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ON EQUIVALENT CHARACTERIZATIONS OF WEAKLY COMPACTLY GENERATED BANACH SPACES

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1. Introduction. Let us consider the following nice

Theorem. *For a Banach space V the following assertions are equivalent*

(a) *V is weakly compactly generated* (w.c.g.),

(b) *V is* GSG *and simultaneously a Vašák* (*i.e., weakly K-countable determined*) *space, and*

(c) *V is* GSG *and moreover* (V^*, w^*) *continuously injects into* $\Sigma(\Gamma)$ *for some set* Γ.

Recall that it has, according to [**18**, Theorem [S8], **20** and 14, proof of Theorem A, Proposition 4.1], an even nicer, topological counterpart

Theorem$'$. *The following assertions are equivalent*

(α) *K is an Eberlein compact,*

(β) *K is simultaneously a Radon Nikodým compact and a Gul'ko compact, and*

(γ) *K is simultaneously a Radon Nikodým compact and a Corson compact.*

In the theorem, the implications (a) $\rightarrow$ (b) $\rightarrow$ (c) are not quite new. In fact, according to the interpolation theorem [**12**, p. 163], every w.c.g. space contains a dense continuous image of a reflexive space and hence is GSG. An observation that every w.c.g. space is Vašák is due to Talagrand [**19**]. Finally, the fact that the dual of a Vašák space endowed with the weak* topology continuously injects into $\Sigma(\Gamma)$ is due

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to Gul'ko [**9**], see also [**13**] an [**6**].

A proof of (c) $\to$ (a), which also works for the proof of (b) $\to$ (a), is quite recent and is due to Orihuela, Schachermayer and Valdivia [**14**] and to Stegall [**18**]. The proof proceeds in two steps. First, with the help of interpolation [**17**], an Asplund space Y is constructed in such a way that Y continuously and densely embeds into V and that either Y is Vasak or (Y^*, w^*) continuously injects into $\Sigma(\Delta)$ for some set Δ. Then, secondly, by a result from [**3**] and [**21**], respectively, it follows that Y is w.c.g. Therefore, so is V.

A main aim of this paper is to present a more direct proof that (b) or (c) implies (a). We shall avoid interpolation as well as gymnastics involving Gul'ko and Corson compacta that can be found in [**14**] and [**18**]. In particular, we shall no longer need a result of Gul'ko [**8**] that a continuous image of a Corson compact is a Corson compact.

A central concept we shall use in our proof will be a slight variant of a projectional generator introduced recently by Orihuela and Valdivia [**15**]: A *projectional generator* on a Banach space V is any at most countable valued mapping Φ: $V^* \to 2^V$ such that $\Phi(B)^\perp \cap \overline{B}^* = \{0\}$ whenever $B \subset V^*$ and $\overline{B}$ is linear. Using this concept we can even obtain a slightly more general equivalence

Proposition. *A Banach space is* w.c.g. *if* (*and only if*) *it is* GSG *and each of its complemented subspaces admits a projectional generator.*

Let us recall that a projectional generator on V exists if V is a Vasak space [**15**, Example after Theorem 1] or if (V^*, w^*) continuously injects into $\Sigma(\Gamma)$, see the proof of [**21**, Theorem 1]. This means that the theorem is included in the above proposition.

Let us mention a few words about the organization of the paper. Section 2 provides some preliminaries. In Section 3 we carefully construct, for our subsequent needs, "long sequences" of nice projections from a projectional generator. Section 4 is devoted to the proof of the proposition. A main idea here is to imitate a proof of the fact that V is w.c.g. if it is Asplund and admits a projectional generator, see [**3, 15, 21**].

2. Preliminaries. The cardinality for a set M is denoted by $\#M$. Ordinal numbers are denoted by $\alpha, \beta, \dots$. The cardinal number corresponding to α is denoted by $\#\alpha$. The letter ω is reserved for the first infinite ordinal. Symbols $\aleph_0, \aleph_1$ are the first infinite and uncountable cardinals, respectively.

The reals are denoted by $\mathbf{R}$. The symbol $\mathbf{N}$ means the set of positive integers. $\mathbf{N}^{\mathbf{N}}$ is considered with the product topology.

Let V be a Banach space. V^* and V^{**} denote the first and the second dual of V, respectively. The closed unit ball of V is denoted by B_V. For $v \in V$ and $v^* \in V^*$ the symbol $\langle v^*, v\rangle$ means the value of v^* at v. If $M \subset V$ then $\overline{M}$, $\overline{\mathrm{sp}}\, M$ and $\operatorname{dens} M$ are used to denote the closure, the closed linear span and the density of M, respectively. Also, for $M \subset Y$ we put $M^0 = \{v^* \in V^* : \sup\langle v^*, M\rangle \le 1\}$. If $M \subset V^*$, then the weak* closure of M is denoted by $\overline{M}^*$. If Y is another Banach space such that $Y \subset V$, and $M \subset Y$, then $\overline{M}^Y$ and $\overline{\mathrm{sp}}^Y M$ mean the Y-closure and the Y-closed linear span of M respectively. If $M \subset V$, then $M^\perp$ denotes the annihilator of M in V^*. Similarly, for $M \subset V^*$, the symbol $M_\perp$ is reserved for the annihilator of M in V. Letters w and w^* denote the weak and weak* topologies, respectively.

