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CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTIONS AND ALGEBRAIC INTEGERS

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1. Introduction. Examples of continuous, nowhere differentiable functions are well known [**1**] and go back to at least 1860, see [**2**, pp. 954-956] for historical discussion. In most of these examples one considers an infinite series $\sum f_n(x)$, where each f_n is small in some norm, but "wiggles a lot." We present a curious example of another construction, based on some properties of algebraic numbers. The basic idea is as follows. Let $\beta > 1$ be a fixed number, β not equal to a rational integer. Every real number $x \in [0,1]$ can be uniquely (in a sense defined below) represented as $x = \sum_{n=1}^{\infty} \varepsilon_n(x)\beta^{-n}$ (representation in base β). Let now $|\alpha| > 1$ be another number, and define a function

$$f_{\alpha,\beta}(x) = \sum_{n=1}^{\infty} \varepsilon_n(x)\alpha^{-n}. \tag{1}$$

When α and β are chosen arbitrarily, the function $f_{\alpha,\beta}$ so defined is generally not continuous. In some special cases, however, when α and β are conjugate algebraic integers, the function $f_{\alpha,\beta}$ defined by (1) turns out to be continuous, but nowhere differentiable. We should remark that not every pair of conjugate algebraic integers leads to such a function, but we will give a condition which allows one to construct plenty of examples.

2. β-Expansions of real numbers. We recall here some basic definitions and results, most of which can be found in A. Renyi [**4**] and W. Parry [**3**]. We change the notation slightly and modify some of the results from these two sources in order to suit our specific needs.

Let $\beta > 1$, β not equal to a rational integer, be fixed. For every x, $0 \le x \le 1$, define a sequence of integer "digits" $\varepsilon_n(x)$ and "remainders"

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$r_n(x)$ by the following recursive formulas:

$$\begin{aligned} \varepsilon_1(x) &= \lfloor \beta x \rfloor, \qquad r_1(x) = \{\beta x\} \\ \varepsilon_{n+1}(x) &= \lfloor \beta r_n(x) \rfloor, \qquad r_{n+1}(x) = \{\beta r_n(x)\}, \qquad n = 1, 2, \dots . \end{aligned} \tag{2}$$

Here $\lfloor . \rfloor$ is the "floor" or the greatest integer function, and $\{.\}$ is the fractional part. Thus, for all t's, $t = \lfloor t \rfloor + \{t\}$. One easily verifies by induction that for each n

$$r_n(x) = \beta^n \left(x - \sum_{j=1}^{n} \frac{\varepsilon_j(x)}{\beta^j} \right). \tag{3}$$

Thus,

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{\beta^n} \tag{4}$$

and the integers $\varepsilon_n(x)$ are called the digits of x in base β. We denote (4) by

$$[x]_\beta = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots). \tag{5}$$

Definition 1. Let $\beta > 1$, β not equal to an integer, be fixed. A number $x \in [0, 1]$ is called *β-terminating* if the expansion (5) is finite, i.e., if $\varepsilon_n(x) = 0$ for large n. A number x is called *β-periodic* if it is not β-terminating and the expansion (5) is eventually periodic. The number β is called *simple* if 1 is a β-terminating number. The number β is called *semisimple* if 1 is β-periodic.

Definition 2. Let $\mathcal{C} = (c_1, c_2, \dots)$ be a bounded sequence of nonnegative integers (finite or not). Define

$$\mathcal{C}(\beta) = (c_1, c_2, \dots)(\beta) = \frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \cdots .$$

A sequence $\mathcal{C}$ is said to be *canonical*, if $\mathcal{C} = [x]_\beta$ for some $x \in [0, 1)$, i.e., if $\mathcal{C}$ is the sequence of digits in β-expansion of some $x \in [0, 1)$.

It is obvious that a necessary condition for $\mathcal{C} = (c_1, c_2, \dots)$ to be canonical is that each $c_i < \beta$. This is not, however, sufficient. For example, if $\beta = (1+\sqrt{5})/2$, then a sequence of 0's and 1's is canonical if and only if it does not contain two successive 1's, see [**4**]. We will shortly give the necessary and sufficient conditions for a sequence $\mathcal{C}$ to be canonical.

Definition 3. Let $\mathcal{C} = (c_1, c_2, \dots)$ and $\mathcal{D} = (d_1, d_2, \dots)$ be two sequences. We say that $\mathcal{C} < \mathcal{D}$ if $\mathcal{C} \neq \mathcal{D}$ and $c_i < d_i$ for the first subscript i for which $c_i \neq d_i$. We apply this definition of inequality to sequences that are finite or infinite. In case one or both sequences are finite, we pad the shorter sequence with 0's at the end, if necessary. Thus, for example, $(1,2) < (1,2,3)$, since $(1,2,0) < (1,2,3)$.

