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ON THE STRUCTURE OF SHIFT-INVARIANT SUBSPACES OF $L^2(T^2, \mu)$

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0. Introduction. Suppose that V_1 and V_2 are commuting isometries on a Hilbert space $\mathcal{H}$. In [**3**] and [**6**], conditions are given for $\mathcal{H}$ to have four-fold Wold and Halmos decompositions with respect to V_1 and V_2. These notions are used in [**4, 5**] to characterize invariant subspaces $\mathcal{M}$ of the Hardy space $H^2(\mathbf{T}^2)$ which are generated by an inner function. Such $\mathcal{M}$ are shown to be those invariant subspaces on which V_1 and V_2 doubly commute, where V_j is now multiplication by the coordinate variable z_j, $j = 1, 2$. More generally, [**1**] describes the invariant subspaces of $L^2(\mathbf{T}^2)$ on which these V_1 and V_2 doubly commute. In this article we explore these ideas in the case $\mathcal{M}$ is an invariant subspace of the weighted space $L^2(\mathbf{T}^2, \mu)$.

1. The univariate case. We begin by considering the univariate analogue. This will shed light on the main problem.

Let μ be a finite nonnegative Borel measure on the unit circle $\mathbf{T}$. Define the isometry V on $L^2(\mu)$ by $(Vf)(z) = zf(z)$. A subspace $\mathcal{M}$ of $L^2(\mu)$ is *invariant* (for V) if $V\mathcal{M} \subseteq \mathcal{M}$. Following [**2**], we describe all of the invariant subspaces of $L^2(\mu)$. This will require the Lebesgue decomposition

$$d\mu = 1_\Gamma w \, d\sigma + 1_{\Gamma^c} \, d\lambda$$

where σ is normalized Lebesgue measure on $\mathbf{T}$, w is a weight function, and $0 = \lambda(\Gamma) = \sigma(\Gamma^c)$.

Theorem 1.1. *A subspace $\mathcal{M}$ of $L^2(\mu)$ satisfies the condition $V\mathcal{M} = \mathcal{M}$ if and only if $\mathcal{M} = 1_\Omega L^2(\mu)$ for some Borel set Ω.*

The proof is very similar to that of [**2**, Theorem 2].

Theorem 1.2. *A subspace $\mathcal{M}$ of $L^2(\mu)$ satisfies the condition*

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$V\mathcal{M} \subsetneq \mathcal{M}$ if and only if w is positive almost everywhere $[\sigma]$, and $\mathcal{M}$ is of the form

$$\mathcal{M} = \mathrm{sp}\,\{z^m h\}_{m=0}^{\infty} \oplus 1_\Omega L^2(\mu),$$

where $|h|^2 = 1_\Gamma w^{-1}$ almost every $[\sigma]$, and $\sigma(\Omega \cap \Gamma) = 0$. In this case, h is unique up to a unimodular constant factor.

Proof. Suppose that $V\mathcal{M} \subsetneq \mathcal{M}$. Let h be a unit vector in $\mathcal{M} \ominus V\mathcal{M}$. Then $\int h\overline{(z^m h)}\, d\mu = \int |h|^2 z^{-m}\, d\mu = 0$ for $m = 1, 2, \dots$. Taking complex conjugates, we find that $\int |h|^2 z^{-m}\, d\mu = 0$ for all nonzero m. Hence $1_{\Gamma^c} h = 0$ and $|h|^2 w = 1_\Gamma$, almost everywhere $[\sigma]$. This forces w to be positive, almost everywhere $[\sigma]$.

Now the orthonormal set $\{z^m h\}_{m=-\infty}^{\infty}$ spans a doubly invariant subspace $\mathcal{N}$. By Theorem 1.1, $\mathcal{N}$ has the structure $1_\Xi L^2(\mu)$ for some Borel set Ξ. Since h in $\mathcal{N}$ is nonvanishing for $z \in \Gamma$, we can take Ξ to be Γ itself. Thus, $\mathrm{sp}\,\{z^m h\}_{m=-\infty}^{\infty} = 1_\Gamma L^2(\mu)$. Note also that $\mathrm{sp}\,\{z^m h\}_{m=0}^{\infty} \subseteq \mathcal{M}$, and $\mathrm{sp}\,\{z^m h\}_{m=-\infty}^{-1} \perp \mathcal{M}$.