A Banach space is called *weakly compactly generated* (w.c.g.) if there is a weakly compact set K in V such that $\overline{\mathrm{sp}}\, K = V$. V is called a *Vašák* space if there exist $\Sigma' \subset \mathbf{N}^{\mathbf{N}}$ and a multivalued upper semicontinuous mapping $\varphi : \Sigma' \to (V, \omega)$ such that $\varphi(\sigma)$ is a nonempty weakly compact set for each $\sigma \in \Sigma'$ and $\cup\{\varphi(\sigma) : \sigma \in \Sigma'\} = V$. A Banach space is called an *Asplund* space if each of its separable subspaces has a separable dual. V is said to be GSG if there exists an Asplund space Y such that $Y \subset V$, $\overline{Y} = V$, and $B_Y \subset B_V$.

For a set Γ we put

$$\Sigma(\Gamma) = \{x \in \mathbf{R}^\Gamma : \#\{\gamma \in \Gamma : x(\gamma) \neq 0\} \le \aleph_0\}$$

and we consider on this space the coordinatewise topology that is inherited from the product topology of $\mathbf{R}^\Gamma$.

Let K be a compact space. K is called an *Eberlein* compact if K is homeomorphic to a weakly compact set of a Banach space. Recall that K is Eberlein if and only if $C(K)$ is w.c.g. [**2**, p. 152]. K is called a *Gul'ko* compact if $C(K)$ is Vašák. It is well known that V is Vašák if and only if (B_{V^*}, w^*) is a Gul'ko compact [**20**]. K is said to be a *Radon Nikodým* compact if it can be found, up to a homeomorphism, in (V^*, w^*) where V is an Asplund space. Finally, K is called a *Corson* compact if it is homeomorphic to a compact subset of $\Sigma(\Gamma)$ for some set Γ.

3. Long sequences of "nice projections" constructed via a projectional generator.

Lemma 1. *Let V, Y be two Banach spaces such that $Y \subset V$ and $\overline{Y} = V$. Assume we have two at most countable valued mappings $\Phi : V^* \to 2^V$ and $\Psi : V \to 2^{V^*}$. Finally, let an infinite cardinal $\aleph$ be given and consider two sets $A_0 \subset Y$, $B_0 \subset V^*$, with $\#A_0 \le \aleph$, $\#B_0 \le \aleph$.*

Then there exist sets $A_0 \subset A \subset Y$, $B_0 \subset B \subset V^*$ such that $\#A \le \aleph$, $\#B \le \aleph$, $\overline{A} \supset \Phi(B)$, $B \supset \Psi(A)$, and that $\overline{A}^Y$, $\overline{B}$ are linear.

Proof. We shall use an old gluing argument due to Mazur. By induction we shall construct two sequences of sets $A_0 \subset A_1 \subset A_2 \subset \cdots \subset Y$ and $B_0 \subset B_1 \subset B_2 \subset \cdots \subset V^*$ as follows. Since $\#\Phi(B_0) \le \max(\aleph_0, \#B_0) \le \aleph$, there is a set $A_0 \subset A_1 \subset Y$, such that $\#A_1 \le \aleph$, $\overline{A}_1 \supset \Phi(B)$, and $\overline{A}_1^Y$ is linear. Similarly, there is a set $B_0 \subset B_1 \subset V^*$ satisfying $\#B_1 \le \aleph$, $B_1 \supset \Psi(A_0)$ and with $\overline{B}_1$ linear. Further, there are sets $A_1 \subset A_2 \subset Y$, $B_1 \subset B_2 \subset V^*$ such that $\#A_2 \le \aleph$, $\#B_2 \le \aleph$, $\overline{A}_2 \supset \Phi(B_1)$, $B_2 \supset \Psi(A_1)$ and that $\overline{A}_2^Y$, $\overline{B}_2$ are linear. Continuing this process, we obtain nondecreasing sequences $\{A_n\}$, $\{B_n\}$ of sets in Y and V^*, respectively, such that for every $n = 1, 2, \ldots$

$$\#A_n \le \aleph, \qquad \#B_n \le \aleph, \qquad \overline{A_{n+1}} \supset \Phi(B_n), \qquad B_{n+1} \supset \Psi(A_n),$$

and with $\overline{A}_n^Y$, $\overline{B}_n$ linear. Now put $A = \cup_{n=1}^{\infty} A_n$, $B = \cup_{n=1}^{\infty} B_n$. Then $\#A \le \aleph$, $\#B \le \aleph$, $\overline{A} \supset \Phi(B)$, $B \supset \Psi(A)$, and it is easy to verify that $\overline{A}^Y$, $\overline{B}$ are linear.

Lemma 2. *Let $(V, \|\ \|)$ be a Banach space having a projectional generator Φ, and let Y be another Banach space such that $Y \subset V$, $\overline{Y} =$*

V, and $B_Y \subset B_V$. For $n = 1, 2, \dots$, let $|| \; ||_n$ denote the (equivalent) norm on V whose unit ball is $\overline{B_Y + (1/n)B_V}$. Let $\Psi : V \to 2^{V^*}$ be an at most countable valued mapping such that

$$||v|| = \sup\{\langle v^*, v\rangle : v^* \in \Psi(v), ||v^*|| = 1\},$$
$$||v||_n = \sup\{\langle v^*, v\rangle : v^* \in \Psi(v), ||v^*||_n = 1\},$$

$n = 1, 2, \dots$. Finally, let $\aleph$ be an infinite cardinal and consider two sets $A_0 \subset Y$, $B_0 \subset V^*$ with $\#A_0 \leq \aleph$, $\#B_0 \leq \aleph$.

Then there exist sets $A_0 \subset A \subset Y$, $B_0 \subset B \subset V^*$ such that $\#A \leq \aleph$, $\#B \leq \aleph$, $\overline{A} \supset \Phi(B)$, $B \supset \Psi(A)$, and that $\overline{A}^Y$, $\overline{B}$ are linear, and moreover, there exists a linear projection $P : V \to V$ satisfying $||P|| = 1$, $PV = \overline{A}$, $P^*V^* = \overline{B}^*$, and $P(B_Y) \subset \overline{B}_Y$.