The following theorem is basic to what follows. Major portions of this theorem were proved in [**3**]; we extend the results from there slightly and give a somewhat different proof:

Theorem 1. *Let $\beta > 1$, β not equal to an integer. The following are necessary and sufficient conditions for a sequence $\mathcal{C} = (c_1, c_2, \dots)$ to be canonical:*

i) *Assume β is not simple and let $[1]_\beta = \mathcal{B} = (b_1, b_2, \dots)$. Then a sequence $\mathcal{C} = (c_1, c_2, \dots)$ of nonnegative integers is canonical if and only if*

$$(6) \qquad (c_n, c_{n+1}, c_{n+2}, \dots) < \mathcal{B} = (b_1, b_2, \dots) \quad \text{for all } n = 1, 2, 3, \dots .$$

ii) *Assume β is simple, let $[1]_\beta = (b_1, b_2, \dots, b_q)$, and let $\overline{\mathcal{B}} = (\bar{b}_1, \bar{b}_2, \dots)$ be the sequence defined by:*

$$(7) \qquad \bar{b}_n = \begin{cases} b_n & \text{if } n \not\equiv 0 \pmod q \\ b_n - 1 & \text{if } n \equiv 0 \pmod q. \end{cases}$$

A sequence $\mathcal{C} = (c_1, c_2, \dots)$ of nonnegative integers is canonical if and only if

$$(8) \qquad (c_n, c_{n+1}, \dots) < \overline{\mathcal{B}} = (\bar{b}_1, \bar{b}_2, \dots) \quad \text{for all } n = 1, 2, \dots .$$

Notice that since β is not an integer, $q > 1$ in part ii)*, and that $\overline{\mathcal{B}}$ is an infinite sequence.*

For the proof we need several simple lemmas.

Lemma 1. *Suppose $\mathcal{C} = (c_1, c_2, \dots)$ is canonical. Then $(c_2, c_3, \dots)$ is also canonical. This applies whether $\mathcal{C}$ is finite or not.*

Proof. Suppose $\mathcal{C}$ is canonical. Let $x = \mathcal{C}(\beta) = (c_1, \dots)(\beta)$. Then, by the definition of β-expansion and (3), $(c_2, c_3, \dots)$ is the β-expansion of $r_1(x) = \beta(x - c_1/\beta) < 1$. Thus, $(c_2, c_3, \dots)$ is canonical. □

Lemma 2. *Suppose β is not simple and $\mathcal{C}$ is a sequence of nonnegative integers for which* (6) *holds. Then $\mathcal{C}(\beta) = (c_1, c_2, c_3, \dots)(\beta) < 1$ and the inequality is strict.*

Proof. Since β is not simple, $b_n > 0$ for infinitely many n's. We first prove the lemma in case $\mathcal{C} = (c_1, c_2, \dots, c_p)$ is finite. If $p = 1$, (6) means $c_1 \le b_1$ so $\mathcal{C}(\beta) = c_1/\beta \le b_1/\beta < 1$. Suppose the assertion is proved for all sequences of length $p < P$, and let $\mathcal{C} = (c_1, c_2, \dots, c_P)$ be a sequence for which (6) holds. There are two cases: i) $c_1 < b_1$ and ii) $c_1 = b_1$. In case i), $c_1 \le b_1 - 1$ and (6) holds for $n = 2, 3, \dots, P$, so by the inductive hypothesis, $(c_2, c_3, \dots, c_P)(\beta) < 1$. Then

$$\begin{aligned}(c_1, c_2, \dots, c_P)(\beta) &= \frac{c_1}{\beta} + \frac{1}{\beta}(c_2, \dots, c_P)(\beta)\\ &\le \frac{b_1}{\beta} - \frac{1}{\beta} + \frac{1}{\beta} < 1.\end{aligned}$$

