ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 4, Fall 1994

RIESZ DECOMPOSITION IN INDUCTIVE LIMIT C^*-ALGEBRAS

K.R. GOODEARL

ABSTRACT. As recently proved by Zhang, the projections in any C^*-algebra of real rank zero enjoy the Riesz decomposition property. Here the Riesz decomposition property is obtained for projections in several types of C^*-algebras with positive real rank, including the inductive limits with slow dimension growth introduced by Blackadar, Dădărlat, and Rørdam. Waiving the dimension restrictions, weaker forms of the Riesz decomposition property are established for general inductive limits of finite direct products of homogeneous C^*-algebras.

1. Introduction and background. The Riesz decomposition property (see below) for projections in C^*-algebras of real rank zero was established by Zhang [**32**, 1.1] and used as a key tool in his investigation of the structure of such algebras and of their multiplier and corona algebras—see, e.g., [**32**, 2.2, 2.3], [**33**, 1.2], [**34**, 1.1]. More recently, Zhang's result has been used by Elliott as one of the key ingredients in his classification of certain C^*-algebras of real rank zero [**10**].

Our aim here is to show that Riesz decomposition is more widespread than Zhang's theorem indicates. We prove it for a large class of C^*-algebras with positive real rank, including all simple inductive limits with slow dimension growth as in [**3**], as well as nonsimple inductive limits satisfying a related form of slow dimension growth. Under somewhat relaxed hypotheses, we obtain weaker forms of Riesz decomposition, sufficient for instance to prove that for any approximately semi-homogeneous C^*-algebra A (see Section 3), the partially ordered abelian group $K_0(A) \otimes \mathbf{Q}$ is a Riesz group.

Our work on inductive limits with slow dimension growth relies on extensions to arbitrary compact Hausdorff spaces of standard cancellation theorems for vector bundles over finite CW-complexes. Various

Received by the editors on November 23, 1992.
1991 *Mathematics Subject Classification.* 46L80, 46M40, 19K14.
This research was partially supported by a grant from the National Science Foundation.

forms of these results exist as folklore, as do their uses in connection with C^*-inductive limits, but they do not seem to be available in the literature. Therefore, we have taken this opportunity to present statements and proofs of sharp versions of these cancellation results.

1.1. We refer the reader to [**1**] for basic information and notation concerning projections in C^*-algebras, to [**13**] for the basic theory of partially ordered abelian groups, and to [**2, 3, 6–8, 10, 21, 24, 25, 27–30**] for other perspectives on inductive limits of finite direct products of homogeneous C^*-algebras. In particular, given a C^*-algebra A, we follow [**1**, 5.1.1] in writing $M_\infty(A)$ for the algebraic direct limit of the matrix algebras $M_n(A)$ under the embeddings

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

and we denote the orthogonal sum of m copies of a projection p by $m.p$. We use $\sim$ to denote Murray–von Neumann equivalence of projections; the notation $p \lesssim q$ means that p is equivalent to a subprojection of q. We shall consider the following conditions on projections in $M_\infty(A)$:

Cancellation. $p \oplus r \sim q \oplus r \Rightarrow p \sim q$;

Riesz decomposition. $p \lesssim q_1 \oplus q_2 \Rightarrow p = p_1 \oplus p_2$ for some projections p_i with $p_i \lesssim q_i$ for $i = 1, 2$;

Riesz interpolation. $p_i \lesssim q_j$ for $i, j = 1, 2 \Rightarrow$ there is a projection r such that $p_i \lesssim r \lesssim q_j$ for $i, j = 1, 2$.

The latter conditions are parallel, but not in general equivalent, to corresponding conditions on the pre-ordered abelian group $K_0(A)$:

Riesz decomposition. $0 \le x \le y_1 + y_2$ with $y_i \ge 0$ for $i = 1, 2 \Rightarrow$ there exist elements $x_i \ge 0$ such that $x = x_1 + x_2$ and $x_i \le y_i$ for $i = 1, 2$;

Riesz interpolation. $x_i \le y_j$ for $i, j = 1, 2 \Rightarrow$ there exists an element z such that $x_i \le z \le y_j$ for $i, j = 1, 2$.

Note that if the projections in $M_\infty(A)$ satisfy cancellation, then $K_0(A)$ is a partially ordered (rather than just pre-ordered) abelian group. Moreover, in this case the Riesz decomposition (interpolation) property for projections in $M_\infty(A)$ is satisfied if and only if Riesz decomposition (interpolation) holds in $K_0(A)$. In $K_0(A)$, these properties are equivalent [**13**, 2.1].

We recall the use of the term *Riesz group* to denote a directed, partially ordered abelian group satisfying the Riesz decomposition (interpolation) property.

1.2. It is well known that the state space of a partially ordered abelian group satisfying Riesz decomposition is a Choquet simplex [**13**, 10.17]. Similar reasoning can be used to reach this conclusion in the setting of pre-ordered abelian monoids, and consequently the state space of K_0 of any unital C^*-algebra whose projections satisfy Riesz decomposition is a Choquet simplex, as stated in the theorem below. When combined with Zhang's theorem, we obtain a new proof of the fact that the state space of K_0 of any unital C^*-algebra with real rank zero is a Choquet simplex. The stably finite case of this corollary is due to Blackadar and Handelman [**4**, III.1.4], and the general case follows easily by considering the maximal stably finite quotient algebra of the algebra in question.

Since the proof below parallels that of the corresponding result for von Neumann regular rings [**11**, 17.5, 17.12], we omit some of the details.

Theorem 1.2. *If A is a unital C^*-algebra such that the projections in $M_\infty(A)$ satisfy Riesz decomposition, then the state space of $(K_0(A), [1_A])$ is a Choquet simplex.*

Proof. As in [**1**, 5.1.2], let $V(A)$ denote the set of all equivalence classes of projections from $M_\infty(A)$; this is an abelian monoid under the addition operation derived from orthogonal sum of projections. There is a natural translation-invariant pre-order $\leq$ on $V(A)$, defined as in $K_0(A)$; namely, $x, y \in V(A)$ satisfy $x \leq y$ if and only if $y = x + x'$ for some $x' \in V(A)$. The hypothesis of Riesz decomposition for projections in $M_\infty(A)$ immediately implies that Riesz decomposition holds in $V(A)$. The equivalence class of 1_A gives an order-unit $u \in V(A)$, and we can define the state space S of $(V(A), u)$ as the set of all monoid homomorphisms $s : V(A) \to \mathbf{R}^+$ such that $s(u) = 1$. Then S is a compact convex subset of $\mathbf{R}^{V(A)}$ (with the product topology), and the natural monoid homomorphism $V(A) \to K_0(A)$ induces an affine homeomorphism from the state space of $(K_0(A), [1_A])$ onto S. Thus, it suffices to show that S is a Choquet simplex.

Let W be the convex cone in $\mathbf{R}^{V(A)}$ with base S. Observe that W consists precisely of all monoid homomorphisms from $V(A)$ to $\mathbf{R}^+$, and that the pointwise ordering on W coincides with the algebraic ordering; that is, for $f, g \in W$ we have $f(v) \leq g(v)$ for all $v \in V(A)$ if and only if $g - f \in W$. Using the Riesz decomposition property in $V(A)$, one verifies that W is a lattice with respect to the pointwise ordering, just as in [**11**, 17.3] or [**13**, 2.27]. Therefore, S is a Choquet simplex. $\square$

1.3. We conclude the introduction with the observation that Riesz decomposition does not hold in general for projections in C^*-algebras of real rank 1. For example, if $A = C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$, then the state space of $(K_0(A), [1_A])$ is a square [**1**, 6.10.4], and hence by Theorem 1.2 the projections in $M_\infty(A)$ do not satisfy Riesz decomposition. To see that A has real rank 1, we first check that it has stable rank 1; then A has real rank at most 1 by [**5**, 1.2], while by Zhang's theorem A cannot have real rank 0.

That A has stable rank 1 appears to be known but unpublished. Several people communicated proofs; we sketch the one suggested by Elliott. As in [**1**, 6.10.4], identify A with the subalgebra of $M_2(C([0,1]))$ consisting of those matrices whose off-diagonal entries vanish at 0 and 1. Since $[0,1]$ is 1-dimensional, $C([0,1])$ has stable rank 1 [**31**, Theorem 7], and we proceed as in Robertson's proof of the fact that stable rank 1 passes up to matrix algebras [**26**, Proposition 3]. Thus let $\varepsilon > 0$ and

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A.$$

There exists an invertible element $u \in C([0,1])$ such that $||u - a|| < \varepsilon$; moreover, there exists an invertible element $v \in C([0,1])$ such that $||v - (d - cu^{-1}b)|| < \varepsilon$. If $w = v + cu^{-1}b$, then $w - cu^{-1}b$ is invertible and $||w - d|| < \varepsilon$. Finally, elementary row reduction shows that the matrix

$$g = \begin{pmatrix} u & b \\ c & w \end{pmatrix}$$

is invertible, and $||g - f|| < \varepsilon$. Since $g \in A$, this proves that the invertible elements are dense in A, as desired.

