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THOMAS'S STRUCTURE BUNDLE FOR CONFORMAL, PROJECTIVE AND RELATED STRUCTURES

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1. Introduction and conventions. Local twistors for four-dimensional conformal spin manifolds were described by Penrose and others [**8, 19, 21**] as a means of applying twistor theory to curved space-times. Numerous authors later realized that the local twistor bundle is an associated vector bundle (via a spin representation) of the Cartan conformal connection [**18**]. We had been using a generalization of this calculus for some time in connection with various problems in conformal and projective geometry when C. Fefferman and C.R. Graham pointed out to us that these methods go back to T.Y. Thomas ([**27**] for the conformal case and [**26**] for the projective), who discovered the "Cartan connections" a little later, but independently of E. Cartan. Thomas's treatment is somewhat difficult for the modern reader since it predates the idea of a vector bundle. His work however is essentially complete, and thus considerably ahead of its time.

The difference between the Cartan approach and the Thomas approach is that the former works with a principal bundle, whereas the latter takes a certain associated vector bundle as the starting point. Of course, principal bundle methods are very powerful, but on the other hand we, at least, do not at an intuitive level think of (for example) a vector field as a function on a principal bundle. We regard Thomas's calculus as the "intuitive" version of the Cartan connections. It has the added advantage that there is a definition of Thomas's vector bundle which is quite direct, in contrast to the construction of the principal bundle for the Cartan connection.

One can perform Thomas's construction for any structure that has a "Cartan connection." Here we treat conformal and projective structures, and mention paraconformal structures [**1**] briefly. We wish to

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present this calculus (with some extensions of our own) which, while it is well known to some, is insufficiently documented. We have recently used it quite extensively in our investigations of invariant differential operators and invariants of the structure for both projective and conformal structures. This is quite an active area at present, and progress is occurring on several fronts. It is our intention to present some results in the near future, and this paper lays the groundwork for this. Calculations with this calculus have also been in the background of other recent work (e.g., [**1, 2**]), and we felt this could best be explained in a self-contained paper on the subject.

There is something of a problem of nomenclature in this area. In the beginning, the authors referred to sections of Thomas's bundles as "local twistors." Penrose's local twistors however are associated to a spin representation of the Cartan connection, and various persons remarked that it is inappropriate to use the term "twistor" for something which, like Thomas's bundle, has no connection with spin. The name "tractor" was eventually suggested, and we have used this firstly because of the etymological connections with "vector" and secondly to honor Tracey Thomas's pioneering work. We have retained the term "local twistor" however for the case of paraconformal manifolds, where one is genuinely dealing with a generalization of Penrose's construction.

We have presented our results for smooth real manifolds. With minor modifications, they carry over into the holomorphic category. Similarly, we consider positive definite Riemannian conformal structures but the transition to the pseudo-Riemannian case is elementary.

1.1. Conventions. We use the *abstract index* notation [**20**] throughout whereby we attach indices to quantities solely as labels to indicate the bundle of which they are a section. Thus v^i is a section of the tangent bundle which we denote $\mathcal{E}^i$ and ϕ_{jk} a section of $\mathcal{E}_{jk}$ (the tensor product of the cotangent bundle with itself), and so on. We use round brackets to denote the totally symmetric part and square brackets to denote the totally anti-symmetric part of both bundles and individual tensors. Thus a 2-form is a section ϕ_{ij} of $\mathcal{E}_{[ij]}$ and satisfies $\phi_{ij} = \phi_{[ij]}$. The natural contraction between the tangent and cotangent bundles is written $v^i\omega_i$, this being the abstract index version of the Einstein summation convention.

Another convention we adopt is the writing of composition series with "+" signs as suggested by Buchdahl. A short exact sequence

$$0 \to C \to A \to B \to 0$$

is equivalent, in this notation, to writing

$$A = B + C.$$

Whilst "+" is not "commutative," it is "associative" and so the expression

$$A = B + C + D$$

is unambiguous. Taking duals of the above we have

$$A^* = D^* + C^* + B^*$$

and under symmetric tensor product we have, for example

$$\bigodot^2 A = \bigodot^2 B + B \otimes C + \begin{matrix} \bigodot^2 C \\ \oplus \\ B \otimes D \end{matrix} + C \otimes D + \bigodot^2 D.$$

A similar formula holds with "$\bigwedge$" replacing "$\bigodot$."

2. Conformal structures.

2.1. Conventions for conformal structures. A conformal Riemannian metric on an n-manifold $\mathcal{M}$ (we assume $n \geq 3$ throughout) can be described by a global tensor field g_{ij} with values in a line bundle which we denote by $\mathcal{E}[2]$. Choose a square root $\mathcal{E}[1]$ for this line bundle. We can thus consider functions and tensors with values in $\mathcal{E}[w]$ for any real w. We say that such objects have *conformal weight* w and write $\mathcal{E}^i[w]$ for $\mathcal{E}^i \otimes \mathcal{E}[w]$, and so on. The tensor g_{ij} gives a canonical isomorphism of $\mathcal{E}^i[w]$ with $\mathcal{E}_i[w+2]$ which is expressed by writing $V_i = g_{ij}V^j$, and so on. We use this isomorphism extensively by raising and lowering indices without comment.