In case ii), let j be the smallest index for which $c_j \ne b_j$ so that $0 \le c_j \le b_j - 1$, in particular $b_j > 0$. If $j > P$, then $\mathcal{C}(\beta) \le (b_1, b_2, \dots, b_P)(\beta) < 1$ since β is not simple. If $j = P$, then $\mathcal{C}(\beta) \le (b_1, b_2, \dots, b_P)(\beta) - 1/\beta^P < 1$ also. If $1 < j < P$, the condition (6) holds for $m = j+1, \dots, P$, so $(c_{j+1}, \dots, c_P)(\beta) < 1$ and

$$\begin{aligned}(c_1, \dots, c_P)(\beta) &= (c_1, \dots, c_{j-1})(\beta) + \frac{c_j}{\beta^j} + (c_{j+1}, \dots, c_P)(\beta)\\ &\le (b_1, \dots, b_{j-1})(\beta) + \frac{b_j}{\beta^j} - \frac{1}{\beta^j} + \frac{1}{\beta^j} < 1\end{aligned}$$

because β is not simple. Hence the Lemma is true for all finite sequences. If $\mathcal{C}$ is an infinite sequence for which (6) holds, then by continuity $\mathcal{C}(\beta) \leq 1$. To show that the inequality is strict, let j be the first subscript for which $c_j \neq b_j$ so that $c_j \leq b_j - 1$. We have

$$(10) \qquad \begin{aligned}(c_1, c_2, \dots)(\beta) &= (b_1, \dots, b_{j-1})(\beta) \\ &\quad + \frac{c_j}{\beta^j} + \frac{1}{\beta^j}(c_{j+1}, \dots)(\beta).\end{aligned}$$

Since $(c_{j+1}, \dots)$ also satisfies (6), $(c_{j+1}, c_{j+2}, \dots)(\beta) \leq 1$, and it follows from (10) that

$$\mathcal{C}(\beta) \leq (b_1, \dots, b_j)(\beta) + \frac{b_j}{\beta^j} - \frac{1}{\beta^j} + \frac{1}{\beta^j} < 1$$

because β is not simple. □

Lemma 3. *Suppose β is simple, let $\beta = [1]_\beta = (b_1, b_2, \dots, b_q)$, and let $\overline{\mathcal{B}} = (\bar{b}_1, \bar{b}_2, \dots)$ be the sequence given by (7). If $\mathcal{C} = (c_1, \dots, c_p)$ is a finite sequence such that*

$$(c_n, \dots, c_p) < (b_1, b_2, \dots, b_q) \qquad \textit{for all } n = 1, 2, \dots, p,$$

then $\mathcal{C}(\beta) < 1$. If $\mathcal{C}$ is an infinite sequence such that

$$(c_n, c_{n+1}, \dots) < (\bar{b}_1, \bar{b}_2, \dots) \qquad \textit{for all } n = 1, 2, \dots,$$

then $\mathcal{C}(\beta) < 1$. In addition,

$$(11) \qquad \overline{\mathcal{B}}(\beta) = (\bar{b}_1, \bar{b}_2, \dots)(\beta) = 1.$$

Proof. The proof is entirely similar to the proof of Lemma 2, and we omit it. In Lemma 2 we repeatedly used the fact that $b_n \neq 0$ for infinitely many n's and $\mathcal{B}(\beta) = 1$. In Lemma 3, by definition, $\bar{b}_n \neq 0$ for infinitely many n's and then we use (11). □

We now proceed with the proof of Theorem 1. Assume first that β is not simple, $[1]_\beta = \mathcal{B} = (b_1, b_2, \dots)$. Suppose $\mathcal{C} = (c_1, c_2, \dots)$ is canonical, i.e., for some $0 \le x < 1$, $[x]_\beta = \mathcal{C}$, and we must show that (6) holds. It is enough to show that (6) holds for $n = 1$, since then we can repeatedly apply Lemma 1 to get the result for an arbitrary n. Let j be the first index for which $c_j \ne b_j$. If $j = 1$, then $c_1 = [\beta x] \le \lfloor \beta \cdot 1 \rfloor = b_1$ so (6) holds. If $j > 1$, then

$$\begin{aligned} r_{j-1}(x) &= \beta^{j-1}\biggl(x - \sum_{k=1}^{j-1} \frac{c_k}{\beta^k}\biggr) \\ &\le \beta^{j-1}\biggl(1 - \sum_{k=1}^{j-1} \frac{b_k}{\beta^k}\biggr) \\ &= r_{j-1}(1). \end{aligned}$$

