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DIMENSION c OF ORBITS AND CONTINUITY OF TRANSLATION FOR SEMIGROUPS

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ABSTRACT. Let S be a semi-topological semigroup and Φ a Banach space on which S acts as a semigroup of linear isometries. Let $\bar{x}\mu$ denote the effect of $x \in S$ on $\mu \in \Phi$. In many cases, we have that either the orbit $\{\bar{x}\mu : x \in S\}$ is nonseparable, or $x \mapsto \bar{x}\mu$ is norm-continuous. We investigate when "nonseparable" can be replaced with "spans a closed subspace of topological dimension at least c." We thus extend results known for the case that S is a group. We given examples and related results.

0. Introduction. Let S be a locally compact group, Φ a Banach space, and $x \mapsto \bar{x}\mu$, $\mu \in \Phi$, $x \in S$ a representation of S that is lower semicontinuous. Then for each $\mu \in \Phi$, $x \mapsto \bar{x}\mu$ is either continuous, or $\{\bar{x}\mu : x \in U\} = \mathcal{O}(\mu, U)$ is nonseparable for each open U; in fact, $\mathcal{O}(\mu, U)$ spans a subspace of topological dimension at least c [**1, 3**].

A semigroup with a topology S is a "semi-topological" semigroup if the semigroup operation is continuous in each variable separately; it is a "topological" semigroup if the semigroup operation is continuous in both variables simultaneously. All semigroup topologies will be assumed to be locally compact.

When S is a topological semigroup, $x \mapsto \bar{x}\mu$ can be discontinuous and $\mathcal{O}(\mu, S)$ separable: we give examples. In this paper we identify conditions under which $\mathcal{O}(\mu, S)$ is nonseparable when $x \mapsto \bar{x}\mu$ is not continuous, apply them to the group case, and provide examples that illustrate the limits of what is possible.

Most of our examples are inspired by what occurs in the case of $\Phi = M(S)$, the space of all regular bounded Borel measures on the

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semi-topological semigroup S, or the space $M_c(S)$, the space of all regular bounded continuous Borel measures on the semi-topological semigroup S.

Let S be a semi-topological semigroup and Φ a Banach space. Suppose that S operates as a semigroup of linear isomorphisms of Φ. We say that S operates "blsc" (bi-lower semicontinuously) if for every $\varepsilon > 0$ and every $\mu \in \Phi$, the set

$$\{(x, y) \in S \times S : ||\bar{x}\mu - \bar{y}\mu|| > \varepsilon\}$$

is open in $S \times S$. The action is "lsc" (lower semicontinuous) if

$$\{x \in S : ||\bar{x}\mu - \mu|| > \varepsilon\} \quad \text{is open for all } \mu \in \Phi, \varepsilon > 0.$$

Let $\mu \in \Phi$ and $\varepsilon > 0$. We say that $x \mapsto \bar{x}\mu$ is "ε-uniformly discontinuous" ("ε-u.d.") on a subset X of S if for all $x \in X$,

$$\limsup_{X \ni y \to x} ||\bar{x}\mu - \bar{y}\mu|| > \varepsilon.$$

We say $x \mapsto \bar{x}\mu$ is "weak ε-u.d." on X if for all $x \in X$

$$\limsup_{y \to x} ||\bar{y}\mu - \bar{x}\mu|| > \varepsilon$$

where the lim sup is taken over $y \in S$ (as opposed to $y \in X$).

Graham, Lau, and Leinert [**3**, 2.4] show that if S is a locally compact group acting lower semicontinuously as a group of isometries of the Banach space Φ, and $\mu \in \Phi$, then either $x \mapsto \bar{x}\mu$ is continuous or there exists a set C of cardinality c and $\varepsilon > 0$ such that $x, y \in C$ and $x \neq y$ imply $||\bar{x}\mu - \bar{y}\mu|| > \varepsilon$. We extend that result by relaxing the requirement that S be a group, at the cost of adding the hypotheses of blsc and ε-u.d. (which always hold in the group case). Our main result appears as Theorem 1.1. Section 1 is devoted to the proof of Theorem 1.1.

In Section 2 we apply Theorem 1.1 to cancellative semigroups, particularly to translation in $M(S)$, thus generalizing [**1, 3**].

Section 3 contains the application of Theorem 1.1 to groups.

In Section 4 we give examples that show that the hypotheses of Theorem 1.1 are necessary and that illustrate other aspects of Theorem

1.1. Example 4.1 shows also that a measure can fail to translate continuously in a commutative cancellative semigroup and still generate a separable subspace. Thus, the result of [**3**, 2.4] does not extend in full generality to commutative cancellative semigroups.