A *conformal scale* is a nowhere vanishing local section τ of $\mathcal{E}[1]$. A conformal scale defines a metric $\tau^{-2}g_{ij}$ in the conformal class, and conversely a metric in the conformal class determines (up to sign) a unique conformal scale. We work with the Levi-Civita connection of this chosen metric. After rescaling our chosen metric by multiplying by a nowhere vanishing smooth function Ω^2, the new connection $\widehat{\nabla}$ is related to the old one ∇ according to (see, e.g., [**20**])

$$\begin{aligned} \widehat{\nabla}_i f &= \nabla_i f + \omega \Upsilon_i f \\ \widehat{\nabla}_i U^j &= \nabla_i U^j + (w+1)\Upsilon_i U^j - U_i \Upsilon^j + U^k \Upsilon_k \delta_i{}^j \\ \widehat{\nabla}_i \omega_j &= \nabla_i \omega_j + (\omega - 1)\Upsilon_i \omega_j - \Upsilon_j \omega_i + \Upsilon^k \omega_k g_{ij} \end{aligned} \tag{1}$$

where δ is the Kronecker delta, $\Upsilon_i = \Omega^{-1}\nabla_i \Omega$, and the quantities f, U^i, ω_i are sections of $\mathcal{E}[w]$, $\mathcal{E}^i[w]$, $\mathcal{E}_i[w]$, respectively. We always use "hats" in this way to denote corresponding quantities after rescaling. (Our convention here differs slightly from [**20, 21**] in the treatment of conformal weights. There, conformally weighted functions are thought of in terms of components so that they would write $\hat{f} = \Omega^w f$, whereas we regard them as sections of a line bundle and write $\hat{f} = f$. The effect is that the factors of Ω^w do not generally appear in formulae.)

The Riemann curvature is defined by

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) U^k = R_{ij}{}^k{}_l U^l$$

and the totally trace-free part of R_{ijkl} is the conformally invariant *Weyl curvature* C_{ijkl}. The Riemann tensor can then be expressed as

$$R_{ijkl} = C_{ijkl} + 2g_{k[i}P_{j]l} + 2g_{l[j}P_{i]k}$$

where the *rho-tensor* P_{ij} is a trace-adjusted multiple of the Ricci tensor $R_{jl} = R_{ij}{}^i{}_l$:

$$R_{ij} = (n-2)P_{ij} + P g_{ij} \quad \text{where} \quad P = P_i{}^i.$$

The rho-tensor appears often in conformal geometry and it has a relatively simple conformal transformation law:

$$\widehat{P}_{ij} = P_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j - \tfrac{1}{2}\Upsilon_k \Upsilon^k g_{ij}. \tag{2}$$

Two consequences of the Bianchi identity $\nabla_{[i} R_{jk]}{}^l{}_m = 0$ which we will need are

$$\nabla_k C_{ij}{}^k{}_l = 2(n-3)\nabla_{[i} P_{j]l}, \qquad \nabla^j P_{ij} = \nabla_i P. \tag{3}$$

(The second of these can be obtained by contracting the former if $n > 3$.)

2.2. The flat model. Let $\mathbf{T}$ denote $\mathbf{R}^{n+2}$ equipped with a bilinear form h of signature $(n+1, 1)$, with matrix given in block form by

$$\begin{pmatrix} \overbrace{0}^{1} & \overbrace{0}^{n} & \overbrace{1}^{1} \\ 0 & \mathrm{Id}_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

It is well known that the space of generators of the null cone of h is the n-sphere with its usual flat conformal structure. Let G denote the identity connected component of $\mathrm{O}(h)$. Then G acts on the n-sphere as the group of all orientation preserving conformal automorphisms. If we fix a point

$$e = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

on the null cone, then the stabilizer of the corresponding point on the sphere is the parabolic subgroup $P \subset G$ consisting of all elements of the block form

$$\begin{pmatrix} \lambda^{-1} & 0 & 0 \\ r & m & 0 \\ -\lambda r^t r/2 & -\lambda r^t m & \lambda \end{pmatrix}$$

where $r \in \mathbf{R}^n$ and r^t denotes the transpose of r.

The sphere S^n is identified with the quotient G/P, and $G \to S^n$ is a principal P-bundle. The Maurer-Cartan forms on G give a connection on this bundle. (Since the forms take values in the Lie algebra of G rather than P, this is *not* a P-connection.) This construction generalizes to an arbitrary conformal n-manifold. One constructs a principal P-bundle as a sub-bundle of the bundle of 2-frames and there is a naturally defined form on this bundle taking values in the Lie algebra of G which defines the "Cartan conformal connection."