Next, suppose that $g \in \mathcal{M} \ominus V\mathcal{M}$. Then $1_{\Gamma^c} g = 0$, and $|g|^2 w$ is constant almost everywhere $[\sigma]$. Furthermore, $\int h\bar{g} z^m\, d\mu = 0$ for all nonzero m. This shows that g is a constant multiple of h. That is, $\mathcal{M} \ominus V\mathcal{M} = (h)$.

Finally, observe that $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_\infty$, where $\mathcal{M}_\infty = \cap_{m=0}^{\infty} V^m \mathcal{M}$, and $\mathcal{M}_0 = \cap_{m=0}^{\infty} \oplus V^m(\mathcal{M} \ominus V\mathcal{M}) = \mathrm{sp}\,\{z^m h\}_{m=0}^{\infty} = 1_\Gamma L^2(\mu)$. Since $\mathcal{M}_\infty$ is doubly invariant, there exists a Borel set Ω such that $\mathcal{M}_\infty = 1_\Omega L^2(\mu)$. Choose any $f \in \mathcal{M}_\infty$. Then f is orthogonal to each $z^m h$, for the latter is in $\mathcal{M}_0$ if $m \geq 0$, or in $\mathcal{M}^\perp$ if $m < 0$. It follows that $\sigma(\Omega \cap \Gamma) = 0$. This demonstrates the necessity assertion and the uniqueness of h.

Conversely, if $w > 0$, almost everywhere $[\sigma]$, and $|h|^2 = 1_\Gamma w^{-1}$, almost everywhere $[\sigma]$, then $h \in L^2(\mu)$. With that, and some Borel subset Ω of Γ^c, the subspace $\mathcal{M}$ given by $\mathrm{sp}\,\{z^m h\}_{m=0}^{\infty} \oplus 1_\Omega L^2(\mu)$ is invariant (since $\Omega \subset \Gamma^c$, the sum is indeed orthogonal). For each

$m = 1, 2, \ldots,$

$$\begin{aligned}\int h\overline{(z^m h)}\,d\mu &= \int |h|^2 z^{-m} w\,d\sigma \\ &= \int z^{-m}\,d\sigma \\ &= 0.\end{aligned}$$

Also, h is orthogonal to $1_\Omega L^2(\mu)$. Thus $h \perp V\mathcal{M}$, and $\mathcal{M}$ must be simply invariant. The sufficiency part is proved. □

Thus an invariant subspace $\mathcal{M}$ has a Wold decomposition which is mirrored by the structure of the measure μ. We therefore expect four-fold a decomposition in the two-variable case, also mirrored by the measure.

2. Preliminaries. Now take μ to be a finite nonnegative Borel measure on the torus, $\mathbf{T}^2$. Henceforth, we adopt the following notation. The torus $\mathbf{T}^2$ is parameterized by (z_1, z_2). For $j = 1, 2$, let V_j be the isometry $f(z_1, z_2) \to z_j f(z_1, z_2)$ on $L^2(\mu)$. As suggested in Section 1, we will need to use the explicit structure of μ. First, define Borel measures μ_1 and μ_2 on $\mathbf{T}$ by

$$\mu_1(S) = \mu(S \times \mathbf{T}), \qquad \mu_2(S) = \mu(\mathbf{T} \times S).$$

Establish the Lebesgue decompositions

$$\begin{aligned}d\mu &= 1_\Gamma w_R d(\sigma \times \mu_2) + 1_{\Gamma^c}\,d\lambda_R \\ d\mu &= 1_\Delta w_T d(\mu_1 \times \sigma) + 1_{\Delta^c}\,d\lambda_T \\ d\mu &= 1_E w\,d\sigma^2 + 1_{E^c}\,d\lambda.\end{aligned}$$

Let A be the collection of $z_2 \in \mathbf{T}$ for which $w_R(z_1, z_2) > 0$, almost everywhere $[\sigma(z_1)]$; B, the collection of z_1 for which $w_T(z_1, z_2) > 0$ almost everywhere $[\sigma(z_2)]$. Throughout, we identify subsets of $\mathbf{T}^2$ whose symmetric difference is μ-null.