Proof. By applying Lemma 1 we obtain the sets $A_0 \subset A \subset Y$, $B_0 \subset B \subset V^*$. It remains to find the projection P. Let us remark that $\overline{A} + B_\perp$ is closed since we have for all $a \in A$ and all $b \in B_\perp$

$$\begin{aligned} ||a + b|| &\geq \sup\{\langle v^*, a + b\rangle : v^* \in B, ||v^*|| = 1\} \\ &= \sup\{\langle v^*, a\rangle : v^* \in B, ||v^*|| = 1\} \\ &\geq \sup\{\langle v^*, \alpha\rangle : v^* \in \Psi(a), ||v^*|| = 1\} \\ &= ||a||. \end{aligned}$$

Further, if $\overline{A} + B_\perp \neq V$, then by the Hahn Banach theorem there is $0 \neq \xi \in V^*$ which is identically zero on $\overline{A} + B_\perp$. But then $\xi \in A^\perp \cap \overline{B}^*$ whence $\xi \in \Phi(B)^\perp \cap \overline{B}^*$ because $\overline{A} \supset \Phi(B)$. Now Φ is a projectional generator, so $\xi = 0$, a contradiction. We have thus shown that V is a direct sum of $\overline{A}$ and $B_\perp$.

Define $P : V \to V$ by

$$P(a + b) = a, \qquad a \in \overline{A}, b \in B_\perp.$$

Then P is a linear projection with $||P|| = 1$, $PV = \overline{A}$, and $P^{-1}(0) = B_\perp$. We shall show that $P^*V^* = \overline{B}^*$. If $\xi \in B$, then for all $v \in V$ we have $\langle P^*\xi, v\rangle = \langle \xi, Pv\rangle = \langle \xi, v\rangle$ as $Pv - v \in B_\perp$; so $\xi \in P^*V^*$. Hence $\overline{B}^* \subset P^*V^*$. Assume now there is $\xi \in P^*V^* \backslash \overline{B}^*$. Then there exists $v \in V$ with $\langle \xi, v\rangle \neq 0$ and $\sup\langle \overline{B}, v\rangle = 0$ since $\overline{B}$ is linear. It follows that $v \in B_\perp$, and so $Pv = 0$. But $0 \neq \langle \xi, v\rangle = \langle \xi, Pv\rangle$, a contradiction.

It remains to show that $P(B_Y) \subset \overline{B_Y}$. We realize that not only $||P|| = 1$ but also $||P||_n = 1$ for each $n = 1, 2, \ldots$. Thus $P(B_Y) \subset P(\overline{B_Y + (1/n)B_V}) \subset \overline{B_Y + (1/n)B_V} \subset B_Y + (2/n)B_V$ for each n and so $P(B_Y) \subset \overline{B_Y}$. □

The next improvement of Lemma 2 will be crucial in the proof of our proposition.

Lemma 3. *The inclusion* $P(B_Y) \subset \overline{B_Y}$ *in Lemma 2 may be replaced by* $P(B_Y) \subset \overline{A \cap B_Y}$.

Proof. We shall proceed by induction. Let A_1, B_1 and P_1 denote, respectively, the A, B, and P found in Lemma 2. Then $||P_1|| = 1$, $P_1V = \overline{A_1}$, $P_1^*V^* = \overline{B_1}^*$, $\Phi(B_1) \subset \overline{A}_1$, $\Psi(A_1) \subset B_1$ and $P_1(B_Y) \subset \overline{B_Y}$. Since $\operatorname{dens} P_1(B_Y) \le \operatorname{dens} P_1V \le \#A_1 \le \aleph$, it follows that there is a set $M \subset B_Y$, with $\#M \le \aleph$, such that $P_1(B_Y) \subset \overline{M}$. In Lemma 2, set $A_0 := A_1 \cup M$ and $B_0 := B_1$. We obtain, then, new A, B, P, denoted by A_2, B_2, P_2, respectively. Then, besides the facts stated in Lemma 2, we have $P_1(B_Y) \subset \overline{A_2 \cap B_Y}$. Continuing this process, we can construct sequences of sets

$$A_0 \subset A_1 \subset A_2 \subset \cdots Y, \qquad B_0 \subset B_1 \subset B_2 \subset \cdots V^*$$

and norm one projections $P_n : V \to V$, $n = 1, 2, \ldots$, such that for all n we have

$$\begin{aligned} &\#A_n \le \aleph, \qquad \#B_n \le \aleph, \qquad \Phi(B_n) \subset \overline{A}_n, \qquad \Psi(A_n) \subset B_n, \\ &P_nV = \overline{A}_n, \qquad P_n^*V^* = \overline{B_n}^*, \quad \text{and} \quad P_n(B_Y) \subset \overline{A_{n+1} \cap B_Y}. \end{aligned}$$

Now put $A = \cup_{n=1}^\infty A_n$, $B = \cup_{n=1}^\infty B_n$. Then all the properties claimed for A and B in Lemma 2 are checked easily. Further, since $\overline{A} \supset \Phi(B)$ and $B \supset \Psi(A)$, we can construct, as in Lemma 2, a linear projection $P : V \to V$ satisfying $||P|| = 1$, $PV = \overline{A}$, and $P^*V^* = \overline{B}^*$. Moreover, as $A_n \subset A_{n+1} \subset A$, $B_n \subset B_{n+1} \subset B$, we have

$$P_nP_{n+1} = P_{n+1}P_n = P_n, \qquad P_nP = PP_n = P_n.$$

From this it easily follows that $||P_nv - Pv|| \to 0$ for each $v \in V$. Therefore, because $P_N(B_Y) \subset \overline{A_{n+1} \cap B_Y} \subset \overline{A \cap B_Y}$, we can conclude that $P(B_Y) \subset \overline{A \cap B_Y}$. □

Lemma 4. *Let V, W, Φ, and Ψ be as in Lemma 3, and let μ be the first ordinal with cardinality* $\operatorname{dens} V$.