2. Inductive limits with slow dimension growth.

2.1. Let A be the C^*-inductive limit of a directed family of C^*-algebras A_i, $i \in I$ and C^*-homomorphisms $\phi_{ji} : A_i \to A_j$, $i \leq j$, where each

$$A_i = \prod_{k=1}^{n_i} C(X_{ik}, M_{t(i,k)}(\mathbf{C}))$$

for some (nonempty) connected compact Hausdorff spaces X_{ik} and positive integers $t(i,k)$. While the A_i are unital, we do not assume that the ϕ_{ji} are unital homomorphisms. Write elements $a \in A_i$ as n_i-tuples with components a_k. Let $e_i = 1_{A_i}$; then the components e_{ik} are the central projections in A_i corresponding to the units in the algebras $C(X_{ik}, M_{t(i,k)}(\mathbf{C}))$. We shall identify $e_{ik}A_i$ with $C(X_{ik}, M_{t(i,k)}(\mathbf{C}))$ whenever convenient. To ensure that A_i has no factors which disappear in the inductive limit, we assume that $\phi_{ji}(e_{ik}) \neq 0$ for $j \geq i$ and $k = 1, \ldots, n_i$.

If p is a projection in $M_\infty(A_i)$ we of course write p as an n_i-tuple of projections p_k from $M_\infty(C(X_{ik}, M_{t(i,k)}(\mathbf{C})))$. Since the X_{ik} are connected, each p_k has constant rank, which we denote by $\operatorname{rank} p_k$.

For indices $i \leq j$ in I, define

$$d_j = \max\{t(j,k)^{-1} \dim X_{jk} \mid k = 1, \ldots, n_j\}$$

$$\mu_{ji} = \min\{t(j,k)^{-1} \operatorname{rank} \phi_{ji}(e_{il})_k \mid k = 1, \ldots, n_j;\ l = 1, \ldots, n_i;\ \phi_{ji}(e_{il})_k \neq 0\},$$

where $\dim X_{jk}$ denotes the *covering dimension* of X_{jk} (see, e.g., [**22**, pp. 44-45]). We shall say that the directed family (A_i, ϕ_{ji}) has *slow dimension growth* in case

$$\lim_{j \geq i} d_j / \mu_{ji} = 0$$

for all $i \in I$.

2.2. The definition of slow dimension growth just given is a modification of the definition given in [**3**], which requires

$$\lim_{j \to \infty} d_j = 0.$$

Since $\mu_{ji} \leq 1$ for $j \geq i$, the condition given in Section 2.1 is stronger in general than the condition in [**3**]. However, in [**3**] slow dimension growth is only used in connection with simple unital inductive limits, in which case the two definitions coincide, as follows.

Indeed, assume that A is simple and that the homomorphisms ϕ_{ji} are all unital, and fix an index $i \in I$. Since $\phi_{ji}(e_{il}) \neq 0$ for $j \geq i$ and $l = 1, \dots, n_i$, it follows from the simplicity of A that there is an index $n \geq i$ such that the projections $\phi_{ni}(e_{il})$ are full. Hence, there exists a positive integer t such that $e_n \lesssim t.\phi_{ni}(e_{il})$ for all l. For any $j \geq n$, we then have $e_j \lesssim t.\phi_{ji}(e_{il})$ for all l, whence $\operatorname{rank} \phi_{ji}(e_{il})_k \geq t(j,k)/t$ for all l, k, and so $\mu_{ji} \geq 1/t$. Therefore, $\lim_{j\to\infty} d_j = 0$ implies $\lim_{j\geq i} d_j/\mu_{ji} = 0$ in this case.

2.3. Blackadar, Dădărlat and Rørdam have shown that any simple unital C^*-algebra which is the C^*-inductive limit of a countable sequence of C^*-algebras with slow dimension growth must have stable rank 1 [**3**, Theorem 1] (cf. also [**7**, 3.6]), and hence such algebras have real rank at most 1 [**5**, 1.2]. In general, however, C^*-inductive limits with slow dimension growth as in Section 2.1 do not always have stable rank 1. For our purposes, cancellation of projections suffices—see Proposition 2.6.

An example in which slow dimension growth does not lead to stable rank 1 can be obtained as follows. Let D be the closed unit disc in the complex plane, and let A be the C^*-inductive limit of the algebras $A_i = M_{2^i}(C(D))$, with block diagonal connecting maps $\phi_{ji} : A_i \to A_j$ for $i \leq j$. Since $\dim D$ is finite and the ϕ_{ji} are unital, this system has slow dimension growth. Now let $z \in A_0$ denote the inclusion map $D \to \mathbf{C}$. If A had stable rank 1, there would be an invertible element $u \in A_n$ for some n such that $||u - \phi_{n0}(z)|| < 1$. We compute that

$$||\phi_{n0}(z)u^* + \phi_{n0}(1 - zz^*) - 1|| \leq ||\phi_{n0}(z)|| \cdot ||u^* - \phi_{n0}(z^*)|| < 1,$$

and so $\phi_{n0}(z)u^* + \phi_{n0}(1 - zz^*)$ is invertible. Set

$$d = \det(\phi_{n0}(z) + \phi_{n0}(1 - zz^*)(u^*)^{-1});$$

then d is an invertible function in $C(D)$. Since $\phi_{n0}(z)$ and $\phi_{n0}(1 - zz^*)$ are just the diagonal matrices z·identity and $(1 - zz^*)$·identity,

$d = z^n + (1-zz^*)f$ for some $f \in C(D)$. Now $g = d/\sqrt{dd^*}$ is a continuous map from D to the unit circle T. Hence, $g|_T$ has winding number 0. But $g|_T = z^n$, which has winding number n. This contradiction shows that A cannot have stable rank 1.

We next give a formula for ranks of projections in directed families of the form in Section 2.1. This formula records the same information as the observation of Martin and Pasnicu that the diagram [**21**, 2.7] is commutative up to an equivalence on projections. We adopt their method of proof since that is simpler than our original version.

Lemma 2.4. *Let (A_i, ϕ_{ji}) be a directed family as in Section* 2.1. *Pick indices $i \le j$, and let $l \in \{1, \dots, n_i\}$ and $k \in \{1, \dots, n_j\}$. Then*

$$\operatorname{rank} \phi_{ji}(f)_k = t(i,l)^{-1} \operatorname{rank} \phi_{ji}(e_{il})_k \operatorname{rank} f$$

for all projections $f \in M_\infty(e_{il}A_i)$.

Proof. As in Section 2.1, we identify the algebras $e_{il}A_i$ and $e_{jk}A_j$ with $C(X_{il}, M_{t(i,l)}(\mathbf{C}))$ and $C(X_{jk}, M_{t(j,k)}(\mathbf{C}))$. We have a nonunital C^*-homomorphism $e_{il}A_i \to e_{jk}A_j$ given by the rule $g \mapsto \phi_{ji}(g)_k$. Given a point $y \in X_{jk}$, it follows from [**21**, 2.5] that there exist points $x_1, \dots, x_n \in X_{il}$ and a unitary $u \in M_{t(j,k)}(\mathbf{C})$ such that

$$\phi_{ji}(g)_k(y) = u(\operatorname{diag}(g(x_1), \dots, g(x_n), 0, \dots, 0))u^*$$

for all $g \in e_{il}A_i$. Consequently, $\operatorname{rank} \phi_{ji}(f)_k = n.\operatorname{rank} f$ for any projection $f \in e_{il}A_i$, and similarly for any projection in $M_\infty(e_{il}A_i)$. In particular, $\operatorname{rank} \phi_{ji}(e_{il})_k = n.t(i,l)$ and therefore $n = t(i,l)^{-1} \operatorname{rank} \phi_{ji}(e_{il})_k$, proving the lemma. $\square$

2.5. In order to make efficient use of slow dimension growth, we shall need some cancellation results for projections in matrix algebras over a homogeneous C^*-algebra $C(X)$ (cf. [**1**, 6.10.3] and [**3**, Lemma D]). While the first two of these results are standard and well known in the case that X is a finite CW-complex, we have not located a detailed reference for the general case, and so we sketch an argument below. Part (c) has been independently observed by N.C. Phillips [unpublished].

Observe that if p is a projection in $M_\infty(C(X))$, then the rule $x \mapsto \operatorname{trace} p(x) = \operatorname{rank} p(x)$ defines a continuous map $X \to \mathbf{Z}$. Hence, X must be a finite union of pairwise disjoint clopen subsets on which p has constant rank.

Theorem 2.5. *Let X be a compact Hausdorff space with $\dim X = d < \infty$, and let p, q, r be projections in $M_\infty(C(X))$.*

(a) *If $\operatorname{rank} p(x) \geq k + (d-1)/2$ for some $k \in \mathbf{N}$ and all $x \in X$, then $k.1_{C(X)} \lesssim p$.*

(b) *If $p \oplus r \sim q \oplus r$ and $\operatorname{rank} p(x) \geq d/2$ for all $x \in X$, then $p \sim q$.*

(c) *If $\operatorname{rank} q(x) - \operatorname{rank} p(x) \geq (d-1)/2$ for all $x \in X$, then $p \lesssim q$.*

Proof. After restricting to suitable clopen subsets of X, there is no loss of generality in assuming that p, q, r have constant rank, say ranks a, b, c. We may also assume that $p, q, r \in M_t(C(X))$ for some $t \in \mathbf{N}$.