Thomas's approach is instead to construct an analogue of the product bundle $S^n \times \mathbf{T} \to S^n$. It turns out that over a conformal n-manifold one can define a rank $n+2$ vector bundle carrying a signature $(n+1,1)$ inner product and a connection which preserves the inner product. Of course, this vector bundle is an associated bundle of Cartan's principal bundle, but it is easier to define it directly.

2.3. Thomas's bundle and connection. Recall that the conformal scale τ defines the metric $\tau^{-2}g_{ij}$ with respect to which the Levi-Civita connection ∇ is defined. Suppose we try to choose a new scale σ for which the metric $\sigma^{-2}g_{ij}$ is Einstein. Write $\sigma = \Omega^{-1}\tau$ so that the metric is rescaled by Ω^2 and we may use equation (2) to conclude that $\sigma^{-2}g_{ij}$ is Einstein if and only if

$$P_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j$$

is pure trace. In the scale defined by τ, the section σ is represented by the function Ω^{-1} whence

$$\nabla_i \nabla_j \sigma = \sigma(\Upsilon_i \Upsilon_j - \nabla_i \Upsilon_j).$$

We see that $\sigma^{-2}g_{ij}$ is Einstein if and only if σ is a solution of the conformally invariant equation

$$\text{Trace-free part of } (\nabla_i \nabla_j + P_{ij})\sigma = 0. \tag{4}$$

Although the equation (4) generally has no solutions even locally, it does define a conformally invariant subspace of the space of 2-jets of sections of $\mathcal{E}[1]$ at each point.

Definition 2.1. The *tractor bundle* $\mathcal{E}^I$ of a conformal manifold is the sub-bundle of the 2-jet bundle $\mathcal{J}^2\mathcal{E}[1]$ defined by the equation (4).

We also use the term tractor for tensor powers of this bundle, sections thereof, etc., by analogy with the use of the term "tensor." To work with tractors we need a more concrete description. Observe first that equation (4) is equivalent to the following pair of equations holding for some μ^i, ρ, sections of $\mathcal{E}^i[-1]$, $\mathcal{E}[-1]$ respectively,

$$\begin{aligned} \nabla_j \sigma - \mu_j &= 0 \\ \nabla_j \mu^i + \delta_j{}^i \rho + P_j{}^i \sigma &= 0. \end{aligned} \tag{5}$$

It is easy to check that under conformal rescaling the quantities above transform according to

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}^i \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu^i + \Upsilon^i \sigma \\ \rho - \Upsilon_j \mu^j - \Upsilon^2 \sigma/2 \end{pmatrix}. \tag{6}$$

This yields the following description of $\mathcal{E}^I$ (on which one could alternatively base the definition):

Proposition 2.2. *For any choice of conformal scale, $\mathcal{E}^I$ is identified with the direct sum*

$$\mathcal{E}^I = \mathcal{E}[1] \oplus \mathcal{E}^i[-1] \oplus \mathcal{E}[-1]$$

and under conformal rescaling σ, μ^i, ρ are identified with their counterparts $\hat{\sigma}$, $\hat{\mu}^i$, $\hat{\rho}$ in the new scale according to (6).

If one differentiates the second of equations (5) once more and contracts a pair of indices, one obtains after a little manipulation that (5) implies

$$\nabla_j \rho - P_{jk} \mu^k = 0. \tag{7}$$

Taken together with (5), this defines a conformally invariant connection on $\mathcal{E}^I$.

Definition 2.3. In a given conformal scale, the *tractor connection* ∇ on $\mathcal{E}^I$ is defined by

$$\nabla_j \begin{pmatrix} \sigma \\ \mu^i \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_j \sigma - \mu_j \\ \nabla_j \mu^i + \delta_j{}^i \rho + P_j{}^i \sigma \\ \nabla_j \rho - P_{ji} \mu^i \end{pmatrix} \tag{8}$$

The conformal invariance follows from the derivation, or it can be checked directly using (6), (1) and (2).

We use ∇ to denote the Levi-Civita connection on conformally weighted tensor fields and also the connection on $\mathcal{E}^I$. We will also use it to denote the induced connection on any tensor product of these bundles. It is of course only conformally invariant on unweighted tensor powers of $\mathcal{E}^J$.

Definition 2.4. The bundle $\mathcal{E}^I$ carries a natural, nondegenerate symmetric form g_{IJ}, the *tractor metric*, defined by

$$g_{IJ}U^I V^J = \mu^i \beta_i + \sigma\gamma + \rho\alpha$$

where

$$U^I = \begin{pmatrix} \sigma \\ \mu^i \\ \rho \end{pmatrix}, \qquad V^J = \begin{pmatrix} \alpha \\ \beta^j \\ \gamma \end{pmatrix}. \tag{9}$$

This metric has signature $(n+1,1)$. (If the underlying manifold is pseudo-Riemannian of signature (p,q), then g_{IJ} will be of signature $(p+1,q+1)$.)