Then there exist long sequences $\{A_\alpha : \omega \le \alpha \le \mu\}$ *and* $\{B_\alpha : \omega \le \alpha \le \mu\}$ *of subsets in Y and V^*, respectively, and a* long sequence $\{P_\alpha : \omega \le \alpha \le \mu\}$ *of linear projections on V such that* $\overline{A_\mu} = V$, $\overline{B_\mu}^* = V^*$, *P_μ is the identity mapping and that for all $\omega < \alpha \le \mu$ the following hold*

(i) $\#A_\alpha \le \#\alpha$, $\#B_\alpha \le \#\alpha$,

(ii) $\overline{A_\alpha} \supset \Phi(B_\alpha)$, $B_\alpha \supset \Psi(A_\alpha)$,

(iii) $\overline{A_\alpha}^Y$, $\overline{B_\alpha}$ *are linear*,

(iv) $||P_\alpha|| = 1$,

(v) $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$, *if* $\beta < \alpha$,

(vi) $\operatorname{dens} P_\alpha V \le \#\alpha$,

(vii) $P_\alpha V = \overline{\cup_{\beta<\alpha} P_{\beta+1} V}$,

(viii) $P_\alpha V = \overline{A_\alpha}$, $P_\alpha^* V^* = \overline{B_\alpha}^*$, *and*

(ix) $P_\alpha(B_Y) \subset \overline{A_\alpha \cap B_Y}$.

Proof. Since $\overline{Y} = V$, there is a set $\{y_\alpha : \omega \le \alpha < \mu\}$ in Y which is dense in V. We shall proceed by transfinite induction on α. Let $A_\omega = \varnothing$, $B_\omega = \varnothing$, $P_\omega \equiv 0$, $\omega \le \gamma \le \mu$ fixed, and assume we have constructed sets $A_\alpha \subset V$, $B_\alpha \subset W$ and projections P_α with the properties stated in the lemma for every $\omega \le \alpha < \gamma$. First, if γ is a limit ordinal, simply put $A_\gamma = \cup_{\alpha<\gamma} A_\alpha$, $B_\gamma = \cup_{\alpha<\gamma} B_\alpha$. Then $\overline{A_\gamma} \supset \Phi(B_\gamma)$, $B_\gamma \supset \Psi(A_\gamma)$. And, as in the proof of Lemma 2, we can assign to the couple A_γ, B_γ a linear projection $P_\gamma : V \to V$. Then (i)–(iv) and (vi)–(viii) hold trivially. Conditions (v) and (ix) are also satisfied since it follows from (vii) that the mapping $\alpha \to P_\alpha v$ is norm continuous at $\alpha = \gamma$. Second, when γ is a nonlimit ordinal, let $A_0 = A_{\gamma-1} \cup \{y_{\gamma-1}\}$, $B_0 = B_{\gamma-1}$. Applying Lemma 3 we get $A_\gamma, B_\gamma, P_\gamma$, and it is straightforward to verify that (i)–(ix) hold.

Finally, if $\gamma = \mu$, then $P_\mu V = \overline{A_\mu} \supset \{y_\alpha : \omega \le \alpha < \mu\} = V$. So P_μ is the identity and $\overline{B_\mu}^* = P_\mu^* V^* = V^*$. □

We recall that a "long sequence" $\{P_\alpha : \omega \le \alpha \le \mu\}$ having the properties (iv)–(vii) is called a *projectional resolution of the identity* (P.R.I.) on V.

4. Proof of the proposition. The proof of the proposition will be preceded by two lemmas. The first one is a deep result of Simons [**16**, Lemma 2] while the second lemma is its consequence.

Lemma 5 (Simons). *Let Γ be a set, $\{f_n\}$ a sequence in the unit ball of $l_\infty(\Gamma)$, and $\Delta \subset \Gamma$. Assume that, for every $\lambda_1, \lambda_2, \dots \ge 0$, with $\lambda_1 + \lambda_2 + \cdots = 1$, there is $\gamma \in \Delta$ such that $\lambda_1 f_1(\gamma) + \lambda_2 f_2(\gamma) + \cdots = \sup ||\lambda_1 f_1 + \lambda_2 f_2 + \cdots||$. Then $\sup_{\gamma\in\Delta} \limsup_{i\to\infty} f_i(\gamma) \ge \inf\{||g|| :$ g is in the convex hull of $\{f_i\}\}$.*

Lemma 6. *Let Y, V be two separable Banach spaces such that Y is Asplund, $Y \subset V$, $\overline{Y} = V$, $B_Y \subset B_V$, and let ρ be a metric defined on V^* by $\rho(\xi,\eta) = \sup\langle \xi - \eta, B_Y\rangle$, $\xi, \eta \in V^*$. Let F be any boundary of B_{V^*}, that is, $F \subset B_{V^*}$ and for each $v \in V$ there is a $\xi \in F$ such that $\langle \xi, v\rangle = ||v||$.*

Then the ρ-closed span of F is equal to V^*.