Finally, in Section 5 we give some open questions.

1. The main result.

Theorem 1.1. *Let S be a semi-topological semigroup, X a compact subset of S, $\varepsilon > 0$, Φ a Banach space on which S acts as a semigroup of linear operators. Let $\mu \in \Phi$ be such that $x \mapsto \bar{x}\mu$ is ε-*u.d. *on X.*

Then the following hold:

(i) *There exists a set $C \subseteq X$ of cardinality at least c such that $x, y \in C$ and $x \neq y$ imply*

$$||\bar{x}\mu - \bar{y}\mu|| \geq \varepsilon/2$$

and

(ii) *The set $\{\bar{x}\mu : x \in X\}$ spans a subspace of dimension at least c.*

Proof. This is a typical Cantor set construction. Since $x \mapsto \bar{x}\mu$ is ε-u.d. on X, there exist $x_{1,1}, x_{1,2} \in X$ such that

$$||\bar{x}_{1,1}\mu - \bar{x}_{1,2}\mu|| > \varepsilon/2.$$

Because $x \mapsto \bar{x}\mu$ is blsc, there exist compact neighborhoods $U_{1,j}$ of $x_{1,j}$, $j = 1, 2$, such that

$$||\bar{u}_{1,1}\mu - \bar{u}_{1,2}|| > \varepsilon/2$$

for all $u_{1,j} \in U_{1,j}$, $j = 1, 2$. We may assume that $U_{1,1} \cap U_{1,2} = \varnothing$.

Suppose that $n \geq 1$ and that we have found $x_{n,j} \in X$ and compact neighborhoods $U_{n,j}$ of $x_{n,j}$ for $1 \leq j \leq 2^n$ such that $u_j \in U_{n,j}$, $1 \leq j \leq 2^n$, imply

$$||\bar{u}_j\mu - \bar{u}_k\mu|| > \varepsilon/2 \quad \text{for } 1 \leq j \neq k \leq 2^n.$$

Because $x \mapsto \bar{x}\mu$ is ε-u.d. on X, for each $1 \leq j \leq 2^n$ there exists $x_{n+1,2j} \in X \cap U_{n,j}$ such that $||\bar{x}_{n,j}\mu - \bar{x}_{n+1,2j}\mu|| > \varepsilon/2$. Set $x_{n+1,2j-1} = x_{n,j}$. Since $x \mapsto \bar{x}\mu$ is blsc, we can find pairwise disjoint compact neighborhoods $U_{n+1,j}$ of $x_{n+1,j}$, $1 \leq j \leq 2^{n+1}$ such that $u_j \in U_{n+1,j}$, $1 \leq j \leq 2^{n+1}$ imply

$$||\bar{u}_j\mu - \bar{u}_k\mu|| > \varepsilon/2 \quad \text{for } 1 \leq j \neq k \leq 2^{n+1}.$$

Let $C_1 = \cap_{n=1}^{\infty} \cup_{j=1}^{2n} U_{n,j}$. Let $x, y \in C_1$, $x \neq y$, be such that there exist distinct sequences $\{j_n\}$, $\{l_n\}$ where $1 \leq j_n$, $l_n \leq 2^n$, $x \in \cap U_{n,j_n}$ and $y \in \cap U_{n,l_n}$. It follows from the construction that $||\bar{x}\mu - \bar{y}\mu|| \geq \varepsilon/2$.

Since there exist c distinct sequences $\{j_n\}$, we can choose $C \subseteq C_1$ having the appropriate cardinality (that uses the axiom of choice). The assertion (ii) follows immediately without the use of the axiom of choice. $\square$

2. Applications to cancellative semigroups.

Lemma 2.1. *Let S be a cancellative semi-topological semigroup and $x \in S$. Suppose that $\Phi = M(S)$ and that the representation of S on $M(S)$ is by convolution: $x \mapsto \delta(x) * \mu = \bar{x}\mu$. Then $||\omega|| = ||\bar{x}\omega||$ for all $\omega \in M(S)$.*