A subspace $\mathcal{M}$ of $L^2(\mu)$ is *invariant* if $V_1\mathcal{M} \subseteq \mathcal{M}$ and $V_2\mathcal{M} \subseteq \mathcal{M}$. We would like to describe these invariant subspaces $\mathcal{M}$ in terms of the structure of μ.

3. Two special cases. We are now equipped to extend the results of Section 1 to two special cases in the bivariate picture.

Theorem 3.1. *A subspace* $\mathcal{M}$ *of* $L^2(\mu)$ *satisfies*

$$V_1\mathcal{M} = \mathcal{M}, \qquad V_2\mathcal{M} = \mathcal{M}$$

if and only if $\mathcal{M} = 1_\Omega L^2(\mu)$ *for some Borel set* Ω.

Again, the proof follows [**2**, Theorem 2].

Theorem 3.2. *A subspace* $\mathcal{M}$ *of* $L^2(\mu)$ *satisfies*

$$V_1\mathcal{M} \subsetneq \mathcal{M}, \qquad V_2\mathcal{M} = \mathcal{M}$$

if and only if the set A *is* μ_2*-nonnull, and* $\mathcal{M}$ *is of the form*

$$\mathcal{M} = \operatorname{sp}\{z_1^m z_2^n h : m \geq 0,\ n \in \mathbf{Z}\} \oplus 1_\Omega L^2(\mu),$$

where, for some μ_2*-nonnull Borel subset* K *of* A, *the function* h *satisfies* $|h| = 1_{\Gamma\cap(\mathbf{T}\times K)} w_R^{-1/2}$, *and* Ω *is a Borel subset of* $\Gamma^c \cup \{\mathbf{T} \times K^c)$.

In this case, h is unique up to a unimodular factor depending only on z_2.

Proof. Let $\{v_0, v_1, v_2, \dots\}$ be an orthonormal basis for $\mathcal{M} \ominus V_1\mathcal{M}$. Put $f_0 = v_0$, and let $Q_0 = \operatorname{sp}\{z_2^n f_0\}_{n=-\infty}^{\infty}$. Having defined $\{f_0, f_1, \dots, f_k\}$ and $\{Q_0, Q_1, \dots, Q_k\}$, let n_{k+1} be the smallest index such that $v_{n_{k+1}}$ does not lie in $Q_0 \vee Q_1 \vee \dots \vee Q_k$. Let f_{k+1} be the component of $v_{n_{k+1}}$ in $(\mathcal{M} \ominus V_1\mathcal{M}) \ominus (Q_0 \vee Q_1 \vee \dots \vee Q_k)$, and let $Q_{k+1} = \operatorname{sp}\{z_2^n f_{k+1}\}_{n=-\infty}^{\infty}$. Now put $\mathcal{L}_k = \operatorname{sp}\{z_1^m z_2^n f_k : m \geq 0,\ n \in \mathbf{Z}\}$, $k = 0, 1, 2, \dots$. Note that the spaces $\mathcal{L}_k$ are subspaces of $\mathcal{M}$, they are pairwise orthogonal, and their direct sum includes $V_1^m\mathcal{M} \ominus V_1^{m+1}\mathcal{M}$ for each $m = 0, 1, 2, \dots$. Moreover, each $\mathcal{L}_k$ is orthogonal to $\mathcal{R}_\infty$, where $\mathcal{R}_\infty = \cap_{m=0}^{\infty} V_1^m\mathcal{M}$. Hence $\mathcal{M}$ has the decomposition $\mathcal{M} = \mathcal{R}_0 \oplus \mathcal{R}_\infty$, with $\mathcal{R}_0 = \sum_{k=0}^{\infty} \oplus \mathcal{L}_k$.

Suppose that j and k are distinct nonnegative integers. Then $\int f_j \bar{f}_k z_1^m z_2^n \, d\mu = 0$ for all m and n. Hence, if Λ_k is exactly the set on which f_k is nonvanishing, $k=0,1,2,\ldots$, then the family $\{\Lambda_k\}_{k=0}^{\infty}$ is pairwise disjoint. Let the f_k be rescaled so that $||f_k||^2=\mu(\Lambda_k)$. We can now define $f=\sum_{k=0}^{\infty} f_k$, where the convergence is in $L^2(\mu)$.

