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ISOMETRIES OF MUSIELAK-ORLICZ SPACES EQUIPPED WITH THE ORLICZ NORM

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ABSTRACT. In [**2, 4**] a characterization of the group of surjective isometries of the complex Musielak-Orlicz space L_Φ equipped with the Luxemburg norm was given. Here it is shown that the same characterization also remains valid for the group of isometries of L_Φ endowed with the Orlicz norm.

1. Introduction. In this paper we will investigate surjective isometries of complex Musielak-Orlicz spaces equipped with the Orlicz norm. The main conclusion is that the groups of isometries with respect to Orlicz or Luxemburg norms are the same. The analogous fact for Orlicz spaces has already been observed in [**13**], however without any proof. Thus the results presented here provide a justification also for the mentioned remark in [**13**].

As a first step we will give a characterization of support functionals of elements in E_Φ. Then, by means of that characterization, we will show that any Hermitian operator H of L_Φ equipped with the Orlicz norm has a diagonal form, that is, $H(f) = h\cdot f$ for some real bounded function h. This description of Hermitian operators appears to be the same as for the Luxemburg norm. Hence we conclude that any surjective isometry preserves disjointness of the supports of functions. Finally, applying the already known criterion of isometries of L_Φ endowed with the Luxemburg norm, we will prove that the criterion is the same if L_Φ is considered with the Orlicz norm.

Although the paper is a continuation of [**2**] and [**4**] and will often refer to the results of those papers, for the convenience of the reader, we recall some definitions and facts on isometries, Hermitian operators and Musielak-Orlicz spaces.

In this paper only surjective isometries will be investigated. Thus we will say a transformation U of a Banach space $(X, || \,||)$ is an *isometry* if it is linear, surjective and if it preserves the norm, that is, $||Ux|| = ||x||$ for all $x \in X$.

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A bounded linear operator H on a complex Banach space $(X, || \, ||)$ is said to be *Hermitian* [**1**], if $x^*(Hx)$ is real for every pair of vectors $x \in X$ and $x^* \in X^*$ such that x^* is a support functional of x, i.e., $||x||^2 = ||x^*||^2 = x^*(x)$.

Let $\mathbf{N}$ be the set of natural numbers, $\mathbf{K}$ the set of real or complex scalars and $\mathbf{R}$ the set of all reals. The measure space (T, Σ, μ) is assumed to be σ-finite, atomless and separable. A function $\Phi(u,t) : \mathbf{R}_+ \times T \to \mathbf{R}_+$ is said to be a *Young function with parameter*, or *Musielak-Orlicz function*, if it is convex and strictly increasing with respect to u, $\Phi(0,t) = 0$ almost everywhere, and is Σ-measurable as a function of t. Throughout the paper we will additionally assume that

$$\lim_{u\to\infty} \frac{\Phi(u,t)}{u} = \infty \quad \text{and} \quad \lim_{u\to 0} \frac{\Phi(u,t)}{u} = 0, \tag{0}$$

for a.a. $t \in T$. By $\Phi'_-(u,t)$ and $\Phi'_+(u,t)$ denote the left and right side derivatives with respect to u. We will also use the notation $\Phi'(u,t)$ for either left or right derivatives in the situations which are valid no matter what derivative is concerned. The conjugate function

$$\Phi^*(u,t) = \sup_{v\geq 0}\{uv - \Phi(v,t)\}$$

is a Musielak-Orlicz function with the properties (0). The assumption (0) implies that $\Phi^*(u,t) < \infty$ and

$$\lim_{u\to\infty} \Phi'(u,t) = \infty \quad \text{and} \quad \lim_{u\to 0} \Phi'(u,t) = 0,$$

for a.a. $t \in T$. To abbreviate long formulas we will use the following notations.

$$\int f = \int f(t) = \int_T f(t)\, d\mu$$

$$I_\Phi(f) = \int \Phi(|f|) = \int_T \Phi(|f(t)|, t)\, d\mu$$

$$\Phi^*(\Phi'(u)) = \Phi^*(\Phi'(u,t), t)$$

$$\int \Phi^*(\Phi'(|f|)) = \int_T \Phi^*(\Phi'(|f(t)|,t),t)\, d\mu.$$

The *Musielak-Orlicz* space L_Φ associated with Φ is the set of all measurable scalar valued functions f such that $I_\Phi(\lambda f) = \int_T \Phi(\lambda |f(t)|, t)\, d\mu <$