It is easy to check the conformal invariance of the definition directly. The metric g_{IJ} provides us with an isomorphism of $\mathcal{E}^I$ with its dual $\mathcal{E}_I$ which we will use implicitly by raising and lowering indices. If U^I is as given by (9) above, then we will write

$$U_I = \begin{pmatrix} \rho & \mu_i & \sigma \end{pmatrix}$$

so that $g_{IJ}U^I V^J = U_J V^J$ is given by matrix multiplication in the usual way. The following proposition is proved by straightforward calculations.

Proposition 2.5. *The tractor metric is preserved by the tractor connection—i.e.,* $\nabla_k g_{IJ} = 0$. *Thus, raising and lowering indices with* g_{IJ} *commutes with the action of* ∇.

It is clear from the form of the inner product in Definition 2.4, together with the conformal rescaling formulae (6) that the structure group of the vector bundle $\mathcal{E}^I$ can be reduced to the parabolic subgroup P discussed in Section 2.2. Note, however, that although ∇ preserves

g_{IJ}, it does not preserve the P-structure (i.e., it does not preserve the composition series (10) below). This reflects the fact that the Cartan conformal connection is not a P-connection.

2.4. Primary and secondary parts. Given U^I as in (9), σ is often called the *primary part* of U^I. Similarly, μ^i is called the *secondary part*, and so on. It is clear from the transformation law (6) that the primary part is conformally invariant, and if this is zero, then the secondary part is invariant, and so on. In the notation for composition series developed in the introduction, this is equivalent to the observation that

$$\mathcal{E}^I = \mathcal{E}[1] + \mathcal{E}^i[-1] + \mathcal{E}[-1]. \tag{10}$$

We refer to the first nonzero part of a given tractor as the *projecting part.*

A similar analysis applies to tractors with several indices. For example,

$$\mathcal{E}^{[KL]} = \mathcal{E}^k + \begin{array}{c}\mathcal{E}^{[kl]}[-2]\\ \oplus \\ \mathcal{E}\end{array} + \mathcal{E}^k[-2].$$

Thus, given a skew tractor S^{IJ}, the first non-zero entry above, counting from the left, is the projecting part. This composition series still holds if we tensor through by, for example, $\mathcal{E}_{[ij]}$ to obtain

$$\mathcal{E}_{[ij]}{}^{[KL]} = \mathcal{E}_{[ij]}{}^k + \begin{array}{c}\mathcal{E}_{[ij]}{}^{[kl]}[-2]\\ \oplus \\ \mathcal{E}_{[ij]}\end{array} + \mathcal{E}_{[ij]}{}^k[-2]. \tag{11}$$

Thus for $\Omega_{ij}{}^{KL}$ which is antisymmetric in both lower and upper case indices, the projecting part is a tensor in $\mathcal{E}_{[ij]}{}^k$, unless this is zero in which case the projecting part is in $\mathcal{E}_{[ij]}{}^{[kl]}[-2] \oplus \mathcal{E}_{[ij]}$, and so on. We will see an example of this below.

The map $\mathcal{E}^I \to \mathcal{E}[1]$ which takes U^I in (9) to σ is given by contraction with a preferred section of $\mathcal{E}^I[1]$ that we shall denote by X^I henceforth. Thus, we have $\sigma = U^I X_I$. In any choice of conformal scale, $X_I = (\,1 \quad 0 \quad 0\,)$. The object X^I also describes the invariant injection $\mathcal{E}[-1] \to \mathcal{E}^I$ according to $\rho \mapsto \rho X^I$. It is easy to check that $X_I X^I = 0$.

2.5. Tractor curvature. The deviation of a conformal structure from the flat model is measured by the *tractor curvature* Ω of ∇ on $\mathcal{E}^I$. This is defined by

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)U^K = \Omega_{ij}{}^K{}_L U^L.$$

On lower tractor indices, we have $(\nabla_i\nabla_j - \nabla_j\nabla_i)V_L = -\Omega_{ij}{}^K{}_L V_K$. Applying this to the tractor metric, since $\nabla_i g_{KL} = 0$, we have $0 = \nabla_{[i}\nabla_{j]}g_{KL} = -\Omega_{ij(KL)}$. Thus, $\Omega_{ij}{}^{KL} = \Omega_{[ij]}{}^{[KL]}$ and so the composition series (11) applies. After some calculation, one gets