Fix a nonnegative integer j. Check that if $m\neq 0$ and $n\in Z$, then $\int f_j\bar{f}z_1^m z_2^n\,d\mu=0$. Then Lemma 3.3 below asserts that $1_{\Gamma^c}|f_j|^2\,d\lambda=0$, and $1_\Gamma|f_j|^2 w_R d(\sigma\times\mu_2)=1_\Gamma d(\sigma\times\xi)$, where ξ is the Borel measure on $\mathbf{T}$ defined by $\xi(S)=\int_{\mathbf{T}\times S}|f_j|^2\,d\mu$. Now $\xi\ll\mu_2$, so for some nonnegative μ_2-integrable function $\Phi_j(z_2)$, we have $|f_j|^2 w_R=\Phi_j$. This forces the set A to be μ_2-nonnull, and Λ_j to be of the form $(\mathbf{T}\times K_j)\cap\Gamma$, where K_j is a μ_2-nonnull subset of A. Since this is true for all j, we conclude that $\mathcal{R}_0\perp L^2(d\lambda)$, $|f|^2 w_R$ is a nonnegative μ_2-integrable function Φ of z_2 only, and f is nonvanishing exactly on $(\mathbf{T}\times K)\cap\Gamma$, K being the disjoint union $\cup_{j=0}^{\infty}K_j$. Thus $\mathcal{R}_0=\mathrm{sp}\,\{z_1^m z_2^n f: m\geq 0,\ n\in\mathbf{Z}\}$. This remains true with f replaced by h, where $h=f\Phi^{-1/2}1_{\Gamma\cap(\mathbf{T}\times K)}$. Note that $|h|^2 w_R=1_{\Gamma\cap(\mathbf{T}\times K)}$ as well.

The space $\mathrm{sp}\,\{z_1^m z_2^n h: m,n\in\mathbf{Z}\}$ must be exactly $1_{\Gamma\cap(\mathbf{T}\times K)}L^2(\mu)$. Furthermore, $z_1^m z_2^n h\perp\mathcal{M}$ whenever $m<0$. Hence $R_\infty=1_\Omega L_2(\mu)$ for some Borel set $\Omega\subseteq\Gamma^c\cup(\mathbf{T}\times K^c)$. This proves the necessity of the representation. Its sufficiency is immediate. □

Lemma 3.3. *Let ν be a finite nonnegative Borel measure on $\mathbf{T}^2$. Define the Borel measure ν_2 on $\mathbf{T}$ by $\nu_2(S)=\nu(\mathbf{T}\times S)$. If $\int z_1^m z_2^n\,d\nu=0$ for all $m\neq 0$ and $n\in\mathbf{Z}$, then $\nu=\sigma\times\nu_2$.*

Proof. Let E and F be open arcs of $\mathbf{T}$. Since $1_F(z_2)$ can be estimated boundedly and pointwise by a polynomial in z_2 and $\bar{z}_2$, we have, for all nonzero m, $\int z_1^m 1_F(z_2)\,d\nu=0$. Now if $p(z_1)$ is a finite sum $\sum a_m z_1^m$, then $\int_{\mathbf{T}\times F}p(z_1)\,d\nu=a_0\nu(\mathbf{T}\times F)=a_0\nu_2(F)$. But $1_E(z_1)$ can be estimated boundedly and pointwise by such sums. Thus

$$\begin{aligned}\nu(E\times F)&=\int_{\mathbf{T}\times F}1_E(z_1)\,d\nu\\&=\int_{\mathbf{T}}1_E(z_1)\,d\sigma(z_1)\cdot\nu_2(z_2)\\&=\sigma(E)\cdot\nu_2(F).\end{aligned}$$

The equality $\nu=\sigma\times\nu_2$ extends to the Borel sets of $\mathbf{T}^2$. □

4. The four-fold Halmos decomposition. We turn our attention to a wider class of invariant subspaces. As in [**3**], we say that $\mathcal{M}$ has a *four-fold Halmos decomposition* if it is of the form

$$\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_\infty,$$

where each component space is invariant (for V_1 and V_2); the operators $V_1 \mid \mathcal{M}_0$, $V_2 \mid \mathcal{M}_0$, $V_1 \mid \mathcal{M}_1$ and $V_2 \mid \mathcal{M}_2$ are pure shifts; and the operators $V_2 \mid \mathcal{M}_1$, $V_1 \mid \mathcal{M}_2$, $V_1 \mid \mathcal{M}_\infty$ and $V_2 \mid \mathcal{M}_\infty$ are unitary.