∞ for some $\lambda > 0$. For $f \in L_\Phi$, define the Orlicz $|| \; ||$ and the Luxemburg norm $|| \; ||_l$ as follows.

$$||f|| = \sup \left\{ \int |fg| : I_{\Phi^*}(g) \leq 1 \right\},$$
$$||f||_l = \inf \{\varepsilon > 0 : I_\Phi(f/\varepsilon) \leq 1\}.$$

The Amemiya formula holds for the Orlicz norm

$$||f|| = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kf)).$$

The Orlicz and Luxemburg norms are equivalent. By E_Φ denote the subspace of L_Φ containing all measurable functions such that $I_\Phi(\lambda f) < \infty$ for all $\lambda > 0$.

We have the following well-known inequalities and equations:

(1) $u\Phi'(u,t) \leq \Phi(2u,t)$

(2) $u \cdot v \leq \Phi(u,t) + \Phi^*(v,t)$

(3) $u\Phi'(u,t) = \Phi(u,t) + \Phi^*(\Phi'(u,t),t)$

(4) $\Phi^*(\lambda\Phi'_-(u) + (1-\lambda)\Phi'_+(u)) = \lambda\Phi^*(\Phi'_-(u)) + (1-\lambda)\Phi^*(\Phi'_+(u))$ for $\lambda \in [0,1]$.

In particular, the properties (3) and (1) imply that if $f \in E_\Phi$, then

$$I_{\Phi^*}(\Phi'(\lambda|f|)) < \infty \tag{5}$$

for all $\lambda \geq 0$.

Now recall some properties of the Orlicz norm $|| \; ||$ in L_Φ. For any $g \in L_\Phi$, set

$$k^* = \inf\left\{k > 0 : \int \Phi^*(\Phi'(k|g|)) \geq 1\right\}$$
$$k^{**} = \sup\left\{k > 0 : \int \Phi^*(\Phi'(k|g|)) \leq 1\right\}.$$

Since $\lim_{u\to\infty} \Phi'(u,t) = \infty$, the numbers k^*, k^{**} are finite.

The proposition below is well known for Orlicz spaces [**8, 11**]. It remains true for Musielak-Orlicz spaces with the proof analogous to that for Orlicz spaces.

Proposition 0.1. *The following facts hold.*

(i) $k^* \le k^{**}$

(ii) k^* *and* k^{**} *do not depend on whether* Φ' *is a left or right derivative.*

(iii) *If* $k \in (k^*, k^{**})$, *i.e., if* $k^* < k^{**}$, *then* $\int \Phi^*(\Phi'(k|g|)) = 1$.

(iv) $\int \Phi^*(\Phi'_+(k^*|g|)) \ge 1$; $\int \Phi^*(\Phi'_-(k^{**}|g|)) \le 1$.

(v) $k \in [k^*, k^{**}]$ *if and only if* $||g|| = (1/k)(1 + I_\Phi(kg))$.

(vi) *If* $\int \Phi^*(\Phi'_-(k^{**}|g|)) < 1 \le \int \Phi^*(\Phi'_+(k^*|g|))$, *then* $k^* = k^{**}$.

More information about Musielak-Orlicz or Orlicz spaces can be found in [**10, 8, 11**].

Finally, recall a characterization of isometries on $(L_\Phi, ||\ ||_l)$ given in [**2, 4**]. Since the measure space (T, Σ, μ) is separable, it is isomorphic to the Lebesgue measure space [**3**], and a regular set isomorphism $\tau : \Sigma \to \Sigma$ appearing in [**2, 4**] is generated by a pointwise measurable transformation $\theta : T \to T$, that is, $\tau(A) = \theta^{-1}[A]$, [**12**].

Let
$$T_0 = \left\{ t \in T : \frac{\Phi'(u,t)}{u} = \text{ constant} \right\}.$$

Theorem 0.2 [**2,4**]. *Assume* $\mu T_0 = 0$.