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\begin{pmatrix}\sigma\\ \mu^k\\ \rho\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 2\nabla_{[i}P_{j]}{}^k & C_{ij}{}^k{}_l & 0\\ 0 & -2\nabla_{[i}P_{j]l} & 0\end{pmatrix}\begin{pmatrix}\sigma\\ \mu^l\\ \rho\end{pmatrix}$$

and thus the square matrix represents the curvature in the given scale. With $\Omega_{ij}{}^K{}_L$ represented by this matrix, the anti-symmetry dictates that the top right entry is zero. The primary part of such an anti-symmetric tractor appears on the "NW–SE" diagonal immediately below this and is zero for the curvature. The secondary part appears on the diagonal below—the projecting part is thus the conformally invariant Weyl curvature. If $n = 3$, then the Weyl curvature tensor necessarily vanishes. Thus, for $n = 3$ the tertiary part $\nabla_{[i}P_{j]k}$ is a conformal invariant. It is usually known as the Cotton-York tensor. Given the first of equations (3), we have:

Proposition 2.6. *The connection* ∇ *on* $\mathcal{E}^I$ *is flat if and only if* $C_{ijkl} = 0$ *for* $n \geq 4$ *or* $\nabla_{[i}P_{j]k} = 0$ *when* $n = 3$.

The conditions in the proposition are exactly the necessary and sufficient conditions for a conformal manifold to be locally equivalent to the flat model. In fact, these methods provide a proof of this fact, as was observed by Penrose [**21**]. We will only outline the idea here.

If the tractor connection is flat, then $\mathcal{E}^I$ can be identified with a product bundle with fiber $\mathbf{R}^{n+2}$ and a signature $(1, n+1)$ inner product. Fix a point p, and let $\mathbf{T}$ denote the fiber of $\mathcal{E}^I$ at p. The fiber of $\mathcal{E}^I$ above any nearby point can be identified with $\mathbf{T}$ by parallel transport. One can then use X^I to map a neighborhood of p to the space of generators of the null-cone in $\mathbf{T}$ (recall that $X^I X_I = 0$). One can check that this provides a conformal equivalence with the flat model.

2.6. The D operator. The D operator, originally described by Thomas, is a conformally invariant means of differentiating conformally weighted tractors.

Definition 2.7. The *operator* $D_I : \mathcal{E}[w] \to \mathcal{E}_I[w-1]$ is defined by

$$D^I f = \begin{pmatrix} w(n+2w-2)f \\ (n+2w-2)\nabla^i f \\ -(\Delta + wP)f \end{pmatrix},$$

where Δ denotes the Laplacian $\nabla_i \nabla^i$.

One can check that this definition is conformally invariant. Furthermore, it remains invariant if f is actually a conformally weighted section of any tensor power of $\mathcal{E}^J$. (Recall that in this case Δ will be the tractor Laplacian formed from the tractor connection.) If $2w = 2-n$, then $\Delta + wP$ is the usual conformally invariant Laplacian (or "Yamabe operator"). For other values of w, it is in some sense a natural modification of the Laplacian since it has a simplified conformal transformation law. In terms of the flat model, D_I is closely related to differentiation with respect to the coordinates on $\mathbf{R}^{n+2}$.

2.7. Hypersurfaces. Let $\mathcal{S} \subset \mathcal{M}$ be a smoothly embedded hypersurface. (In the pseudo-Riemannian or holomorphic categories, we assume also that the normal to the hypersurface nowhere has zero length.) On conformal manifolds, the *unit normal* N^i to a hypersurface is a section over $\mathcal{S}$ of $\mathcal{E}^i[-1]$. Extend N^i to a neighborhood of $\mathcal{S}$ so that it satisfies

$$N^i N_i = 1, \qquad \nabla_{[i} N_{j]} = 0.$$

Our results will be independent of the choice of extension.

Definition 2.8. The *normal tractor* is a tractor of zero conformal weight defined on $\mathcal{S}$ by

$$N^I = \begin{pmatrix} 0 \\ N^i \\ -H \end{pmatrix} \tag{12}$$

where $H = (n-1)^{-1}\nabla_i N^i$ is the mean curvature of $\mathcal{S}$.

It is straightforward to check that this definition is conformally invariant. The normal tractor satisfies $N^I N_I = 1$ and, in the case where $\mathcal{M}$ is the flat model, N_I provides a map from $\mathcal{S}$ to the unit hyperboloid in $\mathbf{R}^{n+2}$; it is, in fact, the *conformal Gauss map* described by Bryant [**7**].

Proposition 2.9. *The normal tractor N^I of a hypersurface $\mathcal{S}$ is constant on the hypersurface if and only if $\mathcal{S}$ is umbilic.*

Proof. Differentiating equation (12), we obtain

$$\nabla_j N^I = \begin{pmatrix} -N_j \\ \nabla_j N^i - H\delta_j{}^i \\ -\nabla_j H - P_{ij}N^i \end{pmatrix}.$$

The primary part vanishes when the "j" index is projected into the surface and the secondary part becomes the trace-free part of the second fundamental form $\mathbf{II}_{ij}$. (Note that this proves the well-known fact that the trace-free part of $\mathbf{II}_{ij}$ is conformally invariant.) Thus, a necessary condition for N^I to be constant on $\mathcal{S}$ is that $\mathbf{II}_{ij} = H\mathbf{I}_{ij}$. In other words, $\mathcal{S}$ is umbilic. In fact, this is a sufficient condition since if $\mathcal{S}$ is umbilic the tertiary part of $\nabla_j N^I$ also vanishes when projected into the surface, as one can show from Codacci's equation. □

2.8. Tractors and curves—conformal circles. Throughout this section, let γ denote a smoothly embedded one-dimensional submanifold of $\mathcal{M}$ smoothly parametrized by a variable t with nowhere vanishing derivative. If we are working in the pseudo-Riemannian or holomorphic categories, then we require in addition that the tangent to γ is nowhere of zero length.