By a theorem of Mardešić [**22**, 27-8], X is homeomorphic to an inverse limit of compact metric spaces X_α with $\dim X_\alpha \leq d$ for all α. The corresponding natural maps $\eta_\alpha : X \to X_\alpha$ induce C^*-homomorphisms $\eta_\alpha^* : C(X_\alpha) \to C(X)$, and the union of the images $\eta_\alpha^*(C(X_\alpha))$ is a *-subalgebra of $C(X)$ which separates points of X and so is dense in $C(X)$. Hence, there exist projections $p_1, q_1, r_1 \in M_t(C(X_\alpha))$ for some α such that $\eta_\alpha^*(p_1)$, $\eta_\alpha^*(q_1)$, $\eta_\alpha^*(r_1)$ lie within $1/t$ of p, q, r; moreover, in case (b) we may assume in addition that $p_1 \oplus r_1 \sim q_1 \oplus r_1$. For $x \in X$, we have

$$\begin{aligned}|\operatorname{rank} p_1(\eta_\alpha(x)) - a| &= |\operatorname{trace}\,(\eta_\alpha^*(p_1)(x) - p(x))| \\ &\leq t||\eta_\alpha^*(p_1)(x) - p(x)|| \leq t||\eta_\alpha^*(p_1) - p|| < 1\end{aligned}$$

and so $\operatorname{rank} p_1(\eta_\alpha(x)) = a$. Likewise, $\operatorname{rank} q_1(\eta_\alpha(x)) = b$ and $\operatorname{rank} r_1(\eta_\alpha(x)) = c$ for all $x \in X$. Consequently, there exists a clopen subset $Y_\alpha \subseteq X_\alpha$ such that $\eta_\alpha(X) \subseteq Y_\alpha$ and p_1, q_1, r_1 have constant ranks a, b, c on Y_α. As a result, it suffices to prove the theorem for the restrictions of p_1, q_1, r_1 to Y_α.

Thus we may assume that X is a compact metric space. In this case a theorem of Freudenthal [**22**, 27-4] shows that X is homeomorphic to the inverse limit of a sequence of finite CW-complexes X_i with $\dim X_i \leq d$ for all i. Arguing as above, we see that it now suffices to prove the theorem for projections over subcomplexes of the X_i. (A reduction from compact Hausdorff spaces directly to CW-complexes is not possible, in view of Mardešić's example of a compact Hausdorff space of dimension 1 which cannot be obtained as an inverse limit of CW-complexes with dimension at most 1 [**22**, 27-12].)

Therefore, we may now assume that X is a finite CW-complex. At this point we quote the Serre-Swan theorem [**20**, I.6.18]: the section functor induces a category equivalence from the category of complex vector bundles over X to the category of finitely generated projective $C(X)$-modules. Statements (a) and (b) now follow from [**18**, 8.1.2 and 8.1.5]. To prove (c), we may assume that $\operatorname{rank} p > 0$, and so $\operatorname{rank} q \geq d/2$. Choose a projection $p' \in M_\infty(C(X))$ such that $p \oplus p' \sim k.1_{C(X)}$ for some $k \in \mathbf{N}$. Then $\operatorname{rank} q \oplus p' \geq k + (d-1)/2$ and so $q \oplus p' \sim k.1_{C(X)} \oplus r' \sim p \oplus p' \oplus r'$ for some projection $r' \in M_\infty(C(X))$, by (a). Since $\operatorname{rank} q \geq d/2$, we conclude from (b) that $q \sim p \oplus r'$, and therefore $p \lesssim q$. □

The following proposition was independently proved for simple C^*-inductive limits with slow dimension growth by Martin and Pasnicu in [**21**, 3.7], using essentially the same method. We use the same technique again in Theorem 2.7 and Lemma 2.8.

Proposition 2.6. *Let A be the C^*-inductive limit of a directed family (A_i, ϕ_{ji}) as in Section* 2.1*, with slow dimension growth. Then the projections in $M_\infty(A)$ satisfy cancellation.*

Proof. It suffices to show that if p, q, r are projections in $M_\infty(A_i)$ for some i such that $p \oplus r \sim q \oplus r$, then there exists an index $j \geq i$ such that $\phi_{ji}(p) \sim \phi_{ji}(q)$.

Now p, q, r are orthogonal sums of projections $p_k, q_k, r_k \in M_\infty(e_{ik}A_i)$, and $p_k \oplus r_k \sim q_k \oplus r_k$ for all k. Since it suffices to find an index $j \geq i$ such that $\phi_{ji}(p_k) \sim \phi_{ji}(q_k)$ for all k, we may concentrate on one component. Thus, there is no loss of generality in assuming that p, q, r all lie in $M_\infty(e_{i1}A_i)$. If $p = 0$, then $r \sim q \oplus r$ and immediately $q = 0$ since A_i is stably finite. Hence, we may also assume that $p \neq 0$.

Using the slow dimension growth hypothesis, there is an index $j \geq i$ such that $d_j/\mu_{ji} \leq 2/t(i,1)$. We show that $\phi_{ji}(p)_k \sim \phi_{ji}(q)_k$ for each $k = 1, \dots, n_j$, from which the desired conclusion $\phi_{ji}(p) \sim \phi_{ji}(q)$ follows. In case $\phi_{ji}(e_{i1})_k = 0$, we have $\phi_{ji}(p)_k = \phi_{ji}(q)_k = 0$ and there is nothing to prove. Now assume that $\phi_{ji}(e_{i1})_k \neq 0$. Then

$$\operatorname{rank} \phi_{ji}(e_{i1})_k \geq t(j,k)\mu_{ji} \geq \frac{1}{2} t(j,k)t(i,1)d_j \geq \frac{1}{2} t(i,1) \dim X_{jk}.$$

Using Lemma 2.4, we obtain

$$\begin{aligned} \operatorname{rank} \phi_{ji}(p)_k &= t(i,1)^{-1} \operatorname{rank} \phi_{ji}(e_{i1})_k \operatorname{rank} p \\ &\geq t(i,1)^{-1} \operatorname{rank} \phi_{ji}(e_{i1})_k \\ &\geq \frac{1}{2} \dim X_{jk}. \end{aligned}$$

Since also $\phi_{ji}(p)_k \oplus \phi_{ji}(r)_k \sim \phi_{ji}(q)_k \oplus \phi_{ji}(r)_k$, we conclude from Theorem 2.5(b) that $\phi_{ji}(p)_k \sim \phi_{ji}(q)_k$, as desired. □

Theorem 2.7. *Let A be the C^*-inductive limit of a directed family (A_i, ϕ_{ji}) as in Section* 2.1, *with slow dimension growth. Then the projections in $M_\infty(A)$ satisfy the Riesz interpolation and decomposition properties, and $K_0(A)$ is a Riesz group.*

Proof. Since the projections in $M_\infty(A)$ satisfy cancellation (Proposition 2.6), it suffices to prove that they also satisfy the Riesz decomposition property. It is enough to show that if p, q_1, q_2 are projections in $M_\infty(A_i)$ for some i such that $p \lesssim q_1 \oplus q_2$, then there exist projections $r_1, r_2 \in M_\infty(A_j)$ for some $j \geq i$ such that $\phi_{ji}(p) \sim r_1 \oplus r_2$ and $r_\alpha \lesssim \phi_{ji}(q_\alpha)$ for each α.

Now p, q_1, q_2 are orthogonal sums of projections $p_k, q_{1k}, q_{2k} \in M_\infty(e_{ik}A_i)$, and $p_k \lesssim q_{1k} \oplus q_{2k}$ for all k. Since it suffices to find projections $r_{1k}, r_{2k} \in M_\infty(A_j)$ for some $j \geq i$ such that $\phi_{ji}(p_k) \sim r_{1k} \oplus r_{2k}$ and $r_{\alpha k} \lesssim \phi_{ji}(q_{\alpha k})$ for all α, k, we may concentrate on one component. Thus, there is no loss of generality in assuming that p, q_1, q_2 all lie in $M_\infty(e_{i1}A_i)$.

Note that $\operatorname{rank} p \leq \operatorname{rank} q_1 + \operatorname{rank} q_2$. If equality holds, it follows that $p \sim q_1 \oplus q_2$, and we are done. Thus, we may assume that $\operatorname{rank} p < \operatorname{rank} q_1 + \operatorname{rank} q_2$. Further, if $\operatorname{rank} q_1 = 0$, then $p \lesssim q_2$ and we are done. Hence, we may also assume that $\operatorname{rank} q_1 > 0$.

Next suppose that $\operatorname{rank} p < \operatorname{rank} q_1$. Because of the slow dimension growth hypothesis, there exists an index $j \geq i$ such that $d_j/\mu_{ji} \leq 1/t(i,1)$. Consider $k \in \{1, \dots, n_j\}$ such that $\phi_{ji}(e_{i1})_k \neq 0$; then

$$\operatorname{rank} \phi_{ji}(e_{i1})_k \geq t(j,k)\mu_{ji} \geq t(j,k)t(i,1)d_j \geq t(i,1)\dim X_{jk}.$$

Using Lemma 2.4, we obtain

$$\begin{aligned}\operatorname{rank} \phi_{ji}(q_1)_k - \operatorname{rank} \phi_{ji}(p)_k &= t(i,1)^{-1}\operatorname{rank} \phi_{ji}(e_{i1})_k(\operatorname{rank} q_1 - \operatorname{rank} p)\\ &\geq t(i,1)^{-1}\operatorname{rank} \phi_{ji}(e_{i1})_k \geq \dim X_{jk}.\end{aligned}$$

Then $\phi_{ji}(p)_k \precsim \phi_{ji}(q_1)_k$ by Theorem 2.5(c). On the other hand, for $k \in \{1, \dots, n_j\}$ such that $\phi_{ji}(e_{i1})_k = 0$, we have $\phi_{ji}(p)_k = \phi_{ji}(q_1)_k = 0$. Therefore, $\phi_{ji}(p) \precsim \phi_{ji}(q_1)$ in this case, and again we are done.