If $\mathcal{M}$ is an invariant subspace of $L^2(\mu)$, then we can construct a candidate for this decomposition as follows. Let

$$\mathcal{R}_\infty = \bigcap_{m=0}^{\infty} V_1^m \mathcal{M}, \qquad \mathcal{T}_\infty = \bigcap_{n=0}^{\infty} V_2^n \mathcal{M}$$
$$\mathcal{R}_0 = \mathcal{M} \ominus \mathcal{R}_\infty, \qquad \mathcal{T}_0 = \mathcal{M} \ominus \mathcal{T}_\infty.$$

Now consider the orthosum

$$\mathcal{M}' = (\mathcal{R}_0 \cap \mathcal{T}_0) \oplus (\mathcal{R}_0 \cap \mathcal{T}_\infty) \oplus (\mathcal{R}_\infty \cap \mathcal{T}_0) \oplus (\mathcal{R}_\infty \cap \mathcal{T}_\infty).$$

As observed in [**3**], $\mathcal{M}$ has a four-fold Halmos decomposition only if $\mathcal{M}'$ is all of $\mathcal{M}$, and its component spaces are invariant. Let us examine these issues more closely. Take $P(\mathcal{N})$ to denote the projection of $L^2(\mu)$ onto the subspace $\mathcal{N}$.

Proposition 4.1. *The following are equivalent.*

(i) *The orthosum for $\mathcal{M}'$ spans all of $\mathcal{M}$.*

(ii) *The spaces $\mathcal{R}_\infty \ominus (\mathcal{R}_\infty \cap \mathcal{T}_\infty)$ and $\mathcal{T}_\infty \ominus (\mathcal{R}_\infty \cap \mathcal{T}_\infty)$ are orthogonal.*

(iii) *The projections $P(\mathcal{R}_\infty)$ and $P(\mathcal{T}_\infty)$ commute.*

Proof. Assume (i). Then $\mathcal{R}_\infty = (\mathcal{R}_\infty \cap \mathcal{T}_0) \oplus (\mathcal{R}_\infty \cap \mathcal{T}_\infty)$, and $\mathcal{T}_\infty = (\mathcal{R}_0 \cap \mathcal{T}_\infty) \oplus (\mathcal{R}_\infty \cap \mathcal{T}_\infty)$. But $(\mathcal{R}_\infty \cap \mathcal{T}_0) \perp (\mathcal{R}_0 \cap \mathcal{T}_\infty)$, giving (ii). If (ii) holds, then $[P(\mathcal{R}_\infty) - P(\mathcal{R}_\infty \cap \mathcal{T}_\infty)][P(\mathcal{T}_\infty) - P(\mathcal{R}_\infty \cap \mathcal{T}_\infty)] = 0$, yielding $P(\mathcal{R}_\infty)P(\mathcal{T}_\infty) = P(\mathcal{R}_\infty \cap \mathcal{T}_\infty)$. Similarly, $P(\mathcal{T}_\infty)P(\mathcal{R}_\infty) = P(\mathcal{R}_\infty \cap \mathcal{T}_\infty)$, and (iii) follows. That (iii) implies (i) is [**3**, Theorem 1.1]. $\square$

This result can be made more descriptive by exhibiting the structure of $\mathcal{R}_\infty$ and $\mathcal{T}_\infty$. (The symbols defined in Section 2 persist.)

Proposition 4.2. *There exist subsets K of A, Λ of B, and Ω of $[\Gamma^c \cup (\mathbf{T}\times K^c)] \cap [\Delta^c \cup (\Lambda^c \times \mathbf{T})]$ such that*

$$\mathcal{R}_\infty = \mathrm{sp}\,\{z_1^m z_2^n h_T : m \in \mathbf{Z},\ n \geq 0\} \oplus 1_\Omega L^2(\mu)$$
$$\mathcal{T}_\infty = \mathrm{sp}\,\{z_1^m z_2^n h_R : m \geq 0,\ n \in \mathbf{Z}\} \oplus 1_\Omega L^2(\mu),$$

where $|h_T| = w_T^{-1/2} 1_{\Delta\cap(\Lambda\times\mathbf{T})}$ and $|h_R| = w_R^{-1/2} 1_{\Gamma\cap(\mathbf{T}\times K)}$. Moreover, $\mathcal{R}_\infty \cap \mathcal{T}_\infty = 1_\Omega L^2(\mu)$.