Proof. Assume the claim is false. Then there exists $\xi_0 \in B_{V^*}$ and a linear continuous functional φ on (V^*, ρ) such that

$$\varphi(\xi_0) > 0 = \varphi(\xi) \qquad \text{for all } \xi \in F.$$

As φ is ρ-continuous, we have $\varphi(B_Y^0) < +\infty$. We may and do assume that $\varphi(B_Y^0) \le 1$. Since $B_Y \subset B_V$, it follows that $B_Y^0 \supset B_V^0$ $(= B_{V^*})$ and so φ must belong to V^{**}. Define $R : V^* \to Y^*$ by $R\xi = \xi|_Y$, $\xi \in V^*$ and $\tilde{\varphi} : R(V^*) \to \mathbf{R}$ by $\tilde{\varphi}(R\xi) = \varphi(\xi)$, $\xi \in V^*$. This $\tilde{\varphi}$ is well defined since $\overline{Y} = V$. Moreover, if $\xi \in V^*$ is such that $\sup\langle \xi, B_Y\rangle \le 1$, then $\xi \in B_Y^0$ and so $\varphi(\xi) \le 1$. Therefore for every $\xi \in V^*$ we have $\varphi(\xi) \le \sup\langle \xi, B_Y\rangle$, i.e.,

$$\tilde{\varphi}(R\xi) \le |R\xi|.$$

Here, and later, $|\cdot|$ denotes the norm on Y and its dual norm on Y^*. Let $\psi \in B_{Y^{**}}$ be any Hahn Banach extension of $\tilde{\varphi}$ from $R(V^*)$ to Y^*. Recall that, by Goldstine's theorem, B_Y is weak* dense in $B_{Y^{**}}$. Since

Y is Asplund, Y^* is separable. Hence, there is a sequence $\{y_k\} \subset B_Y$ such that $y_k \to \psi$ weak* in Y^{**}. Then, especially for each $\xi \in V^*$, we have

$$\langle \xi, y_k \rangle = \langle R\xi, y_k \rangle \to \langle \psi, R\xi \rangle = \tilde{\varphi}(R\xi) = \varphi(\xi).$$

This means that $y_k \to \varphi$ weak* in V^{**}. Hence we may assume that $\langle \xi_0, y_k \rangle > (1/2)\varphi(\xi_0)$ for all $k = 1, 2, \ldots$. Now Lemma 5 applies and so we have

$$\begin{aligned} 0 = \sup\langle \varphi, F \rangle &= \sup\{\lim_k \langle \xi, y_k \rangle : \xi \in F\} \\ &\geq \inf\{||y|| : y \in \operatorname{co}\{y_1, y_2, \ldots\}\} \\ &\geq \{\langle \xi_0, y \rangle : y \text{ is in the convex hull of } \{y_1 y_2, \ldots\}\} \\ &> (1/2)\varphi(\xi_0) > 0, \end{aligned}$$

a contradiction. Hence the result. □

Proof of proposition. The proof is divided into several steps.

1^0. Let $||.||$ denote the norm on V. Find an Asplund space Y such that $Y \subset V$, $\overline{Y} = V$, and $B_Y \subset B_V$. Define a metric ρ on V^* by

$$\rho(\xi, \eta) = \sup\langle \xi - \eta, B_Y \rangle, \qquad \xi, \eta \in V^*.$$

It is easy to check that ρ fragments the weak* topology of V^*, that is, that for every nonempty bounded set M in V^* and every $\varepsilon > 0$ there is a weak* open set (even a weak* open half-space) $W \subset V^*$ such that $W \cap M$ is nonempty and has ρ-diameter less than ε. Let us consider a multivalued mapping D from V into B_{V^*} defined by

$$Dv = \{v^* \in B_{V^*} : \langle v^*, v \rangle = ||v||\}, \qquad v \in V.$$

It is well known, and easy to verify, that D is norm to weak* upper semicontinuous and compact valued. Thus, according to a selection theorem of Jayne-Rogers type [**7**, the desert selection Theorem (A) d)], there are norm to ρ continuous (single-valued) mappings $D_i : V \to B_{V^*}$, $i = 1, 2, \ldots$, such that for every $v \in V$ there is $D_0 v \in Dv$ such that $\rho(D_i v, D_0 v) \to 0$ (it suffices for our purposes that $\inf\{\rho(D_i, D_0 v) : i \in \mathbf{N}\} = 0$.)

2^0. **Claim.** *Given a separable subspace Z of Y, for every $\xi \in (\overline{Z})^*$ and every $\varepsilon > 0$ there are $v_1, \dots, v_m \in \overline{Z}$, $a_1, \dots, a_m \in \mathbf{R}$, and $i_1, \dots, i_m \in \mathbf{N}$ such that*

$$\sup \left\langle \xi - \sum_{j=1}^{m} a_j D_{i_j} v_j | \overline{Z}, B_Z \right\rangle < \varepsilon. \tag{1}$$

Proof. Take a separable subspace $Z \subset Y$ and let $\xi \in (\overline{Z})^*$, $\varepsilon > 0$ be given. Let us remark that the set

$$F = \{D_0 v|_{\overline{Z}} : v \in \overline{Z}\}$$

is a boundary of $B_{(\overline{Z})^*}$. Recalling that Z is Asplund, it then follows, by applying Lemma 6 for $V := \overline{Z}$ and $Y := Z$, that there are $v_1, \dots, v_m \in \overline{Z}$ and $a_1, \dots, a_m \in \mathbf{R}$ such that

$$\sup \left\langle \xi - \sum_{j=1}^{m} a_j D_0 v_j|_{\overline{Z}}, B_Z \right\rangle < \varepsilon.$$