(i) *Let* U *be an isometry of* $(L_\Phi, ||\ ||_l)$. *Then there exists a measurable function* $h : T \to \mathbf{K}$ *and a measurable pointwise transformation* $\theta : T \to T$, *one-to-one and onto up to null sets, such that*

$$(Uf)(t) = h(t) f(\theta(t)), \tag{1}$$
$$\Phi(|h(t)|\lambda, t) = \tau'(t)\Phi(\lambda, \theta(t)) \tag{2}$$

for all $\lambda \ge 0$, *a.a.* $t \in T$, *where* τ' *is the Radon-Nikodym derivative* $d(\mu \circ \theta)/d\mu$.

(ii) *Let* $h : T \to \mathbf{K}$ *be a measurable function and* $\theta : T \to T$ *a measurable transformation one-to-one and onto up to null sets. If the condition* (2) *is satisfied, then* U *given by* (1) *is an isometry of* $(L_\Phi, ||\ ||_l)$.

(iii) *If U is an isometry of $(L_\Phi, \|\ \|_l)$, then U preserves the modular I_Φ, that is, $I_\Phi(Uf) = I_\Phi(f)$ for all $f \in L_\Phi$.*

2. Results. The Musielak-Orlicz spaces L_Φ and their subspaces E_Φ appearing in this paper are always assumed to be spaces of complex valued functions.

Further in the paper we will assume without loss of generality that $\chi_T \in E_\Phi$. In particular it means that any simple function is an element of E_Φ. Indeed, this assumption is not any restriction on Φ, because it is known that there exists a partition $\{T_n\}$ of T such that $\chi_{T_n} \in E_\Phi$ for all $n \in \mathbf{N}$ [**6**].

For $g \in E_\Phi$ and $k \in [k^*, k^{**}]$ assign the function

$$G(t) = \sum_{i=1}^{n} [\lambda_i \Phi'_-(k|g(t)|, t) + (1-\lambda_i)\Phi'_+(k|g(t)|, t)]\chi_{A_i}(t) + \Phi'(k|g(t)|, t)\chi_{A_{n+1}}(t),$$

where λ_i and A_i are arbitrary numbers and sets satisfying the following conditions: $\lambda_i \in [0,1]$, A_i are disjoint and $\cup_{i=1}^{n+1} A_i = T$ and $I_{\Phi^*}(G) = 1$. Since $g \in E_\Phi$, it is always possible to find at least one function G satisfying all of the above requirements. Indeed, if $k \in (k^*, k^{**})$, then setting $G = \Phi'(k|g|)$, we have $I_{\Phi^*}(G) = 1$ by (iii) of Proposition 0.1. Now let $k = k^*$. Then $\int \Phi^*(\Phi'_-(k^*|g|)) \le 1 \le \int \Phi^*(\Phi'_+(k^*|g|))$ by (iv) of Proposition 0.1. Moreover, by virtue of (5), $\int \Phi^*(\Phi'_+(k^*|g|)) < \infty$. Thus there exists a $\lambda \in [0,1]$ such that

$$\begin{aligned} 1 &= \lambda \int \Phi^*(\Phi'_-(k^*|g|)) + (1-\lambda)\int \Phi^*(\Phi'_+(k^*|g|)) \\ &= \int \Phi^*(\lambda\Phi'_-(k^*|g|) + (1-\lambda)\Phi'_+(k^*|g|)), \end{aligned}$$

by equation (4). Then setting $G = \lambda\Phi'_-(k^*|g|) + (1-\lambda)\Phi'_+(k^*|g|)$, we get $I_{\Phi^*}(G) = 1$. If $k = k^{**}$, we can get it analogously.

For $z \in \mathbf{C}$, let $\operatorname{sgn} z = z/|z|$ if $z \ne 0$ and $\operatorname{sgn} z = 0$ if $z = 0$.

Lemma 1. *Let $g \in (E_\Phi, \|\ \|)$. Then*

$$F_g(f) = \|g\| \int f(t) \cdot \operatorname{sgn} g(t) \cdot G(t)\, d\mu$$

is a support functional of g, where G is a function assigned for g.