Proof. The results of Section 3 show that

$$\mathcal{R}_\infty = \mathrm{sp}\,\{z_1^m z_2^n h_T : m \in \mathbf{Z},\ n \geq 0\} \oplus 1_{\Omega_T} L^2(\mu)$$
$$\mathcal{T}_\infty = \mathrm{sp}\,\{z_1^m z_2^n h_R : m \geq 0,\ n \in \mathbf{Z}\} \oplus 1_{\Omega_R} L^2(\mu)$$

with h_R and h_T as claimed, $\Omega_T \subset \Delta^c \cup (\Lambda^c \times \mathbf{T})$, and $\Omega_R \subset \Gamma^c \cup (\mathbf{T}\times K^c)$. We check that $\mathrm{sp}\,\{z_1^m z_2^n h_T : m \in \mathbf{Z},\ n \geq 0\} \perp 1_{\Omega_R} L^2(\mu)$, and $\mathrm{sp}\,\{z_1^m z_2^n h_R : m \in \mathbf{Z}, n \geq 0\} \perp 1_{\Omega_T} L^2(\mu)$. For if $g \in 1_{\Omega_T} L^2(\mu)$ and $m \geq 0$, then

$$\int z_1^{-m-1} z_2^n h_R \bar{g}\, d\mu = \int h_R \overline{(z_1^{-m-1} z_2^{-n} g)}\, d\mu$$

which vanishes since $z_1^{-m-1} z_2^{-n} g \in \mathcal{M}$ and $h_R \perp z_1 \mathcal{M}$. Next, note that R_∞ includes $\cap_{m=0}^\infty V_1^m \mathcal{T}_\infty = 1_{\Omega_R} L^2(\mu)$, and likewise $\mathcal{T}_\infty$ includes $1_{\Omega_T} L^2(\mu)$. This forces $\Omega_R = \Omega_T$. Dropping the subscripts, we see that $\mathcal{R}_\infty \cap \mathcal{T}_\infty = 1_\Omega L^2(\mu)$. This verifies the claims. □

Lemma 4.3. *The orthosum for $\mathcal{M}'$ spans all of $\mathcal{M}$ if and only if $\sigma(K) = 0$ or $\sigma(\Lambda) = 0$, where K and Λ are as in Proposition 4.2.*

Proof. This follows from the equivalence of (i) and (ii) in Proposition 4.1. For (ii) holds if and only if $\int (z_1^m z_2^n h_R)\overline{(z_1^s z_2^t h_T)}\, d\mu = 0$ for all $m \geq 0$, $n \in \mathbf{Z}$, $s \in \mathbf{Z}$, and $t \geq 0$. This, in turn, is equivalent to the condition $1_{\Gamma\cap(\mathbf{T}\times K)} \cdot 1_{\Delta\cap(\Lambda\times\mathbf{T})} = 0$, almost everywhere $[\mu]$; or $1_{\Lambda\times K} = 0$, almost everywhere $[\sigma^2]$. □

With that, the following representations hold.

Theorem 4.4. *The invariant subspace $\mathcal{M}$ has a four-fold Halmos decomposition with $\mathcal{M}_0 = (0)$ if and only if there exist subsets K of A, Λ of B and Ω of $[\Gamma^c \cup (\mathbf{T} \times K^c)] \cap [\Delta^c \cup (\Lambda^c \times \mathbf{T})]$, such that $\sigma(K) \cdot \sigma(\Lambda) = 0$, and*

$$\begin{aligned}\mathcal{M} &= \mathrm{sp}\,\{h_R z_1^m z_2^n : m \geq 0,\ n \in \mathbf{Z}\} \\ &\quad \oplus \mathrm{sp}\,\{h_T z_1^m z_2^n : m \in \mathbf{Z},\ n \geq 0\} \\ &\quad \oplus 1_\Omega L^2(\mu),\end{aligned}$$

where $|h_R| = w_R^{-1/2} 1_{\Gamma \cap (\mathbf{T} \times K)}$, and $|h_T| = w_T^{-1/2} 1_{\Delta \cap (\Lambda \times \mathbf{T})}$.

