm{Der}\,(A[T])$. One has: $D(I^{(1)}) \subset I^{(1)}$, since $I^{(1)} = (X)$ (Remark 1.5). On the other hand, $D(I) \not\subset I$ since $D(XY) \notin I$. Then $D(R_s) \subset R_s$ but $D(R_p) \not\subset R_p$ (see Corollary 1.14a)).

3) $\not\Rightarrow$ 1). Let A, I, R_p, D be as in the above example, and let $J = (X^2, Y)$. If $\langle F \rangle = \{I^n : \langle J \rangle\}$ one has $I : \langle J \rangle = (X)$ (Remark 1.7), then $D(\langle R \rangle) \subset \langle R \rangle$ and $D(R_p) \not\subset R_p$.

Remark 2.3. In general, neither of the implications "1) $\Rightarrow$ 5)" nor "5) $\Rightarrow$ 1)" holds, as we now show.

1) $\not\Rightarrow$ 5). Let $A = k[X,Y]/(Y^p - X^p(1+X)) = k[x,y]$ where k is a field of positive characteristic p, $I = (x)$, $F_p = \{I^n\}$, $F_a = \{(I^n)_a\}$. Further, define D belonging to $\mathrm{Der}\,(A[T])$ by: $D(x) = 0$, $D(y) = 1$, $D(T) = 0$. It is enough to show that $D(I) \subset I$ and $D(I_a) \not\subset I_a$ (see Corollary 1.14). One has $I_a = (x,y)$ (since $x \in I$, $y \in I_a$ and (x,y) is maximal), $D(I_a) \not\subset I_a$ and $D(I) \subset I$.

5) $\not\Rightarrow$ 1). Let $A = k[X,Y]/(Y^p - X^p) = k[x,y]$ (k field, $\mathrm{ch}\,(k) = p$), $I = (x)$, $F_p = \{I^n\}$ and $F_a = \{(I^n)_a\}$. Define $D \in \mathrm{Der}\,(A[T])$ by $D(x) = y$, $D(y) = x$, $D(T) = 0$. One has: $I_a = (x,y)$, $D(I_a) \subset I_a$ and $D(I) \not\subset I$ (since $y \notin I$). The conclusion follows from Corollary 1.14.

The above examples also show that 1) $\not\Rightarrow$ 4) and 4) $\not\Rightarrow$ 1) in Proposition 2.1 (Remark 1.10).

Remark 2.4. We can see that 1) implies 5) when A is a noetherian domain containing a field of characteristic zero (see the following Proposition 2.7). On the contrary, condition 1) does not imply 4) even if A satisfies the above assumption, as the following example shows.

Let $A = k[X, XY, XZ, Y^2, Z^2, YZ^2] \subset k[X,Y,Z]$, k field, $I = (XZ)$, $\Delta = \{(X, XZ)^n, n \geq 1\}$, $F_p = \{I^n\}$ and $F_\Delta = \{(I^n)_\Delta\}$.

Let $D = -(XY)\partial/\partial X + (Y)\partial/\partial Y + (YZ)\partial/\partial Z$. One can see that $D \in \mathrm{Der}\,(A[T])$, $D(A) \subset A$ and $D(I) \subset I$ (so $D(R_p) \subset R_p$, see Corollary 1.14 a)). On the other hand, one has: $XZ^2 \in I_\Delta$, since

$(XZ^2)X \in I(X, XZ)$ and $(XZ^2)(XZ) \in I(X, XZ)$, so $(XZ^2)K \subset IK$ for $K = (X, XZ^2)$. Further, $D(XZ^2) = XYZ^2$ does not belong to I_Δ, otherwise there is an $n \geq 1$ such that $(XYZ^2)X^n \in I(X, XZ)^n = (XZ)(X^n, X^nZ, \dots, X^nZ^n)$, in which case $YZ \in A$ or $Y \in A$ (as one can verify), a contradiction.

Now, more generally, we consider a filtration $F = \{I_n\}$ of ideals of A and the Rees ring R associated to F. Let $\overline{F} = \{\overline{I}_n\}$ in the integral closure $\overline{A}$ of A and $F_a = \{(I_n)_a\}$ in A (see Definition 1.9); from now on, we let $\overline{R}$ (respectively, R_a) denote the Rees ring associated with $\overline{F}$ (respectively with F_a). Further, let $D \in \operatorname{Der}(A[T])$ be such that $D(A) \subset A$. Our aim is to prove that $D(R) \subset R$ implies $D(\overline{R}) \subset \overline{R}$ and $D(R_a) \subset R_a$.