Proof. This is the usual argument. Since S is cancellative, a collection of sets $E_j \subseteq S$, $1 \leq j \leq n$, is pairwise disjoint if and only if $\{zE_j\}$ is a pairwise disjoint, which also implies that $\{z^{-1}E_j\}$ is pairwise disjoint, the preceding holding for all $z \in S$. Hence

$$||\omega|| \geq ||\bar{z}\omega|| \quad (\text{always}) = \sup_{\{E_j\}} \Sigma |(\bar{z} * \omega)(E_j)|$$

where the supremum is taken over all finite collections of pairwise disjoint Borel sets $E_j \subseteq S$. Then

$$\begin{aligned} \sup_{\{E_j\}} \Sigma |(\bar{z} * \omega)(E_j)| &= \sup_{\{E_j\}} \Sigma |\omega(\bar{z}^1 E_j)| \\ &\geq \sup_{\{zF_j\}} \Sigma |\omega(z^{-1} z F_j)| \\ &= \sup_{\{zF_j\}} \Sigma |\omega(F_j)| = ||\omega||. \qquad \square \end{aligned}$$

Lemma 2.2. *Let S be a semi-topological semigroup. Then $x \mapsto \bar{x}\mu$, $\mu \in M(S)$ is* blsc. *In fact, if $x \mapsto \bar{x}f$ is continuous from S to a Banach space X, then $x \mapsto \bar{x}\varphi$ is* blsc *for $\varphi \in X^*$.*

Proof. Let $f \in C_0(S)$. Then

$$(u, v) \mapsto \int [f(uz) - f(vz)]\, d\mu(z) = \int f(z) d(\bar{u}\mu - \bar{v}\mu)$$

is continuous in each variable simultaneously. (Even though $z \mapsto f(uz)$ may not be an element of $C_0(S)$, the integral above is well-defined, since μ is a regular bounded Borel measure.) Therefore $||\bar{u}\mu - \bar{v}\mu||$ is the supremum of continuous functions on $S \times S$, and therefore it is lower semicontinuous on $S \times S$. Therefore $x \mapsto \bar{x}\mu$ is blsc. The proof of the second assertion is identical to that of the first. $\square$

Remarks. (i) Characterizations of semigroup actions that are blsc remain to be found.

(ii) We would like to have been able to say that if S is a cancellative semitopological semigroup and $x \mapsto \bar{x}\mu$ is not continuous at $z \in S$, then $x \mapsto \bar{x}\mu$ is ε-u.d. for some $\varepsilon > 0$ on some compact subset of S. That is false, however (see Example 4.1). The difficulty is that the mapping is weak ε-u.d. but not ε-u.d., and the conclusion of Theorem 1.1 is false.

(iii) We do not know when ε-u.d. must occur. We do, however, know that it must occur for some μ's if S is a topological cancellative semigroup that contains a perfect subset. That is the content of 2.4 below. Only weak ε-u.d. can be guaranteed in general.

Lemma 2.3. *If S is cancellative and $x \mapsto \bar{x}\mu$ is not continuous at z, then for some $\varepsilon > 0$, $x \mapsto \bar{x}\mu$ is weak ε-u.d. on the ideal Sz.*

Proof. Let $\varepsilon = \frac{1}{2} \limsup_{y \to z} ||\bar{y}\mu - \bar{z}\mu||$. By Lemma 2.1, for each $x \in S$

$$\begin{aligned} \limsup_{y \to xz} ||\bar{y}\mu - (xz)^{-}\mu|| &\geq \limsup_{y \to z} ||(xy)^{-}\mu - (xz)^{-}\mu|| \\ &= \limsup_{y \to z} ||\bar{y}\mu - \bar{z}\mu|| > \varepsilon. \end{aligned}$$

Since S is semi-topological, $\lim_{y\to z} xy = xz$. □

Theorem 2.4. *Let S be a cancellative topological semigroup and suppose that S contains a compact perfect set. Then there exists a compact perfect set F and a set E with* $\operatorname{Card} E \geq c$ *such that $x'F \cap xF = \varnothing$ for all distinct $x, x' \in E$.*

Proof. This is a double Cantor set construction. Let $E_0 \subseteq S$ be perfect. Let X, Y be disjoint perfect subsets of E_0. Let $x_{1,1}, x_{1,2} \in X$ be distinct. Choose $y_{1,1}, y_{1,2} \in Y$ such that $x_{1,j}y_{1,k} \neq x_{1,l}y_{1,r}$ except where $(j,k) = (l,r)$. That is possible because S is cancellative. Let U_{1j}, V_{1j} be compact neighborhoods of x_{1j}, y_{1j}, respectively, such that

$$U_{1j}V_{1k} \cap U_{1l}V_{1r} = \varnothing \qquad \text{if } (j,k) \neq (l,r).$$

That is possible because S is a topological semigroup. Let $n \geq 1$, and suppose that we have compact sets $U_{n,j} \subseteq X$, $V_{n,j} \subseteq Y$, $1 \leq j \leq 2^n$, each with nonempty (relative) interior, such that

$$U_{n,j}V_{n,k} \cap U_{n,r}V_{n,s} = \varnothing \qquad \text{if } (j,k) \neq (r,s). \tag{2.1}$$