We choose tangent vectors U^i along γ by requiring that $U^i\nabla_i t = 1$, and we also define $u = \sqrt{U_i U^i}$. We note that u has conformal weight 1. For any field Z on γ (including those with indices) we write dZ/dt for $U^i\nabla_i Z$. The *acceleration vector* of the curve is defined by $A^j = dU^j/dt$. Note for future use that $du/dt = u^{-1}U^i A_i$.

Definition 2.10. The *velocity tractor* U^I and *acceleration tractor* A^I of the parametrized curve γ are defined by

$$U^I = \frac{d}{dt}(u^{-1}X^I), \qquad A^I = \frac{dU^I}{dt}.$$

(Note that X^I has weight 1, and so $u^{-1}X^I$ has weight zero. Thus the definitions are conformally invariant.) A short calculation immediately yields

$$U^I = \begin{pmatrix} 0 \\ u^{-1}U^i \\ -u^{-3}(U_jA^j) \end{pmatrix}$$

and

$$A^I = \begin{pmatrix} -u \\ u^{-1}A^i - 2u^{-3}(U_jA^j)U^i \\ -u^{-3}\left(U_j\dfrac{dA^j}{dt}\right) - u^{-3}A_jA^j + 3u^{-5}(U_jA^j)^2 - u^{-1}P_{ij}U^iU^j \end{pmatrix}.$$

Of the obvious scalar invariants,

$$U^IU_I = 1, \qquad U^IA_I = 0, \tag{13}$$

and after some calculation,

$$A^IA_I = 3u^{-2}A_jA^j + 2u^{-2}U_j\frac{dA^j}{dt} - 6u^{-4}(U_jA^j)^2 + 2P_{ij}U^iU^j.$$

Proposition 2.11. *Given a curve γ (with tangent vector of nowhere zero length), the equation $A^IA_I = 0$, regarded as a condition on the parametrization, determines a preferred family of parametrizations with* *freedom the projective group of the line. We call such parameters* projective.

Proof. The existence of parameters satisfying $A^I A_I = 0$ follows because the equation is an ODE. Let t be such a parameter, with U^a being a tangent vector with $U^a \nabla_a t = 1$ as above. Now let $\tilde{t} = g(t)$ be a new parameter for the curve. Then

$$\widetilde{U}^a = g'^{-1} U^a, \qquad \tilde{u} = g'^{-1} u, \qquad \frac{d}{d\tilde{t}} = g'^{-1} \frac{d}{dt},$$

where the "tilde" denotes the corresponding quantities associated with the new parametrization. Then

$$\widetilde{U}^I = \frac{d}{d\tilde{t}} (\tilde{u}^{-1} X^I) = g'^{-1} \frac{d}{dt} (g' u^{-1} X^I) = U^I + g'^{-1} g'' u^{-1} X^I$$

and so

$$\begin{aligned} \tilde{A}^I &= g'^{-1} \frac{d}{dt} (U^I + g'^{-1} g'' u^{-1} X^I) \\ &= g'^{-1} A^I + g'^{-1} \frac{d}{dt} (g'^{-1} g'') u^{-1} X^I + g'^{-2} g'' U^I. \end{aligned}$$

Recalling (13) and noting that $U^I X_I = X^I X_I = 0$, $A^I X_I = -u$, we get

$$\tilde{A}^I \tilde{A}_I = g'^{-2} A^I A_I + 3 g'^{-4} g''' - 2 g'^{-3} g''^2.$$

The new parameter is thus also projective if and only if it is a solution of the equation formed by equating the last two terms to zero. This is essentially the Schwartzian differential equation, with solutions given by fractional linear transformations, and the result is proved.

The equation $A^I A_I = 0$ is considered (without any derivation) in [**2**] and the result is proved there by direct calculation. □

Proposition 2.12. *The curve γ is a projective parametrized conformal circle if and only if $A^I A_I = 0$ and $dA^I / dt = 0$.*

Definitions of conformal circle can be found in [**30**] and [**2**]; in the latter, being projectively parametrized is part of the definition.