And, as

$$\inf \{\rho(D_i, D_0 v) : i \in \mathbf{N}\} = 0, \qquad j = 1, \dots, m,$$

there are $i_1, \dots, i_m \in \mathbf{N}$ such that (1) holds. □

3^0. **Claim.** *The claim from 2^0 holds for any subspace Z of Y.*

Proof. Let Z be a (nonseparable) subspace of Y and fix some $\xi \in (\overline{Z})^*$, $\varepsilon > 0$. We shall try to convert our situation to the case of a separable subspace of Y and thus 2^0 will be of use. Let A denote the set of all infinite matrices $a = \{a_{ij} : i, j \in \mathbf{N}\}$ with rational entries and such that $a_{ij} = 0$ for all but finitely many of $(i, j) \in \mathbf{N}^2$. Let $Z_1 \neq \{0\}$ be any separable subspace of Z. By induction we shall construct separable subspaces $Z_1 \subset Z_2 \subset \cdots \subset Z$, sequences $\{z_j^1\}, \{z_j^2\}, \dots,$ where $\{z_j^n : j \in \mathbf{N}\}$ is a Z-dense subset of Z_n, and elements $z(n,a) \in B_Z$, $n \in \mathbf{N}$, $a \in A$, such that for all $n = 1, 2, \ldots$ and all $a \in A$

$$\sup \left\langle \xi - \sum_{i,j=1}^{\infty} a_{ij} D_i z_j^n |_{\overline{Z}}, B_Z \right\rangle < \left\langle \xi - \sum_{i,j=1}^{\infty} a_{ij} D_i z_j^n |_{\overline{Z}}, z(n,a) \right\rangle + \frac{1}{n}$$

and

$$Z_{n+1} = \overline{\mathrm{sp}}^Z [Z_n \cup \{z(n,a) : a \in Z\}], \qquad n = 1, 2, \ldots .$$

Finally, put

$$Z_0 = \overline{Z_1 \cup Z_2 \cup \cdots}^Z;$$

then, clearly, Z_0 will be separable. By 2^0, there are $v_1, \ldots, v_m$ in $\overline{Z_0}$, $a_1, \ldots, a_m \in \mathbf{R}$, and $i_1, \ldots, i_m \in \mathbf{N}$ such that

$$\sup \left\langle \xi|_{\overline{Z_0}} - \sum_{k=1}^{m} b_k D_{i_k} v_k |_{\overline{Z_0}}, B_{Z_0} \right\rangle < \frac{\varepsilon}{2}.$$

The continuity of D_{i_k} and the construction of Z_0 ensure that there are $n \in \mathbf{N}$, $n > \varepsilon/2$, and $j_1, \ldots, j_m \in \mathbf{N}$ such that $||v_k - z_{j_k}^n||$ are so small that

$$\sup \left\langle \xi|_{\overline{Z_0}} - \sum_{k=1}^{m} b_k D_{i_k} z_{j_k}^n |_{\overline{Z_0}}, B_{Z_0} \right\rangle < \frac{\varepsilon}{2}.$$

Also, we may assume here that all the b_k are rational. Now put $a = \{a_{ij}\}$, where $a_{i_1 j_1} = b_1, \ldots, a_{i_m j_m} = b_m$ and $a_{ij} = 0$ otherwise. Thus

$$\sup \left\langle \xi|_{\overline{Z_0}} - \sum_{i,j=1}^{\infty} a_{ij} D_i z_j^n |_{\overline{Z_0}}, B_{Z_0} \right\rangle < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned} \sup \left\langle \xi - \sum_{i,j=1}^{\infty} a_{ij} D_i z_j^n |_{\overline{Z}}, B_Z \right\rangle & \\ &< \left\langle \xi - \sum_{i,j=1}^{\infty} a_{ij} D_i z_j^n |_{\overline{Z}}, z(n,a) \right\rangle + \frac{1}{n} \\ &\le \sup \left\langle \xi|_{\overline{Z_0}} - \sum_{i,j=1}^{\infty} a_{ij} D_i z_j^n |_{\overline{Z_0}}, B_{Z_0} \right\rangle + \frac{1}{n} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which means that (1) holds. □

4^0. **Claim.** *There exist a* P.R.I. $\{P_\alpha : \omega \le \alpha \le \mu\}$ *on* V *and a* long sequence $\{Y_\alpha : \omega \le \alpha \le \mu\}$ *of subspaces of* Y *such that for each* $\omega \le \alpha < \mu$ *the following hold*

$$P_\alpha V = \overline{Y_\alpha}, \qquad P_\alpha(B_Y) \subset \overline{B_{Y_\alpha}}, \tag{2}$$

$$P_\alpha^* V^* \supset \bigcup_{n=1}^{\infty} D_n(P_\alpha V). \tag{3}$$

Proof. Define $\Psi_0 : V \to 2^{V^*}$ by $\Psi_0(v) = \{D_1(v), D_2(v), \dots\}$, $v \in V$, and enlarge each $\Psi_0(v)$ to some countable set $\Psi(v)$ such that the assumptions of Lemma 3 are satisfied. Let $\{A_\alpha : \omega \le \alpha \le \mu\}$, $\{B_\alpha : \omega \le \alpha \le \mu\}$, and $\{P_\alpha : \omega \le \alpha \le \mu\}$ correspond to our V, Y, Φ, Ψ by Lemma 4. For each α put $Y_\alpha = \overline{A_\alpha}^Y$. Then trivially $P_\alpha V = \overline{A_\alpha} = \overline{Y_\alpha}$ and

$$P_\alpha(B_Y) \subset \overline{A_\alpha \cap B_Y} \subset \overline{Y_\alpha \cap B_Y} = \overline{B_{Y_\alpha}}.$$

Further, we know that

$$B_\alpha \supset \Psi(A_\alpha) \supset \bigcup_{n=1}^{\infty} D_n(A_\alpha).$$