Proof. Applying (3), (4) and (v) of Proposition 0.1, we get

$$\begin{aligned} F_g(g) &= ||g||\bigg\{ \sum_{i=1}^{n} \int_{A_i} |g(t)|[\lambda_i \Phi'_-(k|g(t)|,t) + (1-\lambda_i)\Phi'_+(k|g(t)|,t)] \\ &\qquad + \int_{A_{n+1}} |g(t)|\Phi'(k|g(t)|,t)\} \\ &= ||g||\frac{1}{k}\bigg\{ \sum_{i=1}^{n} \lambda_i \bigg(\int_{A_i} \Phi(k|g|) + \int_{A_i} \Phi^*(\Phi'_-(k|g|))\bigg) \\ &\qquad + \sum_{i=1}^{n} (1-\lambda_i)\bigg(\int_{A_i} \Phi(k|g|) + \int_{A_i} \Phi^*(\Phi'_+(k|g|))\bigg) \\ &\qquad + \int_{A_{n+1}} \Phi(k|g|) + \int_{A_{n+1}} \Phi^*(\Phi'(k|g|))\bigg\} \\ &= ||g||\frac{1}{k}\{I_\Phi(kg) + I_{\Phi^*}(G)\} \\ &= ||g||\frac{1}{k}\{I_\Phi(kg) + 1\} = ||g||^2. \end{aligned}$$

Moreover, by the Hölder inequality [**10**], $|F_g(f)| \le ||g||\,||f||\,||G||_l^{\Phi^*} \le ||g|| \cdot ||f||$, because $I_{\Phi^*}(G) = 1$, which implies that $||G||_l^{\Phi^*} \le 1$. Thus, $||F_g|| = ||g||$. □

Lemma 2. *Let H be a Hermitian operator in $(L_\Phi, ||\ ||)$. Let G be a function assigned for $f \in E_\Phi$. For any $A, B \in \Sigma$ with $A \cap B = \varnothing$, we have*

$$\int_A Hf\chi_B \cdot \operatorname{sgn} f \cdot G = \int_B \overline{Hf\chi_A} \cdot \operatorname{sgn} \bar{f} \cdot G.$$

Proof. Let $Pf = f\chi_A$ and $Qf = f\chi_B$. Then P and Q are Hermitian projections on L_Φ satisfying $PQ = 0$. Indeed, $|e^{i\alpha P} f| = |f|$ for any $f \in L_\Phi$ and $\alpha \in \mathbf{R}$. Thus the operator $e^{i\alpha P}$ is an isometry on L_Φ, which implies that P is Hermitian [**1, 2**].

By Kalton and Wood's result (Theorems 2.4, 2.6 in [**5**] or Lemma 2.1 in [**4**]), for any $f \in E_\Phi$, $F_f(\chi_A Hf\chi_B + \chi_B Hf\chi_A)$ and $iF_f(\chi_A Hf\chi_B -$

$\chi_B Hf\chi_A)$ are reals. Then

$$F_f(\chi_A Hf\chi_B) = \overline{F_f(\chi_B Hf\chi_A)}.$$

Thus, by the previous lemma,

$$\int_A Hf\chi_B \cdot \operatorname{sgn} f \cdot G = \int_B \overline{Hf\chi_A} \cdot \operatorname{sgn} \bar{f} \cdot G. \qquad \square$$

Lemma 3. *Assume that* $\mu T_0 = 0$. *If* H *is a Hermitian operator of* $(L_\Phi, \|\ \|)$, *then* $\operatorname{supp} H\chi_A \subset A$ *almost everywhere for any* $A \in \Sigma$.