Finally, suppose that $\operatorname{rank} p \geq \operatorname{rank} q_1$. Using the slow dimension growth hypothesis a second time, we again obtain an index $j \geq i$ such that $d_j/\mu_{ji} \leq 1/t(i,1)$. Consider $k \in \{1, \dots, n_j\}$ such that $\phi_{ji}(e_{i1})_k \neq 0$. As in the previous paragraph, it follows from Lemma 2.4 that

$$\begin{gathered}\operatorname{rank} \phi_{ji}(p)_k \geq \operatorname{rank} \phi_{ji}(q_1)_k \geq \dim X_{jk}\\ \operatorname{rank} \phi_{ji}(q_1)_k + \operatorname{rank} \phi_{ji}(q_2)_k - \operatorname{rank} \phi_{ji}(p)_k \geq \dim X_{jk}.\end{gathered}$$

Let m_k be the largest integer not exceeding the difference $\operatorname{rank} \phi_{ji}(q_1)_k - (\dim X_{jk} - 1)/2$, and note that

$$\begin{gathered}0 \leq m_k \leq \operatorname{rank} \phi_{ji}(q_1)_k\\ (\dim X_{jk} - 1)/2 \leq \operatorname{rank} \phi_{ji}(q_1)_k - m_k < (\dim X_{jk} + 1)/2.\end{gathered}$$

Choose a projection $r_{1k} \in M_\infty(e_{jk}A_j)$ with $\operatorname{rank} r_{1k} = m_k$. Then

$$\operatorname{rank} \phi_{ji}(p)_k - \operatorname{rank} r_{1k} \geq \operatorname{rank} \phi_{ji}(q_1)_k - \operatorname{rank} r_{1k} \geq (\dim X_{jk} - 1)/2.$$

By Theorem 2.5(c), $r_{1k} \precsim \phi_{ji}(p)_k$ and $r_{1k} \precsim \phi_{ji}(q_1)_k$. In particular, $\phi_{ji}(p)_k \sim r_{1k} \oplus r_{2k}$ for some projection $r_{2k} \in M_\infty(e_{jk}A_j)$. Further,

$$\begin{aligned}\operatorname{rank} \phi_{ji}(q_2)_k - \operatorname{rank} r_{2k} &= \operatorname{rank} \phi_{ji}(q_2)_k - \operatorname{rank} \phi_{ji}(p)_k + m_k\\ &\geq \dim X_{jk} - \operatorname{rank} \phi_{ji}(q_1)_k + m_k\\ &> \dim X_{jk} - (\dim X_{jk} + 1)/2\\ &= (\dim X_{jk} - 1)/2,\end{aligned}$$

and so $r_{2k} \lesssim \phi_{ji}(q_2)_k$ by Theorem 2.5(c).

For $k \in \{1, \dots, n_j\}$ such that $\phi_{ji}(e_{i1})_k = 0$, we have $\phi_{ji}(p)_k = r_{1k} \oplus r_{2k}$ with $r_{\alpha k} = 0 = \phi_{ji}(q_\alpha)_k$. Collecting all the $r_{\alpha k}$, we therefore obtain projections $r_\alpha = (r_{\alpha 1}, \dots, r_{\alpha, n_j})$ in $M_\infty(A_j)$ such that $\phi_{ji}(p) \sim r_1 \oplus r_2$ and $r_\alpha \lesssim \phi_{ji}(q_\alpha)$ for each α. □

The method used in proving Theorem 2.7 can easily be modified to establish Riesz interpolation. While this results in a slightly shorter proof of the theorem (with fewer separate cases), we have chosen the present proof in order to keep the focus on Riesz decomposition.

We refer the reader to [**2, 3, 14**] for examples of C^*-inductive limits with slow dimension growth which have real rank 1.

After seeing an early version of this paper, Handelman pointed out that the methods used to prove Proposition 2.6 and Theorem 2.7 also show that for C^*-algebras A of the type considered there, $K_0(A)$ satisfies the weakened form of unperforation introduced by Elliott in [**9**]. We include this result because it implies that the Riesz decomposition and interpolation properties carry over from $K_0(A)$ to the quotient of $K_0(A)$ modulo its torsion subgroup (see Theorem 2.11).

Lemma 2.8. *Let A be the C^*-inductive limit of a directed family (A_i, ϕ_{ji}) as in Section* 2.1, *with slow dimension growth. Let p, q, r be projections in $M_\infty(A)$.*

(a) *If $m.p \lesssim m.q$ for some $m \in \mathbf{N}$, then there exist projections $p', p'', q', q'' \in M_\infty(A)$ such that $p = p' \oplus p''$ and $q = q' \oplus q''$ while $p' \lesssim q'$ and $m.p'' \sim m.q''$.*

(b) *If there exist $m, n \in \mathbf{N}$ such that $m.q \sim m.r$ and $r \lesssim n.p \oplus q$, then $r \lesssim p \oplus q$ and $q \lesssim p \oplus r$.*

Proof. (a) It suffices to show that if f, g are projections in $M_\infty(A_i)$ for some i such that $m.f \lesssim m.g$, then there exist projections $f', f'', g', g'' \in M_\infty(A_j)$ for some $j \geq i$ such that $\phi_{ji}(f) = f' \oplus f''$ and $\phi_{ji}(g) = g' \oplus g''$ while $f' \lesssim g'$ and $m.f'' \sim m.g''$. Since it is enough to find such decompositions for the images under ϕ_{ji} of each component of f and g, there is no loss of generality in assuming that $f, g \in M_\infty(e_{i1}A_i)$. It follows from the assumption $m.f \lesssim m.g$ that $\operatorname{rank} f \leq \operatorname{rank} g$.

If $\operatorname{rank} f = \operatorname{rank} g$, then since $m.f \lesssim m.g$, we must have $m.f \sim m.g$. In this case, take $j = i$ and $f' = g' = 0$, while $f'' = f$ and $g'' = g$.

Now suppose that $\operatorname{rank} f < \operatorname{rank} g$. By the slow dimension growth hypothesis, there is an index $j \geq i$ such that $d_j/\mu_{ji} \leq 2/t(i,1)$. For $k \in \{1, \dots, n_j\}$ such that $\phi_{ji}(e_{i1})_k \neq 0$, it follows that $\operatorname{rank} \phi_{ji}(e_{i1})_k \geq (1/2)t(i,1)\dim X_{jk}$. Using Lemma 2.4, we obtain

$$\operatorname{rank} \phi_{ji}(g)_k - \operatorname{rank} \phi_{ji}(f)_k \geq t(i,1)^{-1}\operatorname{rank} \phi_{ji}(e_{i1})_k \geq \frac{1}{2}\dim X_{jk},$$

and hence $\phi_{ji}(f)_k \lesssim \phi_{ji}(g)_k$ by Theorem 2.5(c). On the other hand, for $k \in \{1, \dots, n_j\}$ such that $\phi_{ji}(e_{i1})_k = 0$, we have $\phi_{ji}(f)_k = \phi_{ji}(g)_k = 0$. Thus, $\phi_{ji}(f) \lesssim \phi_{ji}(g)$, and so we may take $f'' = g'' = 0$ while $f' = \phi_{ji}(f)$ and $g' = \phi_{ji}(g)$.

(b) It suffices to show that if f, g, h are projections in $M_\infty(A_i)$ for some i such that $m.g \sim m.h$ and $h \lesssim n.f \oplus g$, then there exists an index $j \geq i$ such that $\phi_{ji}(h) \lesssim \phi_{ji}(f \oplus g)$ and $\phi_{ji}(g) \lesssim \phi_{ji}(f \oplus h)$. There is no loss of generality in assuming that $f, g, h \in M_\infty(e_{i1}A_i)$. Note that $\operatorname{rank} g = \operatorname{rank} h$.

If $f = 0$, then $h \lesssim g$, and since $\operatorname{rank} g = \operatorname{rank} h$ we must have $g \sim h$. In this case take $j = i$.