Proof. It remains only to check that the component spaces are invariant, and this is obvious. □

Theorem 4.5. *The invariant subspace $\mathcal{M}$ has a four-fold Halmos decomposition with $\mathcal{M}_0 \neq (0)$ if and only if the following conditions hold*:

(i) *the weight function w is positive, almost everywhere $[\sigma^2]$*;

(ii) *$\mathcal{M}_0$ is a nontrivial proper subspace of $L^2(w\, d\sigma^2)$ on which V_1 and V_2 are pure shifts*;

(iii) *There exist subsets K of A, Λ of B, and Ω of $[\Gamma^c \cup (\mathbf{T} \times K^c)] \cap [\Delta^c \cup (\Lambda^c \times \mathbf{T})]$, such that $\sigma(K) = \sigma(A) = \sigma^2(\Omega) = 0$, and*

$$\begin{aligned}\mathcal{M} &= \mathcal{M}_0 \oplus \mathrm{sp}\,\{z_1^m z_2^n h_R : m \geq 0,\ n \in \mathbf{Z}\} \\ &\quad \oplus \mathrm{sp}\,\{z_1^m z_2^n h_T : m \in \mathbf{Z},\ n \geq 0\} \\ &\quad \oplus 1_\Omega L^2(\mu),\end{aligned}$$

where $|h_R| = w_R^{-1/2} 1_{\Gamma \cap (\mathbf{T} \times K)}$, and $|h_T| = w_T^{-1/2} 1_{\Delta \cap (\Lambda \times \mathbf{T})}$.

Proof. Suppose that $\mathcal{M}$ has a four-fold Halmos decomposition, and $\mathcal{M}_0 \neq (0)$. Since $\cap_{m=0}^{\infty} V_1^m \mathcal{M}_0 = (0) = \cap_{n=0}^{\infty} V_2^n \mathcal{M}_0$, it must be that $\mathcal{M}_0 \subseteq 1_E L^2(\mu) = L^2(w\, d\sigma^2)$. Choose a nonzero f in $\mathcal{M}_0 \ominus V_1 \mathcal{M}_0$. Observe that $\int |f|^2 z_1^m z_2^n w\, d\sigma^2 = 0$ whenever $m \geq 1$ and $n \geq 0$, and whenever $m \leq 1$ and $n \leq 0$. Hence there is an F in the Hardy space $H^1(\mathbf{T}^2)$ such that $|f(z_1,z_2)|^2 w(z_1,z_2) = \mathrm{Re}\ F(z_1,\bar{z}_2)$, almost everywhere $[\sigma^2]$. Since this is nonnegative, and not identically zero, it cannot vanish on a set of σ^2-positive measure. This verifies conditions (i) and (ii). The pairwise orthogonality of the component spaces forces $\sigma(K)=\sigma(\Lambda)=\sigma^2(\Omega)=0$. The rest of (iii) follows as in Theorem 4.4.

The sufficiency part can be checked by inspection. □

The condition (ii) in Theorem 4.5 is not very satisfying. However, it seems that such $\mathcal{M}_0$ do not admit a simple characterization. Hence, in the next section, we consider invariant subspaces with an additional property.

5. Invariant subspaces of Beurling type. Extending the terminology of [**1**] and [**4**], we say that an invariant subspace $\mathcal{M}$ of $L^2(\mu)$ is of *Beurling type* if $V_1 \mid \mathcal{M}$ commutes with $(V_2 \mid \mathcal{M})^*$, i.e., if V_1 and V_2 are doubly commuting on $\mathcal{M}$. We wish to describe all such invariant subspaces. This was done for $\mu=\sigma^2$ in [**1**], and for $\mathcal{M} \subseteq H^2(\mathbf{T}^2)$ in [**4, 5**].

Let an invariant subspace $\mathcal{M}$ of $L^2(\mu)$ be given, and let $\mathcal{H}$ be the subspace $(\mathcal{M} \ominus V_1,\mathcal{M}) \cap (\mathcal{M} \ominus V_2\mathcal{M})$.