Proof. Let $A \in \Sigma$ and $\mu(A^c) > 0$. Let $0 < \gamma < \beta$ be arbitrary. By the assumption in the paper, that any simple function belongs to E_Φ and by (5), $\int \Phi^*(\Phi'(a)) < \infty$ for any number $a \geq 0$. Thus the nonatomicity of μ implies the existence of a set $A_1 \subset A^c$ with $\mu(A^c \backslash A_1) > 0$ and such that $\int_{A_1} \Phi^*(\Phi'(\beta)) < 1$. Since $\lim_{u\to 0} \Phi'(u,t) = 0$, there exists $\alpha > 0$ such that

$$\int_A \Phi^*(\Phi'_-(\alpha)) + \int_{A_1} \Phi^*(\Phi'(\beta)) \leq 1 \tag{3.1}$$

and

$$\int_A \Phi^*(\Phi'_+(\alpha)) + \int_{A_1} \Phi^*(\Phi'(\beta) \geq 1.$$

Then there exists $\lambda_1 \in [0,1]$ with

$$\lambda_1 \int_A \Phi^*(\Phi'_-(\alpha)) + (1-\lambda_1) \int_A \Phi^*(\Phi'_+(\alpha)) + \int_{A_1} \Phi^*(\Phi'(\beta)) = 1.$$

This implies

$$\int_A \Phi^*(\lambda_1 \Phi'_-(\alpha) + (1-\lambda_1)\Phi'_+(\alpha)) + \int_{A_1} \Phi^*(\Phi'(\beta)) = 1,$$

by (4). Setting $f = \alpha\chi_A + \beta\chi_{A_1}$, and $F(t) = (\lambda_1\Phi'_-(\alpha,t) + (1-\lambda_1)\Phi'_+(\alpha,t))\chi_A(t) + \Phi'(\beta,t)\chi_{A_1}(t)$, we have $I_{\Phi^*}(F) = 1$. Since $\gamma < \beta$, $\int_{A_1}\Phi^*(\Phi'(\gamma)) \le \int_{A_1}\Phi^*(\Phi'(\beta))$. Thus we will find $A_2 \subset (A\cup A_1)^c$ (A_2 may be $\varnothing$) and $\delta > 0$ and $\lambda_2 \in [0,1]$ such that

$$\int_A \Phi^*(\lambda_1\Phi'_-(\alpha) + (1-\lambda_1)\Phi'_+(\alpha)) + \int_{A_1}\Phi^*(\Phi'(\gamma)) + \int_{A_2}\Phi^*(\lambda_2\Phi'_-(\delta) + (1-\lambda_2)\Phi'_+(\delta)) = 1.$$

Now set

$$g = \alpha\chi_A + \gamma\chi_{A_1} + \delta\chi_{A_2}$$

and

$$\begin{aligned} G(t) = {} & [\lambda_1\Phi'_-(\alpha,t) + (1-\lambda_1)\Phi'_+(\alpha,t)]\chi_A(t) \\ & + [\lambda_2\Phi'_-(\delta,t) + (1-\lambda_2)\Phi'_+(\delta,t)]\chi_{A_2}(t) \\ & + \Phi'(\gamma,t)\chi_{A_1}(t). \end{aligned}$$

We have $I_{\Phi^*}(G) = 1$. Applying the previous lemma to f, g and A, A_1, we get

$$\int_A H\beta\chi_{A_1}F = \int_{A_1}\overline{H\alpha\chi_A(t)}\Phi'(\beta,t)$$
$$\int_A H\gamma\chi_{A_1}G = \int_{A_1}\overline{H\alpha\chi_A(t)}\Phi'(\gamma,t)$$

and $F(t) = G(t)$ for $t \in A$. Hence

$$\int_{A_1}\overline{H\alpha\chi_A(t)}\frac{\Phi'(\beta,t)}{\beta} = \int_{A_1}\overline{H\alpha\chi_A(t)}\frac{\Phi'(\gamma,t)}{\gamma},$$

which implies that

$$\int_{A_1}\overline{H\chi_A(t)}\left[\Phi'(\beta,t) - \frac{\beta}{\gamma}\Phi'(\gamma,t)\right] = 0.$$