Hence, $c_j = \lfloor \beta r_{j-1}(x) \rfloor \le \lfloor \beta r_{j-1}(1) \rfloor = b_j$, i.e. $c_j < b_j$, so that (6) holds also. Conversely, suppose that $\mathcal{C} = (c_1, c_2, \dots)$ is a sequence of nonnegative integers for which (6) holds. Let $x = \mathcal{C}(\beta)$; we want to show that $[x]_\beta = \mathcal{C}$. By Lemma 2, $x < 1$ and $\beta x = c_1 + (c_2, c_3, \dots)(\beta)$. Since $(c_2, c_3, \dots)$ also satisfies (6), $(c_2, c_3, \dots)(\beta) < 1$ and so $\varepsilon_1(x) = \lfloor \beta x \rfloor = c_1$ and $r_1(x) = (c_2, c_3, \dots)(\beta)$. Applying the same reasoning to $r_1(x)$ we get $\varepsilon_1(r_1(x)) = \varepsilon_2(x) = c_2$. Continuing in this way we get $[x]_\beta = \mathcal{C}$.

Suppose now that β is simple; let $[1]_b = \mathcal{B} = (b_1, b_2, \dots, b_q)$, b_1 and b_q are both not equal to 0. Let $\mathcal{C} = (c_1, c_2, \dots)$ be a sequence of nonnegative integers for which (8) holds. Put $x = \mathcal{C}(\beta)$. Again, by Lemma 3, $x < 1$ and $\beta x = c_1 + (c_2, c_3, \dots)(\beta)$. By Lemma 3, $\varepsilon_1(x) = c_1$. Continuing as in the case when β was not simple, we get $[x]_\beta = \mathcal{C}$ so that $\mathcal{C}$ is canonical. Conversely, let $0 \le x < 1$ and let $\mathcal{C} = [x]_\beta$; we must show that (8) holds. Again, it is enough to show this for $n = 1$ and then repeatedly apply Lemma 1. Since $x < 1$ and $\overline{\mathcal{B}}(\beta) = 1$, $\mathcal{C} \ne \overline{\mathcal{B}}$, so let j be the first index for which $c_j \ne \bar{b}_j$. We must show that $c_j < \bar{b}_j$. Let k be the integer such that $kq < j \le (k+1)q$. By Lemma 1, $\mathcal{C}' = (c_{kq+1}, c_{kq+2}, \dots) = (c'_1, c'_2 \dots)$ is also canonical and $x' = \mathcal{C}'(\beta) < 1$. Let $i = j - kq$. Since $\overline{\mathcal{B}}$ is periodic with period q, we have $\bar{b}_j = \bar{b}_i$, $c'_i = c_j$ and $c'_m = \bar{b}_m$ for $1 \le m < i$. We must show that $c'_i < \bar{b}_i$. Suppose $c'_i > \bar{b}_i$. If $i = 1$, this means that

$c'_1 = \lfloor \beta x' \rfloor > \bar{b}_1 = b_1 = \lfloor \beta \cdot 1 \rfloor$ so $x' > 1$, a contradiction. If $1 < i < q$,

$$\begin{aligned} r_{i-1}(x') &= \beta^{i-1}\left(x' - \sum_{m=1}^{i-1} \frac{c'_m}{\beta^m}\right) \\ &< \beta^{i-1}\left(1 - \sum_{m=1}^{i-1} \frac{b_m}{\beta^m}\right) \\ &= r_{i-1}(1). \end{aligned}$$

If $c'_i = \lfloor \beta r_{i-1}(x') \rfloor > \bar{b}_i = b_i = \lfloor \beta r_{i-1}(1) \rfloor$, this would imply $r_{i-1}(x') > r_{i-1}(1)$, again a contradiction. Finally, if $i = q$, and $c'_q > \bar{b}_q$, then $c'_q \geq b_q$ and $x' = \mathcal{C}'(\beta) \geq \mathcal{B}(\beta) = 1$, also impossible. This proves Theorem 1. □

We need one more additional result for later.

Lemma 4. *If $\mathcal{C}$ and $\mathcal{D}$ are two canonical sequences, then $\mathcal{C} < \mathcal{D}$ if and only if $\mathcal{C}(\beta) < \mathcal{D}(\beta)$.*

Proof. Let $x = \mathcal{C}(\beta)$ and $y = \mathcal{D}(\beta)$. Suppose first that $\mathcal{C} < \mathcal{D}$, and let j be the first index such that $c_j \neq d_j$. If $j = 1$, then $c_1 = \lfloor \beta x \rfloor] < d_1 = \lfloor \beta y \rfloor$ implies $x < y$. If $j > 1$, then by (3) $r_{j-1}(x) = r_{j-1}(y)$ and again by (3)

$$c_j = \lfloor \beta r_{j-1}(x) \rfloor = \left\lfloor \beta^j \left(x - \sum_{k=1}^{j-1} \frac{c_k}{\beta^k}\right) \right\rfloor$$
$$< d_j = \lfloor \beta r_{j-1}(y) \rfloor = \left\lfloor \beta^j \left(y - \sum_{k=1}^{j-1} \frac{d_k}{\beta^k}\right) \right\rfloor$$

which implies $x < y$ since $c_k = d_k$ for $k = 1, 2, \ldots, j-1$. The proof of the converse is entirely similar and we omit it.