Theorem 5.1. *The invariant subspace $\mathcal{M}$ is of Beurling type and $\mathcal{H}$ is trivial, if and only if there exist Borel subsets K of A, Λ of B, and Ω of $[\Gamma^c \cup (\mathbf{T}\times K^c)] \cap [\Delta^c \cup (\Lambda^c \times \mathbf{T})]$, such that $\sigma(K)\sigma(\Lambda)=0$, and*

$$\begin{aligned}\mathcal{M} &= \mathrm{sp}\,\{h_R z_1^m z_2^n : m \ge 0,\ n \in \mathbf{Z}\}\\ &\oplus \mathrm{sp}\,\{h_T z_1^m z_2^n : m \in \mathbf{Z},\ n \ge 0\}\\ &\oplus 1_\Omega \mathbf{L}^2(\mu),\end{aligned}$$

where $|h_R| = w_R^{-1/2} 1_{\Gamma\cap(\mathbf{T}\times K)}$, and $|h_T| = w_T^{-1/2} 1_{\Delta\cap(\Lambda\times\mathbf{T})}$.

Proof. Sufficiency is established by noting that at least one of V_1 and V_2 is unitary on each component space. Hence, V_1 and V_2 doubly commute on $\mathcal{M}$. Necessity is immediate from (b) ⇒ (c) in [**3**, Theorem 4.2], followed by Theorem 4.4. □

Theorem 5.2. *The invariant subspace $\mathcal{M}$ is of Beurling type, and*

$\mathcal{H}$ is nontrivial, if and only if the following conditions hold:

(i) *the weight function w is positive, almost everywhere* $[\sigma^2]$;

(ii) *there exist subsets K of A, Λ of B and Ω of* $[\Gamma^c \cup (\mathbf{T} \times K^c)] \cap [\Delta^c \cup (\Lambda^c \times \mathbf{T})]$, *such that $\sigma(K) = \sigma(\Lambda) = \sigma^2(\Omega) = 0$, and such that*

$$\begin{aligned} \mathcal{M} &= \mathrm{sp}\,\{z_1^m z_2^n h : m,\ n \geq 0\} \\ &\quad \oplus \mathrm{sp}\,\{z_1^m z_2^n h_R : m \geq 0,\ n \in \mathbf{Z}\} \\ &\quad \oplus \mathrm{sp}\,\{z_1^m z_2^n h_T : m \in \mathbf{Z},\ n \geq 0\} \\ &\quad \oplus 1_\Omega L^2(\mu), \end{aligned}$$

where $|h| = w^{-1/2} 1_E$, $|h_R| = w_R^{-1/2} 1_{\Gamma \cap (\mathbf{T} \times K)}$, *and* $|h_T| = w_T^{-1/2} 1_{\Delta \cap (\Lambda \times \mathbf{T})}$.

Proof. For sufficiency, use unitarity of V_1 or V_2 on the last three component spaces; cite (ii) $\Rightarrow$ (iv) of [**6**, Theorem 1] for the first.

To prove necessity, apply (b) $\Rightarrow$ (c) in [**3**, Theorem 4.2], then Theorem 4.4. This verifies the assertions concerning the last three component spaces, and gives

$$\mathcal{M}_0 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \oplus V_1^m V_2^n \mathcal{H}.$$

As in the proof of [**4**, Theorem 2], we have that $\mathcal{H}$ is spanned by a single unit vector h, and $\int |h|^2 z_1^m z_2^n w \, d\sigma = 0$ unless $m = n = 0$. This shows that $|h| = w^{-1/2} 1_E$. $\square$

Remark 5.3. Take $\mu = \sigma^2$.

(a) Let $\mathcal{M} = \mathrm{sp}\,\{z_1^m z_2^n : m \geq 0 \text{ or } n \geq 0\}$. Then $\mathcal{M}' = \mathcal{M}$. However, $\mathcal{R}_0 \cap \mathcal{T}_\infty$ and $\mathcal{R}_\infty \cap \mathcal{T}_0$ fail to be invariant (see [**3**, p. 187]).

(b) Let $\mathcal{M} = \mathrm{sp}\,\{z_1^m z_2^n : m + n \geq 0\}$. Then $\mathcal{R}_\infty = (0)$ and $\mathcal{T}_\infty = (0)$. Yet, $\mathcal{H} = \mathrm{sp}\,\{z_1^m z_2^{-m} : m \in \mathbf{Z}\}$, which is infinite dimensional.

So complications arise even in the Lebesgue case.

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