Repeating the same procedure for any $B \subset A_1$, we get

$$\int_B \overline{H\chi_A(t)}\left[\Phi'(\beta,t) - \frac{\beta}{\gamma}\Phi'(\gamma,t)\right] = 0 \qquad (3.2)$$

for all $B \subset A_1$. Since $\int_{A^c} \Phi^*(\Phi'(\beta)) < \infty$, there exists a finite sequence $A^1, \dots, A^k$ of disjoint sets such that $A^c = \cup_{i=1}^k A^i$ and $\int_{A^i} \Phi^*(\Phi'(\beta)) < 1$ and $\mu((A \cup A^i)^c) > 0$. Now repeating the process which starts at (3.1) for each A^i instead of A, we get (3.2) for all $B \subset A^i$. Hence we have equation (3.2) for all $B \subset A^c$. However, it implies that

$$\overline{H\chi_A(t)}\left[\Phi'(\beta, t) - \frac{\beta}{\gamma}\Phi'(\gamma, t)\right] = 0$$

for a.a. $t \in A^c$ and all $0 < \gamma < \beta$. Thus $\Phi'(\beta, t)/\beta = \Phi'(\gamma, t)/\gamma$ whenever $H\chi_A(t) \neq 0$ and $t \in A^c$. Then by the assumption $\mu T_0 = 0$, we have that $\operatorname{supp} H\chi_A \subset A$ almost everywhere. □

Applying the above lemma and following the proof of Theorem 6 in [**2**] we get the characterization of Hermitian operators on $(L_\Phi, \|\ \|)$.

Theorem 4. *If $\mu T_0 = 0$, then any Hermitian operator of $(L_\Phi, \|\ \|)$ is of the form $H(f) = h \cdot f$ where h is a real bounded function. The sets of Hermitian operators of L_Φ equipped either with the Luxemburg or Orlicz norm coincide.*

Now, by standard methods (compare [**9**] or [**2**]), the above characterization allows us to show that an isometry of $(L_\Phi, \|\ \|)$ has disjoint support property. More precisely, the following theorem holds.

Theorem 5. *Assume that $\mu(T_0) = 0$. If U is an isometry of $(L_\Phi, \|\ \|)$, then*

$$(Uf)(t) = h(t)f(\theta(t))$$

for some measurable function $h : T \to \mathbf{K}$ and a measurable transformation $\theta : T \to T$ which is one-to-one and onto up to null sets.

Recall that the dual space $(L_\Phi)^*$ is a direct sum of regular and singular functionals, that is, $(L_\Phi)^* = (L_\Phi)^*_r \oplus (L_\Phi)^*_s$. If $F \in (L_\Phi)^*$, then $F = F^r + F^s$, where $F^r \in (L_\Phi)^*_r$ and $F^s \in (L_\Phi)^*_s$. The space $(L_\Phi)^*_r$ may be naturally (isometrically) identified with the space L_{Φ^*} equipped with the Luxemburg norm. There exists $g \in L_{\Phi^*}$ such that $F^r(f) = \int gf$, $f \in L_\Phi$, and $\|F^r\| = \|g\|_l^{\Phi^*}$ [**7, 10**].

Proposition 6. *Let $\mu T_0 = 0$. If U is an isometry of $(L_\Phi, \|\ \|)$, then $U^*_{|L_{\Phi^*}}$ is an isometry of $(L_{\Phi^*}, \|\ \|_l^{\Phi^*})$.*

Proof. The adjoint U^* is an isometry of $(L_\Phi)^*$. So it is enough to show that any regular functional is transformed into a regular one. Let $F = F^r \in (L_\Phi)^*_r$. There exists a $g \in L_{\Phi^*}$ such that $F^r(f) = \int gf$ for all $f \in L_\Phi$. Thus, applying Theorem 0.2 and change of variables formula $U^*(F^r)(f) = F^r(Uf) = \int gUf = \int ghf(\theta) = \int g(\theta^{-1})(h(\theta^{-1})/\tau'(\theta^{-1}))f < \infty$ for all $f \in L_\Phi$. Hence the function $g(\theta^{-1})(h(\theta^{-1})/\tau'(\theta^{-1})) \in L_{\Phi^*}$, and $U^*(F) \in (L_\Phi)^*_r$. □.