Now suppose that $f \neq 0$, and choose an index $j \geq i$ such that $d_j/\mu_{ji} \leq 2/t(i,1)$. For $k \in \{1, \dots, n_j\}$ such that $\phi_{ji}(e_{i1})_k \neq 0$, it follows using Lemma 2.4 that

$$\operatorname{rank} \phi_{ji}(f \oplus g)_k - \operatorname{rank} \phi_{ji}(h)_k \geq \frac{1}{2}\dim X_{jk}$$
$$\operatorname{rank} \phi_{ji}(f \oplus h)_k - \operatorname{rank} \phi_{ji}(g)_k \geq \frac{1}{2}\dim X_{jk}.$$

Then $\phi_{ji}(h)_k \lesssim \phi_{ji}(f \oplus g)_k$ and $\phi_{ji}(g)_k \lesssim \phi_{ji}(f \oplus h)_k$ by Theorem 2.5(c). On the other hand, for $k \in \{1, \dots, n_j\}$ such that $\phi_{ji}(e_{i1})_k = 0$, we have $\phi_{ji}(g)_k = \phi_{ji}(h)_k = 0$. Therefore, $\phi_{ji}(h) \lesssim \phi_{ji}(f \oplus g)$ and $\phi_{ji}(g) \lesssim \phi_{ji}(f \oplus h)$. □

2.9. Let G be a partially ordered abelian group, and let $\operatorname{tor} G$ denote its torsion subgroup. Then G is *weakly unperforated* provided

(a) Whenever $x \in G$ and $m \in \mathbf{N}$ such that $mx \geq 0$, there exists $y \in \operatorname{tor} G$ such that $x + y \geq 0$ and $my = 0$.

(b) Whenever $x \in G^+$, $y \in \operatorname{tor} G$, and $n \in \mathbf{N}$ such that $nx + y \geq 0$, then $x \pm y \geq 0$.

For the equivalence of this definition with the weak unperforation condition given by Elliott in [**9**], see [**15**, 8.1].

Proposition 2.10. *Let A be the C^*-inductive limit of a directed family (A_i, ϕ_{ji}) as in Section 2.1, with slow dimension growth. Then $K_0(A)$ is weakly unperforated.*

Proof. First, let $x \in K_0(A)$ and $m \in \mathbf{N}$ such that $mx \geq 0$. Write $x = [q] - [p]$ for some projections $p, q \in M_\infty(A)$. Since the projections in $M_\infty(A)$ satisfy cancellation (Proposition 2.6), the hypothesis $mx \geq 0$ implies that $m.p \lesssim m.q$. By Lemma 2.8(a), there exist projections $p', p'', q', q'' \in M_\infty(A)$ such that $p = p' \oplus p''$ and $q = q' \oplus q''$ while $p' \lesssim q'$ and $m.p'' \sim m.q''$. Set $y = [p''] - [q'']$. Then $my = 0$ and $x + y = [q'] - [p'] \geq 0$.

Second, let $x \in K_0(A)^+$, $y \in \operatorname{tor} K_0(A)$, $n \in \mathbf{N}$ such that $nx + y \geq 0$. Write $x = [p]$ and $y = [q] - [r]$ for some projections $p, q, r \in M_\infty(A)$. Then $r \lesssim n.p \oplus q$ and $m.q \sim m.r$ for some $m \in \mathbf{N}$. By Lemma 2.8(b), $r \lesssim p \oplus q$ and $q \lesssim p \oplus r$, and therefore $x \pm y \geq 0$. □

Theorem 2.11. *Let A be the C^*-inductive limit of a directed family (A_i, ϕ_{ji}) as in Section 2.1, with slow dimension growth. Then $K_0(A)/\operatorname{tor} K_0(A)$ is an unperforated Riesz group.*

Proof. Since $K_0(A)$ is directed, so is $K_0(A)/\operatorname{tor} K_0(A)$. By [**15**, 8.1], $K_0(A)/\operatorname{tor} K_0(A)$ is unperforated. Since $K_0(A)$ satisfies Riesz decomposition by Theorem 2.7, it follows from Proposition 2.10 and [**9**, 4.5] that $K_0(A)/\operatorname{tor} K_0(A)$ satisfies Riesz decomposition. □

2.12. Under the hypotheses of Theorem 2.11, it follows from [**15**, 8.5b] that, whenever $x, y \in K_0(A)^+$ and $m \in \mathbf{N}$ such that $(m+1)x \leq my$, then $x \leq y$. Since the projections in $M_\infty(A)$ satisfy cancellation, this result implies that whenever p and q are projections in $M_\infty(A)$ with $(m+1).p \lesssim m.q$ for some $m \in \mathbf{N}$, then $p \lesssim q$. The latter result can also be proved directly, using the methods of Proposition 2.6, Theorem

2.7 and Lemma 2.8.

3. Approximately semi-homogeneous C^*-algebras. Without any assumptions such as slow dimension growth, the projections in an inductive limit as in Section 2.1 need not satisfy Riesz decomposition. However, a weak form of Riesz decomposition still holds, as we will see in this section. Here we work in a more general setting than Section 2.1. First, rather than dealing directly with inductive limits, we work with a C^*-algebra A which has "enough" sub-C^*-algebras isomorphic to finite direct products of homogeneous C^*-algebras. Second, we make no connectedness assumptions on the spectra of the homogeneous C^*-algebras that occur. While conceivably this "approximately semi-homogeneous" setting might not be any more general than that of arbitrary inductive limits, we prefer it for the sake of notational convenience.

3.1. For the sake of convenient terminology, let us say that a C^*-algebra B is *semi-homogeneous* provided B is isomorphic to a finite direct product of homogeneous C^*-algebras of the particular form $C_0(X, M_t(\mathbf{C}))$ where X is a locally compact Hausdorff space and t is a positive integer. Any such C^*-algebra B can be uniquely presented in the form

$$C_0(X_1, M_{t_1}(\mathbf{C})) \times \cdots \times C_0(X_n, M_{t_n}(\mathbf{C}))$$

where the X_i are nonempty locally compact Hausdorff spaces and $t_1 < \cdots < t_n$. We shall refer to t_1 as the *minimum matrix rank* of B, and to t_n as the *maximum matrix rank* of B.

3.2. We call a C^*-algebra A *approximately semi-homogeneous* provided that for each $\varepsilon > 0$ and each finite subset $F \subset A$, there is a semi-homogeneous sub-C^*-algebra $B \subseteq A$ such that each element of F lies within ε of (an element of) B.

Lemma 3.2. *Let A be an approximately semi-homogeneous C^*-algebra, let $t_1, \dots, t_n \in \mathbf{N}$, and for $j = 1, \dots, n$ let p_j be a projection in $M_{t_j}(A)$. Let $\varepsilon > 0$, let F be a finite subset of A, and let (a_{ij}) and (b_{ij}) be $m \times n$ matrices of nonnegative integers such that $a_{i1}.p_1 \oplus \cdots \oplus a_{in}.p_n \sim b_{i1}.p_1 \oplus \cdots \oplus b_{in}.p_n$ for all i. Then there exist a semi-homogeneous sub-C^*-algebra $B \subseteq A$ and projections $q_j \in M_{t_j}(B)$ for each j such that*

(a) $||p_j - q_j|| < \varepsilon$ *and* $p_j \sim q_j$ *for all* j.

(b) $a_{i1}.q_1 \oplus \cdots \oplus a_{in}.q_n \sim b_{i1}.q_1 \oplus \cdots \oplus b_{in}.q_n$ *in* $M_\infty(B)$ *for all* i.

(c) *Each element of* F *lies within* ε *of* B.

Moreover, if A *is unital, then* B *can be chosen to be a unital subalgebra.*

Proof. We may assume that $\varepsilon < 1/12$. Choose partial isometries $x_i \in M_\infty(A)$ such that $x_i x_i^* = a_{i1}.p_1 \oplus \cdots \oplus a_{in}.p_n$ and $x_i^* x_i = b_{i1}.p_1 \oplus \cdots \oplus b_{in}.p_n$ for each i. There exists a semi-homogeneous sub-C^*-algebra $B \subseteq A$ such that each x_i lies within ε of some $y_i \in M_\infty(B)$, each p_j lies within $\varepsilon/16$ of some $z_j \in M_{t_j}(B)$, and each element of F lies within ε of B. Without loss of generality, each z_j is self-adjoint.

For each j, observe that $||z_j^2 - z_j|| < \varepsilon/4$. Hence, there exists a projection $q_j \in M_{t_j}(B)$ such that $||z_j - q_j|| < \varepsilon/2$ (see, e.g., [**12**, 19.8]), and so $||p_j - q_j|| < \varepsilon$. In particular $||p_j - q_j|| < 1$ and thus $p_j \sim q_j$. Furthermore,

$$||(a_{i1}.p_1 \oplus \cdots \oplus a_{in}.p_n) - (a_{i1}.q_1 \oplus \cdots \oplus a_{in}.q_n)|| < \varepsilon$$
$$||(b_{i1}.p_1 \oplus \cdots \oplus b_{in}.p_n) - (b_{i1}.q_1 \oplus \cdots \oplus b_{in}.q_n)|| < \varepsilon$$

for all i. Next, observe for each i that

$$||(a_{i1}.q_1 \oplus \cdots \oplus a_{in}.q_n)y_i(b_{i1}.q_1 \oplus \cdots \oplus b_{in}.q_n) - y_i|| < 4\varepsilon < 1/3$$
$$||y_i y_i^* - (a_{i1}.q_1 \oplus \cdots \oplus a_{in}.q_n)|| < 4\varepsilon < 1/3$$
$$||y_i^* y_i - (b_{i1}.q_1 \oplus \cdots \oplus b_{in}.q_n)|| < 4\varepsilon < 1/3,$$

from which it follows that $a_{i1}.q_1 \oplus \cdots \oplus a_{in}.q_n \sim b_{i1}.q_1 \oplus \cdots \oplus b_{in}.q_n$ in $M_\infty(B)$ [**12**, 19.7].