Proof. When $A^I A_I = 0$, the tertiary entry in A^I becomes just $A_j A^j/(2u^3)$. A straightforward calculation of dA^I/dt shows that the primary part vanishes and that the vanishing of the secondary part is, after some manipulation, equivalent to

$$\frac{dA^i}{dt} = 3u^{-2}(U_j A^j)A^i - \tfrac{3}{2}u^{-2}(A_j A^j)U^i + u^2 U^j P^i{}_j - 2U^j U^k P_{jk} U^i$$

which is the conformal circle equation as given in [**2**]. This proves that the equation is conformally invariant since it is the vanishing of the secondary part of a tractor with zero primary part. Note that since $A^I A_I = 0$, we have by differentiating that $A_I dA^I/dt = 0$ and therefore if the secondary part of dA^I/dt is zero, then so must the tertiary part be, which completes the proof. □

As an application, we give

Proposition 2.13. *An umbilic hypersurface $\mathcal{S}$ is totally conformally circular—i.e., a conformal circle whose 2-jet at a point lies in $\mathcal{S}$, lies entirely in $\mathcal{S}$.*

Proof. Fix a point $p \in \mathcal{S}$. It suffices to show that if γ is a projectively parametrized conformal circle through p with $u^{-1}U^i N_i = 0$ and $d(u^{-1}U^i N_i)/dt = 0$ at p then $d^2(u^{-1}U^i N_i)/dt^2 = 0$ at p also as a consequence of the conformal circle equation. (The "u^{-1}" factor is included for convenience in what follows.) Since the 2-jet of γ lies in $\mathcal{S}$, we have that to second order along γ, $U^I N_I = u^{-1}U^i N_i$. We know that

$$0 = \frac{d(u^{-1}U^i N_i)}{dt} = \frac{d(U^I N_I)}{dt} = A^I N_I + U^I \frac{dN_I}{dt} = A^I N_I$$

at p. (The last equality follows because if $\mathcal{S}$ is umbilic then N^I is constant on $\mathcal{S}$ by Proposition 2.9.) We now calculate:

$$\frac{d^2(u^{-1}U^i N_i)}{dt^2} = \frac{d^2(U^I N_I)}{dt^2} = N_I \frac{d^2 U^I}{dt^2} + 2\left(\frac{dN_I}{dt}\right)\left(\frac{dU^I}{dt}\right) + U^I \frac{d^2 N_I}{dt^2} = 0$$

at p—the first term vanishing as a consequence of the conformal circle equation and the last two because N^I is constant on $\mathcal{S}$. □

2.9. The D operator and invariants. The authors were initially led to "rediscovering" Thomas's D-operator in their work on conformally invariant differential operators and invariants of conformal structures. This is an active subject at the moment and it would take us too far afield to make more than one or two passing remarks. A review of conformally invariant operators can be found in [**6**], and an extended discussion of most of what is now known in [**25**].

Clearly, a conformally invariant calculus, such as that provided by tractors and D_I, is a useful tool for writing down conformally invariant differential operators. For example,

$$(D_I f)(D^I f) = (n + 2w - 2)((n + 2w - 2)(\nabla^i f)(\nabla_i f) \\ - 2wf(\nabla + wP)f)$$

is a conformally invariant operator on functions of weight w. Invariants can arise not only as "complete contractions" such as the above (and linear combinations thereof), but also as the projecting part of a tractor—for example, on functions f of conformal weight $w = 1 - n/2$, the projecting part of $D^I f$ is the *conformally invariant Laplacian* $(\Delta + (1 - n/2)P)f$.

In the flat case, the classification of invariant *linear* operators is a straightforward exercise in representation theory. (In the flat case, "invariant" means G-equivariant between G-homogeneous bundles, where G is as in Section 2.2.) An interesting question (the complete answer to which seems to be unknown at present) is which invariant linear operators in the flat model have *curved analogues*—i.e., linear operators with the same symbol but with the addition of lower order "curvature corrections" which are then conformally invariant on a general conformal manifold. The paradigm for this is the conformally invariant Laplacian mentioned above. A less trivial example is given by iterating the Laplacian; Δ^2 is conformally invariant in the flat model on functions f of weight $2 - n/2$. One can construct a curved analogue of this operator by taking $D^I D^J f$. A somewhat tedious calculation shows that all except the "last" possible part of this tractor vanish, and so the projecting part is a conformally invariant operator which turns out to be:

$$\Delta^2 f + 4P^{ij}\nabla_i\nabla_j f - (n-2)P\Delta f - (n-6)(\nabla^j P)\nabla_j f \\ + (n-4)\left(\tfrac{n}{4}P^2 - P_{ij}P^{ij} - \tfrac{1}{2}\Delta P\right)f.$$

In dimension four, this is the operator described in [**10**]. There is a general discussion of the construction of curved analogues in dimension four using local twistors in [**9**]. A construction of curved analogues of a large range of so-called "standard" operators can be found in [**13, 14**]. The only proof we know of for the nonexistence of a curved analogue can be found in [**15**] where it is shown that Δ^3, a conformally invariant operator in the flat model on functions of weight $3-n/2$, has no curved analogue in dimension four.