By the cancellative property of S, there exist $x_{n+1,2j-1}, x_{n+1,2j} \in U_{n,j}, y_{n+1,2j-1}, y_{n+1,2j} \in V_{n,j}$, $1 \leq j \leq 2^n$ such that

$$x_{n+1,j}y_{n+1,k} \neq x_{n=1,r}y_{n+1,s} \qquad \text{if } (j,k) \neq (r,s).$$

Since the multiplication in S is continuous, there are compact relative neighborhoods $U_{n+1,j}$, $V_{n+1,j}$ of $x_{n+1,j}$, $y_{n+1,j}$, respectively, such that (2.1) holds with $n+1$ in place of n.

Let $F = \cap_n \cup_j V_{n,j}$. Let $E_1 = \cap_n \cup_j U_{n,j}$. As in the proof of Theorem 1.1, but here using (2.1), if $x, y \in E_1$ have $x \in \cap U_{n,j_n}$, $y \in \cap U_{n,l_n}$ and $\{j_n\} \neq \{l_n\}$, then $xF \cap yF = \varnothing$. Using the axiom of choice, we find $E \subseteq E_1$ as in the proof of Theorem 1.1. □

The set of translates of a discrete measure typically generates a subspace of dimension at least c. For many continuous measures the same is true, as the following corollary asserts.

Corollary 2.5. *Let S be a cancellative topological semigroup with a perfect compact subset. Then the following hold.*

(i) *There exists* $\mu \in M_c(S)$ *such that* $\{\bar{s}\mu : s \in S\}$ *spans a space of dimension at least* c; *and*

(ii) $\{\bar{s}\mu : s \in S\}$ *is not separable for at least* c *continuous measures on* S.

Proof. (i) Let E, F be given by Theorem 2.4. Let μ be a continuous measure concentrated on E. Such a measure exists because E is perfect and compact. By the conclusion of Theorem 2.4, $\bar{x}\mu$ and $\bar{x}'\mu$ have disjoint supports for all x, x' distinct in F. Since cardinality of F is at least c, (i) follows.

(ii) The compact perfect set E supports a family of c mutually singular continuous measures. (E contains a subset that has a metrizable quotient set W that is perfect, metrizable, and totally disconnected.) The set W is homeomorphic to $(Z_2)^\infty$, so $M_c(W)$ has dimension c. From that follows the assertion about $M_c(E)$.) The argument of (i) applied to each of the measures in that family now applies, and (ii) follows. □

Remark . Example 4.1 shows that there exists a (necessarily) cancellative sub-semigroup S of the plane and $\mu \in M(S)$ such that $s \mapsto \bar{s}\mu$ discontinuously, but $\{\bar{s}\mu : s \in S\}$ is nevertheless separable.

3. Applications to locally compact groups. We show in this section that the hypotheses of Theorem 1.1 are satisfied when S is a locally compact topological group and Φ is any Banach space on which S acts isometrically and lower semicontinuously (these are the hypotheses of [**3**]). The result [**3**, 2.4] follows immediately, which we restate as Corollary 3.2.

In this section S will always be a locally compact group.

Lemma 3.1. *Suppose that* S *acts as a group of isometries on the Banach space* Φ.

(i) *If* $x \mapsto \bar{x}\mu$ *is not continuous at* $x_0 \in S$, *then for some* $\varepsilon > 0$, $x \mapsto \bar{x}\mu$ *is* ε-u.d.

(ii) *If the action is* lsc, *then it is* blsc.

Proof. (i) Suppose that

$$\limsup_{y\to x_0} ||\bar{y}\mu - \bar{x}_0\mu|| = \varepsilon > 0.$$

Then for each $z \in S$, we have

$$\limsup_{y\to z} ||\bar{y}\mu - \bar{z}\mu|| = \limsup_{x_0 z^{-1} y\to x_0} ||\overline{x_0 z^{-1} y}\mu - \bar{x}_0\mu|| = \varepsilon.$$

(ii) Since the action is lsc,

$$\{x : ||\bar{x}\mu - \mu|| > \varepsilon\}$$

is open, for every $\varepsilon > 0$, and $\mu \in \Phi$. But then, by the continuity of multiplication in S,

$$\{(x,y) \in S \times S : ||(\overline{y^{-1}x})\mu - \mu|| > \varepsilon\}$$

is open. Since S acts isometrically, $||(\overline{y^{-1}x})\mu - \mu|| = ||\bar{x}\mu - \bar{y}\mu||$ and (ii) follows. □

Corollary 3.2. *Let S be a locally compact group. If $\mu \in \Phi$ and $x \mapsto \bar{x}\mu$ is not continuous, then $\{\bar{x}\mu : x \in S\}$ spans a subspace of dimension $\geq c$.*

Proof. Immediate from Lemma 3.1 and Theorem 1.1. □

Remarks. Corollary 3.2 also holds for left cancellative stips. See [**2**] for the definition of a stip.