Now suppose that A is unital. In this case we can choose B as above such that B also contains a self-adjoint element w satisfying $||1 - w|| < 1/16$. It follows that there exists a projection $e \in B$ such that $||1 - e|| < 1$, and then $e = 1$. □

Observe that this lemma also applies to subequivalences, since for instance any subequivalence of the form $a_1.p_1 \oplus \cdots \oplus a_n.p_n \lesssim b_1.p_1 \oplus \cdots \oplus b_n.p_n$ can be rewritten as

$$a_1.p_1 \oplus \cdots \oplus a_n.p_n \oplus 1.p_{n+1} \sim b_1.p_1 \oplus \cdots \oplus b_n.p_n \oplus 0.p_{n+1}$$

for some projection p_{n+1}.

Corollary 3.3. *Every approximately semi-homogeneous C^*-algebra A is stably finite.*

Proof. Consider projections $p_1, p_2 \in M_\infty(A)$ such that $p_1 \sim p_1 \oplus p_2$. By Lemma 3.2, there exist a semi-homogeneous sub-C^*-algebra $B \subseteq A$ and projections $q_1, q_2 \in M_\infty(B)$ such that $p_j \sim q_j$ for each j while also $q_1 \sim q_1 \oplus q_2$. Since B is stably finite, $q_2 = 0$, and therefore $p_2 = 0$. □

Proposition 3.4. *Let A be an approximately semi-homogeneous C^*-algebra.*

(a) *If p, q, r are projections in $M_\infty(A)$ such that $p \oplus r \sim q \oplus r$, then there exists a positive integer m such that $m.p \sim m.q$.*

(b) *Let p_1, p_2, q_1, q_2 be projections in $M_\infty(A)$ such that $p_i \lesssim q_j$ for all i, j. Then there exist a projection $r \in M_\infty(A)$ and a positive integer m such that $m.p_i \lesssim r \lesssim m.q_j$ for all i, j.*

Proof. (a) In view of Lemma 3.2, there exist a semi-homogeneous sub-C^*-algebra $B \subseteq A$ and projections $p', q', r' \in M_\infty(B)$ such that p', q', r' are equivalent to p, q, r while also $p' \oplus r' \sim q' \oplus r'$ in $M_\infty(B)$. Since it suffices to show that $m.p' \sim m.q'$ for some $m \in \mathbf{N}$, there is no loss of generality in assuming that A itself is semi-homogeneous. We then immediately reduce to the case that $A = C_0(X)$ for some nonempty locally compact Hausdorff space X and p, q, r all have constant rank. Moreover, since there is nothing to prove if $r = 0$, we may further assume that $r \neq 0$. Then $||r(x)|| = 1$ for all $x \in X$, which forces X to be compact.

By, e.g., [**23**, Theorem VII.3], X is homeomorphic to an inverse limit of finite CW-complexes X_α. As in the proof of Theorem 2.5, A is thus isomorphic to the C^*-inductive limit of the algebras $C(X_\alpha)$, and we can reduce the problem to the case of projections in $M_\infty(C(X_\alpha))$ for some α. Hence, we may now assume that X is a finite CW-complex. In particular, $\dim X < \infty$.

If $p = 0$, then $r \sim q \oplus r$ and immediately $q = 0$. Now assume that $p \neq 0$, and choose $m \in \mathbf{N}$ such that $\operatorname{rank} m.p \geq (1/2)\dim X$. Since $m.p \oplus m.r \sim m.q \oplus m.r$, we conclude from Theorem 2.5(b) that $m.p \sim m.q$.

(b) As in the proof of (a), we may reduce to the case that $A = C(X)$ for some finite CW-complex X and p_1, p_2, q_1, q_2 all have constant rank.

If $\operatorname{rank} p_s = \operatorname{rank} q_t$ for some s, t, then since $p_s \lesssim q_t$ we obtain $p_s \sim q_t$, and consequently $p_i \lesssim p_s \lesssim q_j$ for all i, j.

Now suppose that $\operatorname{rank} p_i < \operatorname{rank} q_j$ for all i, j. Set $a = \max\{\operatorname{rank} p_1, \operatorname{rank} p_2\}$ and $b = \min\{\operatorname{rank} q_1, \operatorname{rank} q_2\}$, and choose $m \in \mathbf{N}$ such that $m(b-a) \geq 2\dim X$. Then choose a projection $r \in M_\infty(A)$ with constant rank $ma + \dim X$. Since

$$\operatorname{rank} r - \operatorname{rank} m.p_i \geq \dim X \quad \text{and} \quad \operatorname{rank} m.q_j - \operatorname{rank} r \geq \dim X$$

for all i, j, we therefore conclude from Theorem 2.5(c) that $m.p_i \lesssim r \lesssim m.q_j$ for all i, j. $\square$

3.5. Given any partially ordered abelian group G, the tensor product $G \otimes \mathbf{Q}$ has a natural partially ordered abelian group structure (cf. [**16**]), with positive cone (generated by) the set $\{x \otimes a \mid x \in G^+; a \in \mathbf{Q}^+\}$. It is advantageous to write the elements of $G \otimes \mathbf{Q}$ as fractions x/n ($= x \otimes n^{-1}$) where $x \in G$ and $n \in \mathbf{N}$. Such a fraction x/n is zero in $G \otimes \mathbf{Q}$ if and only if x is a torsion element of G, and $x/n \geq 0$ in $G \otimes \mathbf{Q}$ if and only if there is some $m \in \mathbf{N}$ such that $mx \geq 0$ in G. Note that $G \otimes \mathbf{Q}$ is automatically unperforated.

We next prove that for any approximately semi-homogeneous C^*-algebra A, the rationalized K_0-group $K_0(A) \otimes \mathbf{Q}$ is a Riesz group. (In the case that A is homogeneous, this can easily be obtained from known general results, as pointed out by Blackadar.)

Theorem 3.6. *Let A be an approximately semi-homogeneous C^*-algebra.*

(a) *Given any $x_1, x_2, y_1, y_2 \in K_0(A)$ such that $x_i \leq y_j$ for all i, j, there exist $z \in K_0(A)$ and $m \in \mathbf{N}$ such that $mx_i \leq z \leq my_j$ for all i, j.*

(b) *The partially ordered abelian group $K_0(A) \otimes \mathbf{Q}$ is a Riesz group.*

(c) *If A is unital, then the state space $S(K_0(A), [1_A])$ is a Choquet simplex.*

Proof. (a) After translating x_1, x_2, y_1, y_2 by some common element of $K_0(A)$, we may assume that there are projections p_1, p_2, q_1, q_2 in $M_\infty(A)$ such that each $x_i = [p_i]$ and each $y_j = [q_j]$. Then there is a projection $f \in M_\infty(A)$ such that $p_i \oplus f \lesssim q_j \oplus f$ for all i, j. Hence, after a further translation of x_1, x_2, y_1, y_2 by $[f]$, we may assume that $p_i \lesssim q_j$ for all i, j. Now Proposition 3.4(b) provides us with a projection $r \in M_\infty(A)$ and a positive integer m such that $m.p_i \lesssim r \lesssim m.q_j$ for all i, j. Therefore $mx_i \leq [r] \leq my_j$ in $K_0(A)$ for all i, j.

(b) This is clear from (a).

(c) The natural map $K_0(A) \to K_0(A) \otimes \mathbf{Q}$ induces an affine homeomorphism from the state space $S(K_0(A) \otimes \mathbf{Q}, [1_A]/1)$ onto $S(K_0(A), [1_A])$. Since the state space of any Riesz group with order-unit is a Choquet simplex [**13**, 10.17], we are done. □

3.7. For unital inductive limits of semi-homogeneous C^*-algebras, Theorem 3.6(c) can be proved using several standard facts. First, the state space of K_0 of any unital commutative ring is a Choquet simplex by [**17**, 3.9b], from which it immediately follows that the state space of K_0 of any unital semi-homogeneous C^*-algebra is a Choquet simplex. To pass to inductive limits, recall that K_0 preserves inductive limits [**12**, 19.9], that the state space functor converts inductive limits to inverse limits [**13**, 6.14], and that inverse limits of Choquet simplices are Choquet simplices [**13**, 11.7].

3.8. Let D denote the unital AF C^*-algebra obtained as the C^*-inductive limit of the matrix algebras $M_n(\mathbf{C})$ for $n \in \mathbf{N}$ and the unital block diagonal maps $M_m(\mathbf{C}) \to M_n(\mathbf{C})$ for $m \mid n$. (This algebra is sometimes called the *universal Glimm* (*UHF*) *algebra*.) Then $K_0(A \otimes D) \cong K_0(A) \otimes \mathbf{Q}$ for any C^*-algebra A.