Now the D_n are norm to ρ continuous and $P_\alpha V = \overline{A_\alpha}$. So

$$D_n(P_\alpha V) \subset \overline{D_n(A_\alpha)}^\rho \subset \overline{D_n(A_\alpha)}^* \subset \overline{B_\alpha}^*$$

as $D_n(V) \subset B_{V^*}$. But we know that $\overline{B_\alpha}^* = P_\alpha^* V^*$. This finishes the proof of the claim. □

5^0. **Claim.** *For each limit* $\omega \le \alpha \le \mu$

$$P_\alpha^* V^* = \overline{\cup_{\beta<\alpha} P_\beta V^*}^\rho. \tag{4}$$

Proof. Fix one such α, take any $v^* \in P_\alpha^* V^*$, and let $\varepsilon > 0$. Recall that, by (2), $P_\alpha V = \overline{Y_\alpha}$. Then using Claim 3^0 with $Z := Y_\alpha$, there are $v_1, \dots, v_m \in P_\alpha V$, $a_1, \dots, a_m \in \mathbf{R}$ and $i_1, \dots, i_m \in \mathbf{N}$ such that $\sup\langle v^* - \sum_{j=1}^m a_j D_{i_j} v_j, B_{Y_\alpha}\rangle = \sup\langle v^*|_{P_\alpha V} - \sum_{j=1}^m a_j D_{i_j} v_j|_{P_\alpha V}, B_{Y_\alpha}\rangle < \varepsilon$. Since $P_\alpha V = \overline{\cup_{\beta<\alpha} P_\beta V}$ and D_{i_j} are norm to ρ continuous, we can find $\gamma < \alpha$ and $u_1, \dots, u_m \in P_\gamma V$ so that

$$\sup\left\langle v^* - \sum_{j=1}^m a_j D_{i_j} u_j, B_{Y_\alpha}\right\rangle < \varepsilon.$$

Now, $v^* \in P_\alpha^* V^*$ and we know by (3) that $D_{i_j} u_j \in P_\gamma^* V^* \subset P_\alpha^* V^*$. Hence it follows with the help of (2) that

$$\begin{aligned}\rho\left(v^*, \sum_{j=1}^m a_j D_{i_j} u_j\right) &= \sup\left\langle v^* - \sum_{j=1}^m a_j D_{i_j} u_j, B_Y\right\rangle \\ &= \sup\left\langle v^* - \sum_{j=1}^m a_j D_{i_j} u_j, P_\alpha(B_Y)\right\rangle \\ &\le \sup\left\langle v^* - \sum_{j=1}^m a_j D_{i_j} u_j, B_{Y_\alpha}\right\rangle \\ &< \varepsilon.\end{aligned}$$

And, as $\varepsilon > 0$ was arbitrary and $\gamma < \alpha$, we can conclude that $v^* \in \overline{\cup_{\beta<\alpha} P_\beta^* V^*}^\rho$. □

6^0. **Claim.** *For each limit $\omega < \alpha \le \mu$ and each $v^* \in V^*$*

$$\rho(P_\beta^*\xi, P_\alpha^*\xi) \to 0 \quad \textit{as } \beta \uparrow \alpha.$$

Proof. Fix one such α and ξ, and let any $\varepsilon > 0$ be given. By 5^0 there are $\gamma < \alpha$ and $\eta \in V^*$ such that

$$\rho(P_\alpha^*\xi, P_\gamma^*\eta) < \varepsilon/2.$$

Then for $\gamma \le \beta < \alpha$ we have

$$\begin{aligned}\rho(P_\beta^*\xi, P_\gamma^*\eta) &= \sup\langle P_\beta^*\xi - P_\gamma^*\eta, B_Y\rangle \\ &= \sup\langle P_\beta^*(P_\alpha^*\xi - P_\gamma^*\eta, B_Y\rangle \\ &= \sup\langle P_\alpha^*\xi - P_\gamma^*\eta, P_\beta(B_Y)\rangle \\ &\leq \sup\langle P_\alpha^*\xi - P_\gamma^*\eta, B_Y\rangle \\ &= \rho(P_\alpha^*\xi, P_\gamma^*\eta) \\ &< \varepsilon/2\end{aligned}$$

because of (2), $P_\beta(B_Y) \subset \overline{B_Y}$. Hence, for these β

$$\rho(P_\alpha^*\xi, P_\beta^*\xi) \leq \rho(P_\alpha^*\xi, P_\gamma^*\eta) + \rho(P_\gamma^*\eta, P_\beta^*\xi) < \varepsilon. \qquad \square$$

7^0. Now we are prepared to conclude the proof of our proposition, that is, to find a weakly compact subset of V which would generate the whole V. We shall proceed by transfinite induction and show the following statement.

If every complemented subspace of a Banach space V admits a projectional generator and Y is an Asplund space such that $Y \subset V$, $\overline{Y} = V$, and $B_Y \subset B_V$, then there exists a weakly compact set K in $\overline{B_Y}$ such that $\overline{\mathrm{sp}}\, K = V$.

Proof. If V is separable, then there is almost nothing to prove. Otherwise, let an uncountable cardinal $\aleph$ be given and assume the statement holds whenever $\operatorname{dens} V < \aleph$. Now suppose that V has density equal to $\aleph$. Let $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ be a P.R.I. on V constructed in Claim 4^0. Then, for every $\omega \leq \alpha < \mu$ we have

$$\operatorname{dens} P_\alpha V \leq \#\alpha < \#\mu = \aleph, \qquad P_\alpha V = \overline{Y_\alpha}, \qquad B_{Y_\alpha} \subset B_{P_\alpha V},$$

and Y_α is Asplund. So, by the induction assumption, for each $\omega \leq \alpha < \mu$ there is a weakly compact set $K_{\alpha+1}$ in $\overline{B_{Y\alpha+1}}$ such that $\overline{\mathrm{sp}}\, K_{\alpha+1} = P_{\alpha+1} V$. Consider the set