Corollary 3.2 does not hold for noncancellative stips. Indeed, let $S = \{0, 1/2, 1/3, \dots\}$ be the commutative semigroup with multiplication given by taking the maximum. Then the unit point mass $\mu = \bar{0}$ at 0 operates discontinuously at 0 and $\{\bar{x}\mu \in S\}$ spans a separable subspace.

4. Three examples.

4.1. *Weak ε-u.d. is not enough.* Let $T = [3,4] \times \{0, 1/2, 1/4, 1/8, \dots\}$ and let S be the subsemigroup of $\mathbf{R}^2$ generated by T. Let μ denote the product $\mu = \chi_{[3,4]} m_{\mathbf{R}} \times \delta(0)$ of Lebesgue measure on [3,4] with the unit point mass at 0. Then the orbit of μ in $M(S)$ (under the usual translation) is separable. For $X_1 = [3,4] \times \{0\}$, $x \mapsto \bar{x}\mu$ is weak ε-u.d. but for no $\varepsilon > 0$ and no compact perfect $X \subseteq S$ is $x \mapsto \bar{x}\mu$ ε-u.d.

4.2. *Cardinalities may vary.* Let e, f be any cardinalities $\geq c$, and let G, H be compact abelian groups having cardinalities e, f, respectively. Let $\mu = m_G \times \delta(0)$ denote the product of Haar measure on G with the unit point mass at the identity of H. Then $x \mapsto \bar{x}\mu$ (usual translation) is ε-u.d. on $S = G \times H$ for $0 < \varepsilon < 2$. But $\max \operatorname{Card} \{C : x, y \in C, x \neq y \Rightarrow ||\bar{x}\mu - \bar{y}\mu|| > \varepsilon\} = f$. Thus c is not the maximum cardinality we can hope for, in general, in Theorem 1.1 and Corollary 3.2.

4.3. *Many translators with separable orbits.* Define $M(S)_a = \{\mu \in M(S) : x \mapsto \bar{x}\mu \text{ is continuous}\}$. Define $\mathcal{E}$ to be the set of $\mu \in M(S) \cap M(S)_a^{\perp}$ such that $x \mapsto \bar{x}\mu$ is not continuous but $\{\bar{x}\mu : x \in S\}$ is separable. It is not necessarily the case that $\mathcal{E}$ is contained in $L^1(\omega)$ for some σ-finite ω. Indeed, let $S = \{0,1\}^A$, where A is uncountable and $\{0,1\}$ has the maximum operation. Let S_f denote the elements of S such that $\{i \in A : f(i) = 0\}$ is countable infinite, finite, or empty. Then S is a compact semigroup with an identity. Let $\mathcal{F} = \{\bar{g} : g \in S_f\}$. Then $\mathcal{F} \subseteq \mathcal{E}$ and there is no σ-finite measure ω with $\mathcal{F} \subseteq L^1(\omega)$. The required conclusion follows from [**2**, 3.10(a)], which states that for a compact semigroup S, $xS \cup Sx$ is finite if and only if $\bar{x}$ translates continuously.

5. Questions. Example 4.1 shows that $\{\bar{s}\mu : s \in S\}$ can be separable without $s \mapsto \bar{s}\mu$ being continuous, even when μ is a continuous measure. (Recall that a measure μ is continuous if it vanishes on all singletons.) That suggests the following questions.

Question 5.1. When does $\{\bar{s}\mu : s \in S\}$ separable for *all* continuous μ imply S is countable?

That was addressed in Theorem 2.4, but Theorem 2.4 did not cover all cases. By the "nonseparability-continuous dichotomy for an open set" we mean: either $x \mapsto \bar{x}\mu$ is such that $x \mapsto \bar{x}\mu$ is continuous for all $x \in U$, an open set, or $\{\bar{x}\mu : x \in U\}$ spans a subspace which is nonseparable.

Question 5.2. For which S, U, Φ does that dichotomy hold?

Question 5.3. For which actions of S on Φ does lsc imply blsc?

Question 5.4. Can 2.3 be extended beyond cancellative topological semigroups?

Question 5.5. Does 2.4 hold for semi-topological semigroups?

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