First, we show the following facts

Lemma 2.5. *Let A be a noetherian domain containing a field of characteristic zero, I an ideal of A, and $D \in \operatorname{Der}(A)$. Under the same notation as in Definition* 1.9*, if $D(I) \subset I$, then $D((I^n)_a) \subset (I^n)_a$ and $D(\overline{I^n}) \subset \overline{I^n}$ for all $n \geq 1$.*

Proof. Let $D(I) \subset I$; by putting $D(T) = 0$, we obtain a derivation $D \in \operatorname{Der}(A[T])$ such that $D(R_p) \subset R_p$ (Corollary 1.14 a)). Let $\overline{R}$ be the integral closure of R. Then, $D(\overline{R}) \subset \overline{R}$ (see [**13**, 5]), since R is a noetherian domain containing a field of characteristic zero. On the other hand, $\overline{R} = \overline{A} \oplus \overline{I}T \oplus \cdots \oplus \overline{I^n}T^n \oplus \cdots$ (see, e.g., [**12**, p. 126]). It follows that $D(\overline{I^n}) \subset \overline{I^n}$ for all $n \geq 1$ (see Proposition 1.1 I)); then we have also $D((I^n)_a) \subset (I^n)_a$ since $(I^n)_a = \overline{I^n} \cap A$. □

Lemma 2.6. *Let A be a noetherian domain, α, β ideals of A, $\overline{A}$ the integral closure of A. One has*:

1) $(\alpha :_A \beta)\overline{A} \subset \bar{\alpha} :_{\overline{A}} \bar{\beta}$
2) $\alpha :_A \beta \subset \alpha_a :_A \beta_a$.

Proof. 1) Since $(\alpha :_A \beta)\overline{A} \subset \alpha\overline{A} :_{\overline{A}} \beta\overline{A}$, it is enough to show that $\alpha\overline{A} :_{\overline{A}} \beta\overline{A} \subset \bar{\alpha} :_{\overline{A}} \bar{\beta}$. We recall that, for each ideal α of A, one has: $\bar{\alpha} = (\cap \alpha V) \cap \overline{A}$, where the intersection is taken over all the valuation overrings V of $\overline{A}$ (see, e.g., [**14**, Vol. II, Appendix 4, Theorem 1]). Then, if $x \in \overline{A}$ is such that $x(\beta\overline{A}) \subset \alpha\overline{A}$, then (for each V as before) $x\bar{\beta} \subset x[(\beta V) \cap \overline{A}] \subset (x\beta V) \cap \overline{A} \subset (\alpha V) \cap \overline{A}$, so that $x\bar{\beta} \subset \cap[(\alpha V) \cap \overline{A}] = \bar{\alpha}$.

2) One can proceed as in 1) by recalling that for each ideal α of A one has: $\alpha_a = (\cap \alpha V) \cap A$, where $V \in \{\text{valuation overrings of } A\}$. □

Now we can prove

Proposition 2.7. *Let A be a noetherian domain containing a field of characteristic zero, $F = \{I_n\}$ a filtration in A, $\overline{F} = \{\overline{I}_n\}$, $F_a = \{(I_n)_a\}$, $R, \overline{R}, R_a$ as before. Further, let $D \in \mathrm{Der}\,(A[T])$ be such that $D(A) \subset A$. Then if $D(R) \subset R$ one has $D(\overline{R}) \subset \overline{R}$ and $D(R_a) \subset R_a$.*

Proof. Let $D(R) \subset R$. One has $D(I_n) \subset I_n$ for each n (Proposition 1.1 I)) then $D(\overline{I}_n) \subset \overline{I}_n$ and $D((I_n)_a) \subset (I_n)_a$ for all I_n (see Lemma 2.5), i.e., condition I) of Proposition 1.1 holds (for $\overline{R}$ and R_a). Now let $D(T) = \Sigma_j q_j T^j$, $j = 0, \ldots, r$. According to the assumption, Proposition 1.1 and Corollary 1.2 (i), we have $q_j \in \cap_{k \geq j}(I_k : I_{k-j+1})$ for all $j \geq 1$. Then $q_j \in \cap_{k \geq j}(\overline{I}_k : \overline{I}_{k-j+1})$ and $q_j \in \cap_{k \geq j}((I_k)_a : (I_{k-j+1})_a)$ for all $j \geq 1$ (Lemma 2.6). So $D(\overline{R}) \subset \overline{R}$ and $D(R_a) \subset R_a$ (see Proposition 1.1 and Corollary 1.2 (i)). □

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Dipartimento di matematica, Università di Genova, 16132 Genova, Italy