$$K = \bigcup_{\alpha<\mu} (P_{\alpha+1} - P_\alpha)(K_{\alpha+1}) \cup \{0\}.$$

Then $\overline{\text{sp}}\, K = V$. In fact, it follows from the properties of P.R.I. that $V = \overline{\text{sp}} \cup_{\alpha<\mu} (P_{\alpha+1} - P_\alpha)(V)$ and

$$\begin{aligned}(P_{\alpha+1} - P_\alpha)(V) &= (P_{\alpha+1} - P_\alpha)(P_{\alpha+1}V)\\ &= (P_{\alpha+1} - P_\alpha)(\overline{\text{sp}}\, K_{\alpha+1})\\ &\subset \overline{\text{sp}}(P_{\alpha+1} - P_\alpha)(K_{\alpha+1}).\end{aligned}$$

Finally, we shall show that K is weakly compact. So, let there be a sequence $\{a_i\} \subset [\omega, \mu)$ and $k_i \in K_{\alpha_i+1}$, $i = 1, 2, \dots$. If $\alpha_i = \alpha$ for infinitely many i, then we are done since $K_{\alpha+1}$ is weakly compact. Second, assume $\{\alpha_i\}$ forms an infinite set and, for brevity, suppose that $\alpha_1 < \alpha_2 < \cdots < \alpha_i \hat{|} \lambda$. Fix $\xi \in V^*$. Then

$$\begin{aligned}|\langle \xi (P_{\alpha_i+1} - P_{\alpha_i})k_i\rangle| &= |\langle (P^*_{\alpha_i+1} - P^*_{\alpha_i})\xi, k_i\rangle|\\ &\le \sup\langle (P^*_{\alpha_i+1} - P^*_{\alpha_i})\xi, B_{Y_{\alpha_i+1}}\rangle\\ &\le \sup\langle (P^*_{\alpha_i+1} - P^*_{\alpha_i})\xi, B_Y\rangle\\ &= \rho(P^*_{\alpha_i+1}\xi, P^*_{\alpha_i}\xi)\\ &\le \rho(P^*_{\alpha_i+1}\xi, P^*_\lambda\xi)\\ &\quad + \rho(P^*_\lambda\xi, P^*_{\alpha_i}\xi) \to 0\end{aligned}$$

by 6^0. This shows that $(P_{\alpha_i+1} - P_{\alpha_i})k_i \to 0$ weakly and, consequently, K is weakly compact. Moreover, by (2) we know that $K \subset 2\overline{B_Y}$, so $K/2$ is the desired weakly compact set. □

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REFERENCES

1. R. Deville and G. Godefroy, *Some applications of projective resolutions of identity*, preprint.

2. J. Diestel, *Geometry of Banach spaces, selected topics*, Lect. Notes in Math. **485** (1975), .

3. M. Fabian, *Each weakly countably determined Asplund space admits Frechet differentiable norm*, Bull. Austral. Math. Soc. **36** (1987), 367–374.

4. ———, *Projectional resolution of the identity sometimes implies weakly compact generating*, Bull. Polish. Acad. Sci. **38** (1990), 117–120.

5. M. Fabian and G. Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity*, Studia Math. **91** (1988), 144–151.

6. M. Fabian and S. Troyanski, *A Banach space admits a locally uniformly rotund norm if its dual is a Vašák space*, Israel J. Math. **69** (1990), 214–224.

7. N. Ghoussoub, B. Maurey and W. Schachermayer, *Slicings, selections and their applications*, preprint.

8. S.P. Gul'ko, *On properties of subsets of* Σ*-products*, Doklady Akad. Nauk SSSR **237** (1977), 505–507.

9. ———, *The structure of spaces of continuous functions and their hereditary paracompactness*, Russian Math. Surveys **34** (1979), 36–44.

10. J.E. Jayne and C.A. Rogers, *Borel selectors for upper semicontinuous set valued mappings*, Acta Math. **155** (1985), 41–79.

11. K. John and V. Zizler, *Duals of Banach spaces which admit nontrivial smooth functions*, Bull. Austral. Math. Soc. **11** (1974), 161–166.

12. ———, *Smoothness and its equivalents in weakly compactly generated Banach spaces*, J. Funct. Anal. **15** (1974), 1–11.

13. S. Mercourakis, *On weakly countably determined Banach spaces*, Trans. Amer. Math. Soc. **300** (1987), 307–327.

14. J. Orihuela, W. Schachermayer and M. Valdivia, *Every Radon-Nikodym Corson compact space is Eberlein compact*, Studia Math. **98** (1991), 157–174.

15. J. Orihuela and M. Valdivia, *Projective generators and resolutions of identity in Banach spaces*, Congress on Functional Analysis (Madrid 1988), Rev. Math. Univ. Compl. Madrid **2** (1989), 179–199.

16. S. Simons, *A convergence theorem with boundary*, Pacific J. Math. **40** (1972), 703–708.

17. Ch. Stegall, *The Radon-Nikodym property in conjugate Banach spaces* II, Trans. Amer. Math. Soc. **264** (1981), 507–519.

18. ———, *More facts about conjugate Banach spaces with the Radon-Nikodym property*, preprint.

19. M. Talagrand, *Sur un conjecture de H.H. Corson*, Bull. Sci. Math. **99** (1975), 211–212.

20. ———, *Espaces de Banach faiblement* K*-analytiques*, Ann. of Math. **110** (1979), 407–438.

21. M. Valdivia, *Resolutions of the identity in certain Banach spaces*, Collect. Math. **39** (1988), 1–14.

22. L. Vašák, *On one generalization of weakly compactly generated Banach spaces*, Studia Math. **70** (1981), 11–19.

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