Lemma 7. *Let $k > 0$ and $t, s \in T$. If, for all $v \geq 0$, $\Phi(kv, t) = (\Phi(k,t)/\Phi(1,s))\Phi(v,s)$, then*

$$\Phi^*\left(\frac{ku\Phi(1,s)}{\Phi(k,t)}, s\right) = \frac{\Phi^*(u,t)}{\Phi(k,t)}\Phi(1,s).$$

Proof. By the definition of Φ^*, we have

$$\begin{aligned}\Phi^*\left(\frac{ku\Phi(1,s)}{\Phi(k,t)}, s\right) &= \sup_{v\geq 0}\left\{v\frac{ku\Phi(1,s)}{\Phi(k,t)} - \Phi(v,s)\right\}\\ &= \sup_{w\geq 0}\left\{uw - \Phi\left(\frac{\Phi(k,t)}{\Phi(1,s)}wk^{-1}, s\right)\right\}.\end{aligned}$$

For any $\lambda \geq 0$,

$$\Phi(\lambda, t) = \Phi(\lambda k k^{-1}, t) = \frac{\Phi(k,t)}{\Phi(1,s)}\Phi(\lambda k^{-1}, s),$$

so

$$\Phi(\lambda k^{-1}, s) = \frac{\Phi(\lambda,t)\Phi(1,s)}{\Phi(k,t)}.$$

Hence

$$\begin{aligned}\Phi^*\left(\frac{ku\Phi(1,s)}{\Phi(k,t)}, s\right) &= \sup_{w\geq 0}\left\{uw - \frac{\Phi(\frac{\Phi(k,t)}{\Phi(1,s)}w, t)\Phi(1,s)}{\Phi(k,t)}\right\}\\ &= \frac{\Phi(1,s)}{\Phi(k,t)}\Phi^*(u,t). \qquad \square\end{aligned}$$

Theorem 8. *Let $\mu T_0 = 0$. The groups of isometries of L_Φ in both Luxemburg and Orlicz norms coincide.*

Proof. If U is an isometry of $(L_\Phi, \|\,\|_l)$, then U preserves the modular I_Φ (Theorem 0.2.III). Thus, by the Amemiya formula of the Orlicz norm, U is an isometry with respect to this norm.

Now let U be an isometry of $(L_\Phi, \|\,\|)$. Then, by Proposition 6, the adjoint U^* is an isometry of $(L_{\Phi^*}, \|\,\|_l^{\Phi^*})$. Then applying Theorem 0.2 to L_{Φ^*}, there exists a measurable function h and a measurable transformation θ such that

$$U^* g = h g(\theta)$$

for all $g \in L_{\Phi^*}$ and

$$\Phi^*(|h(t)|\lambda, t) = \tau'(t)\Phi^*(\lambda, \theta(t)) \tag{8.1}$$

for all $\lambda \geq 0$ and a.a. $t \in T$, where $\tau' = d(\mu \circ \theta)/d\mu$. By the integral representation of functionals on L_Φ we have that

$$\int h f g(\theta) = \int (Uf) g$$

for all $f \in L_\Phi$ and $g \in L_{\Phi^*}$. The change of variables formula then implies that

$$\int (Uf) g = \int \frac{h(\theta^{-1}) f(\theta^{-1})}{\tau'(\theta^{-1})} g.$$

Hence

$$Uf = \frac{h(\theta^{-1}) f(\theta^{-1})}{\tau'(\theta^{-1})}.$$

Thus, to prove that U is an isometry for the Luxemburg norm, it is enough to show the following equation

$$\Phi\left(\left|\frac{h(\theta^{-1}(t))}{\tau'(\theta^{-1}(t))}\right| \lambda, t\right) = \frac{1}{\tau'(\theta^{-1}(t))} \Phi(\lambda, \theta^{-1}(t)). \tag{8.2}$$

By substituting 1 for λ in (8.1), we get

$$\tau'(t) = \frac{\Phi^*(|h(t)|, t)}{\Phi^*(1, \theta(t))}.$$

Now the equation (8.2) is equivalently transformed into

$$\Phi(|h(t)|\frac{\Phi^*(1,\theta(t))}{\Phi^*(|h(t)|,t)}\lambda,\theta(t)) = \frac{\Phi^*(1,\theta(t))}{\Phi^*(|h(t)|,t)}\Phi(\lambda,t). \tag{8.3}$$

Applying Lemma 7 to Φ^* instead of Φ (it is known that $\Phi^{**} = \Phi$), $k = |h(t)|$, $v = \lambda$ and $s = \theta(t)$, equation (8.1) simply implies (8.3), which completes the proof. □

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