Corollary 3.8. *If A is an approximately semi-homogeneous C^*-algebra, then $K_0(A \otimes D)$ is a Riesz group. Furthermore, the projections in $M_\infty(A \otimes D)$ satisfy cancellation, and the Riesz interpolation and decomposition properties.*

Proof. That $K_0(A \otimes D)$ is a Riesz group is immediate from Theorem 3.6(b). Using Proposition 3.4(a), we infer that the projections in

$M_\infty(A \otimes D)$ satisfy cancellation, and then Riesz interpolation and decomposition follow. □

4. Simple approximately semi-homogeneous C^*-algebras. For a simple unital approximately semi-homogeneous C^*-algebra A, the weak form of Riesz decomposition obtained in Theorem 3.6 can be improved somewhat. In particular, it can be improved sufficiently to show that Riesz decomposition in $K_0(A)$ will follow from strict unperforation, the advantage being that strict unperforation is an "equational" condition rather than an "existential" condition. It follows that Riesz decomposition for projections holds in some simple inductive limits which do not satisfy slow dimension growth, such as those investigated in [**14**]. Although the inductive limits used in [**14**] can be modified to a form in which slow dimension growth holds, it seems unlikely that such a modification is possible in general.

4.1. Suppose that A is the C^*-inductive limit of some directed family of unital semi-homogeneous C^*-algebras A_i and injective unital C^*-homomorphisms. It is a well-known piece of folklore that if A is simple and infinite-dimensional, the minimum matrix ranks of the A_i must be unbounded. An analogous result holds for approximately semi-homogeneous C^*-algebras, as follows.

Lemma 4.1. *Let A be a simple, unital, infinite-dimensional, approximately semi-homogeneous C^*-algebra. Let $m \in \mathbf{N}$, let $\varepsilon > 0$, and let F be a finite subset of A. Then there exists a unital semi-homogeneous sub-C^*-algebra $B \subseteq A$ such that the minimum matrix rank of B is greater than m and each element of F lies within ε of B.*

Proof. Assume that the conclusion fails. We claim that the *maximum* matrix rank of any unital semi-homogeneous sub-C^*-algebra $C \subseteq A$ is at most m.

If n is the maximum matrix rank of C, there exists a nonzero projection $p \in C$ such that $n.p \lesssim 1_C$. By simplicity, p is full in A, and hence $1_A \lesssim k.p$ for some $k \in \mathbf{N}$. According to Lemma 3.2, there exist a unital semi-homogeneous sub-C^*-algebra $D \subseteq A$ and a projection $q \in D$ such that $p \sim q$ in A and $n.q \lesssim 1_D \lesssim k.q$ in D while also each element of F lies within ε of D. Since $1_D \lesssim k.q$ in D, the projection q is full in D. Hence, it follows from the relation $n.q \lesssim 1_D$ that the minimum matrix rank of D is at least n. On the other hand, since the conclusion of the lemma was assumed to fail, the minimum matrix rank of D cannot be greater than m. Thus $n \leq m$ and the claim is proved.

Let s_t denote the standard identity of degree $t = m^2 + 1$, that is, the t-variable function given by the rule

$$s_t(x_1, \dots, x_t) = \sum_{\sigma \in S_t} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(t)}$$

where S_t denotes the symmetric group of degree t. By, e.g., [**19**, Example (4), pp. 13-14], s_t vanishes on (t-tuples from) any matrix ring of rank m or less over any unital commutative ring. (Actually s_{2m} vanishes on all such matrix rings, but this fact—the Amitsur-Levitzki theorem—is a much deeper result.) In view of the claim just proved, s_t vanishes on all unital semi-homogeneous sub-C^*-algebras of A. Since s_t is uniformly continuous, it therefore vanishes on A.

Now A is a primitive ring with center $\mathbf{C}$, and we have just seen that A satisfies the polynomial identity s_t. But then the Kaplansky-Amitsur theorem [**19**, p. 17] implies that A is finite-dimensional over $\mathbf{C}$. This contradiction establishes the lemma. □

4.2. Under the conditions of Lemma 4.1, we can of course prove a strengthened version of Lemma 3.2, in which the semi-homogeneous sub-C^*-algebra B has minimum matrix rank greater than a pre-assigned integer. The following statement contains as much of the strengthened version as we shall need.

Lemma 4.2. *Let A be a simple, unital, infinite-dimensional, approximately semi-homogeneous C^*-algebra. Let $m_0, t_1, \dots, t_n \in \mathbf{N}$, and for $j = 1, \dots, n$ let p_j be a projection in $M_{t_j}(A)$. Let (a_{ij}) and (b_{ij}) be $m \times n$ matrices of nonnegative integers such that $a_{i1}.p_1 \oplus \cdots \oplus a_{in}.p_n \sim b_{i1}.p_1 \oplus \cdots \oplus b_{in}.p_n$ for all i. Then there exist a unital semi-homogeneous sub-C^*-algebra $B \subseteq A$ and projections $q_j \in M_{t_j}(B)$ for each j such that B has minimum matrix rank greater than m_0 and $p_j \sim q_j$ for all j, while also $a_{i1}.q_1 \oplus \cdots \oplus a_{in}.q_n \sim b_{i1}.q_1 \oplus \cdots \oplus b_{in}.q_n$ in $M_\infty(B)$ for all i.*

The notation $p \prec q$, for projections p and q, means that p is equivalent to a proper subprojection of q. In the presence of stable finiteness, $p \prec q$ if and only if $p \lesssim q$ but $p \not\sim q$.

Lemma 4.3. *Let $m_0 \in \mathbf{N}$, let X be a locally compact Hausdorff space, and let p_1, p_2, q_1, q_2 be projections in $M_\infty(C_0(X))$ such that $\operatorname{rank} q_j(x) - \operatorname{rank} p_i(x) \geq m_0 + 1$ for all i, j and all $x \in X$. Then there exist a projection $r \in M_\infty(C_0(X))$ and a positive integer m such that $m.p_i \prec mm_0.r \prec m.q_j$ for all i, j.*

Proof. Invoking again the reductions used in the proof of Proposition 3.4, we may assume without loss of generality that X is a finite CW-complex and that p_1, p_2, q_1, q_2 have constant rank. Set $a = \max\{\operatorname{rank} p_1, \operatorname{rank} p_2\}$ and $b = \min\{\operatorname{rank} q_1, \operatorname{rank} q_2\}$. Since $b - a \geq m_0 + 1$ by hypothesis, there exists $c \in \mathbf{N}$ such that $a < m_0 c < b$. choose a projection $r \in M_\infty(C(X))$ with constant rank c, and set $m = \max\{1, \dim X\}$. The desired conclusions follow from Theorem 2.5(c). $\square$

Proposition 4.4. *Let A be a simple, unital, infinite-dimensional, approximately semi-homogeneous C^*-algebra. Let $m_0 \in \mathbf{N}$, and let p_1, p_2, q_1, q_2 be projections in $M_\infty(A)$ such that $p_i \prec q_j$ for all i, j. Then there exist a projection $r \in M_\infty(A)$ and a positive integer m such that $m.p_i \prec mm_0.r \prec m.q_j$ for all i, j.*

Proof. For each i, j, there is a nonzero projection $s_{ij} \in M_\infty(A)$ such that $q_j \sim p_i \oplus s_{ij}$. Since A is simple, the s_{ij} are full, and hence there is some $n \in \mathbf{N}$ such that $1_A \lesssim n.s_{ij}$ for all i, j. By Lemma 4.2, there exist a unital semi-homogeneous sub-C^*-algebra $B \subseteq A$ and projections $p'_i, q'_j, s'_{ij} \in M_\infty(B)$ such that B has minimum matrix rank greater than $(m_0+1)n$ and the projections p'_i, q'_j, s'_{ij} are equivalent to p_i, q_j, s_{ij} while also $q'_j \sim p'_i \oplus s'_{ij}$ and $1_B \lesssim n.s'_{ij}$ in $M_\infty(B)$ for all i, j. Now

$$B \cong B'' = C(X_1, M_{t_1}(\mathbf{C})) \times \cdots \times C(X_h, M_{t_h}(\mathbf{C}))$$

for some compact Hausdorff spaces X_k and some integers $t_k > (m_0 + 1)n$. Let p''_i, q''_j, s''_{ij} denote the projections corresponding to p'_i, q'_j, s'_{ij} under the induced isomorphism of $M_\infty(B)$ onto $M_\infty(B'')$.