3. Main results. Let $\beta > 1$, β not equal to an integer, be given. For every α such that $|\alpha| > 1$, define a function

$$f_{\alpha,\beta}(x) = \sum_{n=1}^{\infty} \frac{c_n}{\alpha^n} \quad \text{if } [x]_\beta = (c_1, c_2, \ldots). \tag{12}$$

We now give conditions for α and β for $f_{\alpha,\beta}(x)$ to be a continuous, nowhere differentiable function on $[0,1]$.

Definition 4. Suppose that $\beta > 1$ is simple, $[1]_\beta = \mathcal{B} = (b_1, b_2, \dots, b_q)$, $b_q \neq 0$. The characteristic equation of β is $\mathcal{B}(x) = 1$, or

$$1 = \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots + \frac{b_q}{x^q} \tag{13}$$

which can be rewritten as

$$x^q - (b_1 x^{q-1} + b_2 x^{q-2} + \cdots + b_q) = 0. \tag{14}$$

Suppose that β is semisimple, so that $[1]_\beta = \mathcal{B} = (b_1, b_2, \dots, b_q, \overline{a_1, a_2, \dots, a_p})$, where the overline indicates the periodic part. The characteristic equation of β is again $\mathcal{B}(x) = 1$, or

$$1 = \frac{b_1}{x} + \cdots + \frac{b_1}{x^q} + \frac{1}{x^q}\left(\frac{a_1}{x} + \cdots + \frac{a_p}{x^p}\right)\left(1 + \frac{1}{x^p} + \frac{1}{x^2 p} + \cdots\right), \tag{15}$$

which can be rewritten as

$$(x^p - 1)(x^q - (b_1 x^{q-1} + \cdots + b_q)) - a_1 x^{p-1} + \cdots + a_p = 0. \tag{16}$$

Theorem 2. *If $\beta > 1$ is either simple or semisimple and α is another root of the characteristic equation of β such that $1 < |\alpha| < \beta$, then the function $f_{\alpha,\beta}(x)$ is continuous, nowhere differentiable on $[0,1]$. If α is not a root of the characteristic equation of β, then $f_{\alpha,\beta}(x)$ is not left continuous at terminating points.*

We prove Theorem 2 by a sequence of lemmas.

Definition 5. Let $\beta > 1$ be fixed, and let x be a terminating point, i.e., $[x]_\beta = (c_1, c_2, \dots, c_n)$, $c_n \neq 0$. The number n is called the *rank* of x and is denoted by $\rho(x)$. If x is not β-terminating, we set $\rho(x) = \infty$.

Lemma 5. *Suppose x and y are two successive β-terminating numbers of rank at most n. This means that $\rho(x) \leq n$ and $\rho(y) \leq n$,* *and if* $x < t < y$, *then* $\rho(t) > n$. *Then, for all* $x < t < y$, *the first* n β*-digits of* x *and* t *coincide*: $\varepsilon_j(x) = \varepsilon_j(t)$, *for* $j = 1, 2, \dots, n$.

Proof. Let $[x]_\beta = (c_1, \dots c_n)$, $[y]_\beta = (d_1, \dots, d_n)$, and $[t]_\beta = (e_1, e_2, \dots)$. Let j be the first index so that $c_j \neq e_j$ and assume that $j \leq n$. By Lemmas 2 and 3, the sequence $(e_1, \dots, e_j)$ is canonical and $(e_1, \dots, e_j)(\beta)$ is of rank at most n. If $c_j < e_j$, then by Lemma 4, $x < (e_1, \dots, e_j)(\beta) \leq t < y$, so x and y are not two successive β-terminating numbers of rank at most n. If $e_j < c_j$, then by $t < x$ by Lemma 4, this is contrary to the hypothesis. □

Corollary 1. *For any* $|\alpha| > 1$, $\beta > 1$, *the function* $f_{\alpha,\beta}(x)$ *is continuous at every nonterminating* $x \in [0, 1]$ *and is continuous from the right at every* $x \in [0, 1)$.

Proof. Suppose that t is not β-terminating. Let $\varepsilon > 0$ be given. By Lemmas 2 and 3, truncation of canonical sequences are canonical, hence the set of β-terminating numbers is dense in $[0, 1]$. Choose n so that $\gamma = \sum_{k=n}^{\infty} 2\beta/|\alpha|^k < \varepsilon$, and two successive β-terminating numbers x and y of rank at least n such that $x < t < y$. If t' is another number in the interval (x, y), then, by Lemma 5, the first n β-digits of t and t' are the same (they are the same as the digits of x). It follows that $|f_{\alpha,\beta}(t) - f_{\alpha,\beta}(t')| < \gamma < \varepsilon$. Suppose now that t is β-terminating, $0 \leq t < 1$. For a given $\varepsilon > 0$, choose n and γ as before and let y be the smallest β-terminating number of rank at least n which is greater than t. Arguing as before, if $t < t' < y$, then $|f_{\alpha,\beta}(t) - f_{\alpha,\beta}(t')| < \varepsilon$. □

Lemma 6. *Suppose that* t *is a* β*-terminating number,* $[t]_\beta = (c_1, c_2, \dots, c_N)$, $c_N \neq 0$. *If* β *is not simple,* $[1]_\beta = \mathcal{B} = (b_1, b_2, \dots)$, $b_n \neq 0$ *for infinitely many* n*'s, then for every integer* r *the sequence* $\mathcal{T}_r = (c_1, c_2, \dots, c_{N-1}, c_N - 1, b_1, b_2, \dots, b_r)$ *is canonical, and* $t_r = \mathcal{T}_r(\beta) \uparrow t$ *as* $r \to \infty$. *If* β *is simple and* $\overline{\mathcal{B}} = (\bar{b}_1, \bar{b}_2, \dots)$ *is defined by* (7), *then again, for every* r *the sequence* $\overline{\mathcal{T}}_r = (c_1, c_2, \dots, c_{N-1}, c_N - 1, \bar{b}_1, \bar{b}_2, \dots, \bar{b}_r)$ *is canonical and* $\bar{t}_r = \overline{\mathcal{T}}_r(\beta) \uparrow t$ *as* $t \to \infty$.

Proof. First of all, it is clear that both t_r and $\bar{t}_r$ are increasing, and an easy calculation shows that both sequences converge to t as

$r \to \infty$. We must show that $\mathcal{T}_r$ satisfies condition (6) and $\overline{\mathcal{T}}_r$ satisfies (8). We present the arguemnt for the sequence $\mathcal{T}_r$ only, the other case is completely analogous. Let

$$\begin{aligned}\mathcal{T}_r &= (c_1, c_2, \dots, c_N - 1, c_N - 1, b_1, b_2, \dots, b_r)\\ &= (d_1, d_2, \dots, d_{N+r}),\end{aligned}$$

let m be fixed, and let $\mathcal{D} = (d_m, d_{m+1}, \dots, d_{N+r}) = (d'_1, d'_2, \dots, d'_{N+r-m+1})$. We must show that $\mathcal{D}' < \mathcal{B}$. If $m > N$, we are done, since in that case $\mathcal{D}'$ is just a finite portion of the sequence $\mathcal{B}$, hence it is canonical and (6) holds by Theorem 1. We may thus assume that $1 \le m \le N$. This means that

$$\begin{aligned}\mathcal{D}' &= (d'_1, d'_2, \dots, d'_{N+r-m+1})\\ &= (d_m, d_{m+1}, \dots, d_{N+r})\\ &= (c_m, c_{m+1}, \dots, c_{N-1}, c_N - 1, b_1, b_2, \dots, b_r).\end{aligned}$$

Let j be the first subscript for which $d'_j \ne b_j$. We claim that d'_j is among the c's, i.e., $j \le N - m + 1$. Indeed, if $c_m = b_1$, $c_{m+1} = b_2, \dots, c_{N-1} = b_{N-m}$, and $c_N - 1 = b_{N-m+1}$, then $c_N > b_{N-m+1}$ so $(c_m, \dots, c_{N-1}, c_N) > (b_1, b_2, \dots)$, contradicting the fact that $(c_1, c_2, \dots, c_N)$ is canonical. Thus, d'_j is among the c's. Since $(c_m, \dots, c_{N-1}, c_N) < (b_1, b_2, \dots)$, (6) holds for the sequence $\mathcal{D}'$. □