Since $1_{B''} \lesssim n.s''_{ij}$ for all i, j and $t_k > (m_0 + 1)n$ for all k, we see that the components s''_{ijk} of each s''_{ij} satisfy rank $s''_{ijk}(x) > m_0 + 1$ for all i, j, k and all $x \in X_k$. Therefore, by Lemma 4.3 there exist a projection $r'' \in M_\infty(B'')$ and a positive integer m such that $m.p''_i \prec mm_0.r'' \prec m.q''_j$ for all i, j. The projection $r \in M_\infty(B)$ corresponding to r'' satisfies the desired conditions. □

Theorem 4.5. *Let A be a simple, unital, infinite-dimensional, approximately semi-homogeneous C^*-algebra.*

(a) *Given $m_0 \in \mathbf{N}$ and $x_1, x_2, y_1, y_2 \in K_0(A)$ such that $x_i < y_j$ for all i, j, there exist $z \in K_0(A)$ and $m \in \mathbf{N}$ such that $mx_i < mm_0 z < my_j$ for all i, j.*

(b) *Define a new ordering $\leq_1$ on $K_0(A)$ by declaring $x \leq_1 y$ if and only if either $x = y$ or there exists $m \in \mathbf{N}$ such that $mx < my$. Then $(K_0(A), \leq_1)$ satisfies the Riesz interpolation and decomposition properties.*

(c) *Given $x, y_1, y_2 \in K_0(A)^+$ such that $x \leq y_1 + y_2$, there exist $m \in \mathbf{N}$ and $x_1, x_2 \in K_0(A)$ such that $x = x_1 + x_2$ and $0 \leq mx_i \leq my_i$ for each i.*

Proof. (a) After translating x_1, x_2, y_1, y_2 by a suitable common element of $K_0(A)$, we may assume that each $x_i = [p_i]$ and each $y_j = [q_j]$ for some projections $p_1, p_2, q_1, q_2 \in M_\infty(A)$ such that $p_i \prec q_j$ for all i, j. By Proposition 4.4, there exist a projection $r \in M_\infty(A)$ and a positive integer m such that $m.p_i \prec mm_0.r \prec m.q_j$ for all i, j. Then $mx_i \leq mm_0[r] \leq my_j$ for all i, j. For each i, we have $mm_0.r \sim m.p_i \oplus e_i$ for some nonzero projection $e_i \in M_\infty(A)$. Since A is stably finite (Corollary 3.3), $[e_i] \neq 0$ in $K_0(A)$, and hence $mx_i < mm_0[r]$. Similarly, $mm_0[r] < my_j$ for each j.

(b) It is clear that $(K_0(A), \leq_1)$ is a partially ordered abelian group. Consider x_1, x_2, y_1, y_2 in $K_0(A)$ such that $x_i \leq_1 y_j$ for all i, j. If $x_s = y_t$ for some s, t, then $x_i \leq_1 x_s \leq_1 y_j$ for all i, j. If $x_i <_1 y_j$ for all i, j, there exists $m_0 \in \mathbf{N}$ such that $m_0 x_i < m_0 y_j$ for all i, j. By (a), there exist $z \in K_0(A)$ and $m \in \mathbf{N}$ such that $mm_0 x_i < mm_0 z < mm_0 y_j$ for all i, j, whence $x_i <_1 z <_1 y_j$ for all i, j. Therefore, $(K_0(A), \leq_1)$ satisfies Riesz interpolation, and Riesz decomposition follows.

(c) Observe that $x, y_1, y_2 \in (K_0(A), \leq_1)^+$ and that $x \leq_1 y_1 + y_2$. In view of (b), there exist $x_1, x_2 \in K_0(A)$ such that $x = x_1 + x_2$ and $0 \leq_1 x_i \leq_1 y_i$ for each i. Then there exists $m \in \mathbf{N}$ such that $0 \leq mx_i \leq my_i$ for each i. □

4.6. We note that a "fragment" of Riesz decomposition can be obtained from Theorem 4.5(c) in case the projections in $M_\infty(A)$ satisfy cancellation. Namely, if p, q_1, q_2 are projections in $M_\infty(A)$ such that $p \lesssim q_1 \oplus q_2$, then there exist $m \in \mathbf{N}$ and projections $p_1, p_2, r_1, r_2 \in M_\infty(A)$ such that $p \oplus r_1 \oplus r_2 \sim p_1 \oplus p_2$ and $m.r_i \lesssim m.p_i \lesssim m.q_i \oplus m.r_i$ for each i.

Better results follow from Theorem 4.5 under the assumption that $K_0(A)$ is *strictly unperforated*, meaning that whenever $m \in \mathbf{N}$ and $x \in K_0(A)$ with $mx > 0$, it follows that $x > 0$. (See, e.g., [**1**, 10.11.2] or [**14**, Example 12] for examples of simple C^*-algebras A for which $K_0(A)$ is strictly unperforated but still perforated.) Namely:

Corollary 4.7. *Let A be a simple, unital, infinite-dimensional, approximately semi-homogeneous C^*-algebra.*

(a) *If $K_0(A)$ is strictly unperforated, then it satisfies the Riesz interpolation and decomposition properties.*

(b) *If $K_0(A)$ is strictly unperforated and the projections in $M_\infty(A)$ satisfy cancellation, then the projections in $M_\infty(A)$ satisfy the Riesz interpolation and decomposition properties.*

Acknowledgment. We would like to thank B. Blackadar, G. Elliott, D. Handelman, C. Pasnicu and S. Zhang for their helpful comments.

REFERENCES

1. B. Blackadar, *K-theory for operator algebras*, M.S.R.I. Publ. **5**, Springer-Verlag, New York, 1986.

2. B. Blackadar, O. Bratteli, G.A. Elliott, and A. Kumjian, *Reduction of real rank in inductive limits of C^*-algebras*, Math. Ann. **292** (1992), 111–126.

3. B. Blackadar, M. Dădărlat and M. Rørdam, *The real rank of inductive limit C^*-algebras*, Math. Scand. **69** (1991), 211–216.

4. B. Blackadar and D. Handelman, *Dimension functions and traces on C^*-algebras*, J. Func. Anal. **45** (1982), 297–340.

5. L.G. Brown and G.K. Pedersen, *C^*-algebras of real rank zero*, J. Func. Anal. **99** (1991), 131–149.

6. M. Dădărlat, *Inductive limits of C^*-algebras related to some coverings*, Indiana Univ. Math. J. **37** (1988), 135–143.

7. M. Dădărlat, G. Nagy, A. Némethi and C. Pasnicu, *Reduction of topological stable rank in inductive limits of C^*-algebras*, Pacific J. Math. **153** (1992), 267–276.

8. M. Dădărlat and A. Némethi, *Shape theory and* (*connective*) *K-theory*, J. Operator Theory **23** (1990), 207–291.

9. G.A. Elliott, *Dimension groups with torsion*, Internat. J. Math. Math. Sci. **1** (1990), 361–380.

10. ———, *On the classification of C^*-algebras of real rank zero*, I, J. reine angew. Math. **443** (1993), 179–219.

11. K.R. Goodearl, *Von Neumann regular rings*, Pitman, London, 1979.

12. ———, *Notes on real and complex C^*-algebras*, Shiva, Nantwich (Cheshire), 1982.

13. ———, *Partially ordered Abelian groups with interpolation*, Math. Surveys Monographs **20**, Amer. Math. Soc., Providence, 1986.

14. ———, *Notes on a class of simple C^*-algebras with real rank zero*, Publ. Mat. (Barcelona) **36** (1992), 637–654.

15. ———, *K_0 of multiplier algebras of C^*-algebras with real rank zero*, K-Theory, to appear.

16. K.R. Goodearl and D.E. Handelman, *Tensor products of dimension groups and K_0 of unit-regular rings*, Canadian J. Math. **38** (1986), 633–658.

17. K.R. Goodearl and R.B. Warfield, Jr., *State spaces of K_0 of noetherian rings*, J. Algebra **71** (1981), 322–378.

18. D. Husemoller, *Fibre bundles*, (Second Ed.), Graduate Texts Math. **20**, Springer-Verlag, Berlin, 1975.

19. N. Jacobson, *PI-algebras, An introduction*, Lecture Notes Math. **441**, Springer-Verlag, Berlin, 1975.

20. M. Karoubi, *K-theory, An introduction*, Grundlehren Math. Wiss. **226**, Springer-Verlag, Berlin, 1978.

21. M. Martin and C. Pasnicu, *Some comparability results in inductive limit C^*-algebras*, J. Operator Theory, to appear.

22. K. Nagami, *Dimension theory*, Pure Appl. Math. **37**, Academic Press, New York, 1970.

23. J.-I. Nagata, *Modern general topology*, North-Holland, Amsterdam, 1968.

24. C. Pasnicu, *On certain inductive limit C^*-algebras*, Indiana Univ. Math. J. **35** (1986), 269–288.

25. ———, *On inductive limits of certain C^*-algebras of the form $C(X) \otimes F$*, Trans. Amer. Math. Soc. **310** (1988), 703–714.

26. A.G. Robertson, *Stable range in C^*-algebras*, Math. Proc. Cambridge Phil. Soc. **87** (1980), 413–418.

27. K. Thomsen, *Inductive limits of homogeneous C^*-algebras*, Proc. London Math. Soc. (3) **55** (1987), 157–180.

28. ———, *Approximately trivial homogeneous C^*-algebras*, J. reine angew. Math. **383** (1988), 109–146.

29. ———, *On isomorphisms of inductive limit C^*-algebras*, Proc. Amer. Math. Soc. **113** (1991), 947–953.

30. ———, *Inductive limits of interval algebras: the tracial state space*, Amer. J. Math. **116** (1994), 605–620.

31. L.N. Vaserstein, *Stable rank of rings and dimensionality of topological spaces*, Func. Anal. Appl. **5** (1971), 102–110.

32. S. Zhang, *A Riesz decomposition property and ideal structure of multiplier algebras*, J. Operator Theory **24** (1990), 204–225.

33. ———, *C^*-algebras with real rank zero and the internal structure of their corona and multiplier algebras*, Part III, Canadian J. Math. **42** (1990), 159–190.

34. ———, *Matricial structure and homotopy type of simple C^*-algebras with real rank zero*, J. Operator Theory **26** (1991), 283–312.

Department of Mathematics, University of California, Santa Barbara, CA 93106