Lemma 7. *Suppose β is either simple or semisimple, and t is β-terminating. If α is another root of the characteristic equation of β, $|\alpha| > 1$, then the function $f_{\alpha,\beta}(x)$ is continuous at t. If $|\alpha| > 1$ is not a root of the characteristic equation of β, then $f_{\alpha,\beta}(x)$ is not left continuous at t.*

Proof. Let $[t]_\beta = \mathcal{C} = (c_1, c_2, \dots, c_N)$, $c_N > 0$, and let $\mathcal{D}_r = (d_1, d_2, \dots, d_r)$ be equal to $(b_1, b_2, \dots, b_r)$ if β is semisimple, or $(\bar{b}_1, \bar{b}_2, \dots, \bar{b}_r)$ if β is simple. In either case, $\mathcal{D}_r(x) \uparrow \mathcal{B}(x)$ as $r \to \infty$. By Lemma 6, the sequence $\mathcal{F}_r = (c_1, c_2, \dots, c_{N-1}, c_N - 1, d_1, d_2, \dots, d_r)$ is canonical, so $t_r = \mathcal{F}_r(\beta)$ is β-terminating of rank at most $N + r$. If α is not a root of $\beta(x) = 1$, then

$$f_{\alpha,\beta}(t) - f_{\alpha,\beta}(t_r) = \frac{1}{\alpha^N}(1 - \mathcal{D}_r(\alpha)) \to \frac{1}{\alpha^N}(1 - \mathcal{B}(\alpha)) \ne 0.$$

Hence, $f_{\alpha,\beta}$ is not left continuous at t. Suppose now that α is a root of the characteristic equation. We claim that if s is a β-terminating number of rank at most $N+r$, $s \neq t, t_r$, then s lies outside the interval $[t_r, t]$. Suppose to the contrary that $t_r < s < t$, and let $\mathcal{L} = [s]_\beta = (s_1, s_2, \dots, s_p)$, $p \leq N+r$. By Lemma 4, $\mathcal{F}_r < \mathcal{L} < \mathcal{C}$. Let j be the first index for which $s_j \neq c_j$. If $1 \leq j \leq N-1$, then $s_j < c_j$, so $\mathcal{L} < \mathcal{F}_r$ which, by Lemma 4, implies that $s < t_r$, contrary to assumption. If $j > N$, then clearly $s > t$. Assume then that $j = N$ and $s_N \leq c_N - 1$. By Lemma 2 and Theorem 1, $\mathcal{G} = (s_{N+1}, s_{N+2}, \dots, s_p)$ is also canonical, and $\mathcal{G} < \mathcal{B}$ or $\overline{\mathcal{B}}$, depending on whether β is simple or semisimple. Since the length of $\mathcal{G}$ is at most r, $\mathcal{G} \leq \mathcal{D}_r$. This implies

$$s = \mathcal{L}(\beta) \leq \mathcal{C}(\beta) + \frac{1}{\beta^N}(-1 + \mathcal{D}_r(\beta)) = t_r,$$

contradicting the hypothesis that $t_r < s$. Let now $\varepsilon > 0$ be given. Choose r such that

$$\frac{1}{|\alpha|^N}\bigg(-1 + \mathcal{D}_r(\alpha) + \sum_{m=N+r+1}^{\infty} \frac{\beta}{|\alpha|^m} \bigg) = \delta_r < \varepsilon.$$

By Lemma 5, for any $t_r < s < t$, the first $N+r+1$ digits of s coincide with those of t_r. Thus,

$$\begin{aligned} |f_{\alpha,\beta}(s) - f_{\alpha,\beta}(t)| &= \frac{1}{|\alpha|^N}\bigg(-1 + \mathcal{D}_r(\alpha) + \sum_{m=N+r+1}^{\infty} \frac{\beta}{|\alpha|^m} \bigg) \\ &= \delta_r < \varepsilon. \end{aligned}$$

This shows that $f_{\alpha,\beta}(x)$ is left continuous on t and proves Lemma 7. □

Lemma 8. *If $1 < |\alpha| < \beta$, then $f_{\alpha,\beta}(x)$ is not differentiable on* $[0, 1]$.

Proof. Let $t \in [0, 1]$ be fixed. Assume first that t is nonterminating, and let $[t]_\beta = \mathcal{C} = (c_1, c_2, \dots)$ so there are infinitely many subscripts i_j for which $c_{i_j} > 0$. Let $\mathcal{C}_j$ be the sequence obtained from $\mathcal{C}$ by replacing c_{i_j} with $c_{i_j} - 1$. Clearly, $\mathcal{C}_j$ is canonical and $t_j = \mathcal{C}_j(\beta) \to t$. But

$$\left| \frac{f_{\alpha,\beta}(t) - f_{\alpha,\beta}(t_j)}{t - t_j} \right| = \left| \frac{\beta}{\alpha} \right|^{i_j} \to \infty \quad \text{since } \beta > \alpha. \tag{17}$$

Suppose next that t is terminating, $t < 1$, and $[t]_\beta = \mathcal{C} = (c_1, c_2, \dots, c_r)$. We claim that one can choose a sequence of integers $r < i_1 < i_2 < \dots$ such that each of the sequences $\mathcal{C}_j = (c_1, c_2, \dots, c_r, 0, 0, \dots, 0, 1)$ is canonical (1 stands in the i_jth position). This will certainly show that $f_{\alpha,\beta}(x)$ is not differentiable at t, since with $t_j = \mathcal{C}_j(\beta)$, the relation (17) holds again. Such a sequence i_j can be chosen fairly arbitrarily, provided only that $i_{j+1} > i_j + r$ and that $i_1 - r > r$. A moment of reflection will show that if that is done, the sequence $\mathcal{C}_j$ satisfies (6) or (8) (depending on the nature of β), hence it is canonical. The only case not covered yet is $t = 1$ and t is terminating, i.e., β is simple. In this case $\mathcal{C} = \mathcal{B} = (b_1, b_2, \dots, b_q) = [1]_\beta$. If α is not a root of the characteristic equation $\mathcal{B}(x) = 1$, the function $f_{\alpha,\beta}(x)$ is not left continuous at 1 (Lemma 7), let alone differentiable. We may thus assume that α is a root of $\mathcal{B}(x) = 1$. Let $\overline{\mathcal{B}} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_q)$ be the sequence defined by (7), and put $\mathcal{B}_j = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_{jq})$. By Theorem 1, $\overline{\mathcal{B}}_r$ is canonical, and by direct computation we have for every x:

$$\begin{aligned}\mathcal{B}_j(x) &= \left(1 - \frac{1}{x^q}\right) + \frac{1}{x^q}\left(1 - \frac{1}{x^q}\right) + \frac{1}{x^{2q}}\left(1 - \frac{1}{x^q}\right)\\ &\quad + \dots + \frac{1}{x^{q(j-1)}}\left(1 - \frac{1}{x^q}\right)\\ &= 1 - \frac{1}{x^{qj}}\end{aligned}$$

so

$$\begin{aligned}t_j &= \overline{\mathcal{B}}_j(\beta) = 1 - \beta^{-jq},\\ f_{\alpha,\beta}(1) &= \mathcal{B}(\alpha) = 1, \quad \text{and}\\ f_{\alpha,\beta}(t_j) &= \overline{\mathcal{B}}_j(\alpha) = 1 - \alpha^{-jq}.\end{aligned}$$

Thus,

$$\left|\frac{f_{\alpha,\beta}(1) - f_{\alpha,\beta}(t_j)}{1 - t_j}\right| = \left|\frac{\beta}{\alpha}\right|^{qj} \to \infty \quad \text{also.}$$

This finishes the proofs of Lemma 8 and Theorem 2. □

4. Explicit examples. In this section we briefly indicate how to construct plenty of explicit examples of functions such as those described in Theorem 2. The construction is based on the following result from W. Parry [**3**]:

Theorem 3. *If* $\mathcal{B} = (b_1, b_2, \ldots, b_q)$, $q > 1$ *is a sequence of nonnegative integers such that* $(b_n, b_{n+1}, \ldots, b_q) < \mathcal{B}$ *for each* $n = 2, 3, \ldots, q$, *then the unique solution* $\beta > 1$ *of* $\mathcal{B}(x) = 1$ *is simple and* $[1]_\beta = \mathcal{B}$. *Moreover, all the other solutions* α *of this equation satisfy* $|\alpha| < 2$.

Thus it is enough to find a finite sequence $\mathcal{B}$ such that $\mathcal{B}(2) > 1$, $\mathcal{B}(-1) > 1$, and $\mathcal{B}(-2) < 1$ and for which the condition of Theorem 3 holds. For example, $\mathcal{B} = (3, 2, 0, 2, 0, 2)$ fits the bill: $\mathcal{B}(2) = 2.15625$, $\mathcal{B}(-1) = 3$, $\mathcal{B}(-2) = -0.84375$ giving $\beta = 3.601448365\ldots$, and $\alpha = -1.14028814\ldots$.

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