The tractor calculus can also be used to generate invariants of conformal structures. As an example, we compute a new conformal invariant of oriented conformal 4-manifolds. This invariant is odd (meaning that it changes sign under orientation reversal) and is related to a certain "exceptional invariant" in parabolic invariant theory (see [**3**] for parabolic invariant theory and its relationship to conformal structures, and [**4**] for the "linearized" version of this invariant). The invariant is given by

$$I = D_I D_J(\varepsilon^{abcd}\Omega_{ab}{}^{IK}\Omega_{cd}{}^{J}{}_{K}),$$

where ε^{abcd} denotes the (conformal weight -4) volume form on the 4-manifold. A somewhat tedious calculation shows that (up to a constant), I is given by

$$\varepsilon^{abcd}\begin{pmatrix} (\Delta{+}2P)(C_{abij}C_{cd}{}^{ij}){+}6\nabla^k\nabla_i(C^i{}_{jab}C^j{}_{kcd}){-}12\nabla_i(C^i{}_{jab}\nabla_l C^{lj}{}_{cd}) \\ -12(\nabla_i C^i{}_{jab})(\nabla_l C^{lj}{}_{cd}) + 12P_i{}^k C^i{}_{jab}C^j{}_{kcd} \end{pmatrix}.$$

2.10. The Fefferman-Graham connection. One can replace ε^{abcd} by $g^{ac}g^{bd}$ in both equations above. The resulting even invariant is then up to scale and the addition of a term cubic in the undifferentiated Weyl curvature, the invariant that Fefferman and Graham obtained in [**12**, Proposition 3.4]. In fact,

$$J = D_I D_J(g^{ac}g^{bd}\Omega_{ab}{}^{IK}\Omega_{cd}{}^{J}{}_{K}),$$

is clearly an invariant in all dimensions and Graham has shown [**16**] that for each n

$$J = (n-8)[(n-6)\mathrm{FG} + \text{constant} \times C^{ab}{}_{de}C^{fg}{}_{ab}C^{de}{}_{fg}]/2,$$

where FG is the Fefferman-Graham invariant.

The relationship between J and the Fefferman-Graham invariant was established by using the close connection between the tractor calculus and the geometric machinery developed in [**11**] to study invariants. There Fefferman and Graham show how to construct (in the sense of formal power series extension off of the total space of $\mathcal{E}[-1]$) and $n+2$-dimensional pseudo-Riemannian Ricci-flat manifold $\widetilde{\mathcal{M}}$ from a given conformal manifold. (This generalizes the fact that the flat model is the space of generators of the null cone in $\mathbf{T}$.) Riemannian invariants on $\widetilde{\mathcal{M}}$ then yield conformal invariants on the original manifold. To relate their constructions to the tractor calculus it is convenient to view $\mathcal{M}$ as an equivalence class of sections of $\mathcal{E}[-1]$. There is a special mapping on $\widetilde{\mathcal{M}}$ which extends Lie dragging up the fibers of $\mathcal{E}[-1]$. Functions which are homogeneous with respect to this correspond to densities on $\mathcal{M}$ and, similarly, appropriately homogeneous tangent vectors on $\widetilde{\mathcal{M}}$ determine tractors on $\mathcal{M}$. As Graham has pointed out to us, D_I is closely related to the Levi-Civita connection ∇_I on $\widetilde{\mathcal{M}}$. For most degrees of homogeneity densities and tractors f on $\mathcal{M}$ may be extended harmonically onto $\widetilde{\mathcal{M}}$. In this case $D_I f$ and $\nabla_I f$ agree, up to scale, on $\mathcal{E}[-1]$. In some sense, therefore, the D-calculus is providing an intrinsic way of working with the Fefferman-Graham construction. This may turn out to be important, because in even dimensions the Fefferman-Graham construction is obstructed at finite order.

3. Projective structures.

Our discussion here takes advantage of the similarity between conformal and projective structures by closely paralleling our discussion of the former. We will often be a little briefer, leaving the interested reader to fill in details by analogy.

3.1. Conventions for projective structures.

Projective structures are discussed from our point of view in [**24**]. A *projective structure* on an n-manifold $\mathcal{M}$ (we assume $n \geq 2$ throughout) is an equivalence class of torsion-free affine connections which have the same geodesics (considered as unparametrized curves). In more concrete terms, the transformations allowed are precisely those of the form

$$\begin{aligned}\hat{\nabla}_i U^j &= \nabla_i U^j + \Upsilon_i U^j + U^k \Upsilon_k \delta_i{}^j \\ \hat{\nabla}_i \omega_j &= \nabla_i \omega_j - \Upsilon_i \omega_j - \Upsilon_j \omega_i\end{aligned} \tag{14}$$

where the 1-form Υ_i is arbitrary. We use "hats" to denote the transformed quantities, just as in the conformal case.

We calculate by choosing a particular connection in the projective class. The curvature of this chosen connection is defined by

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) U^k = R_{ij}{}^k{}_l U^l$$

and it can be written uniquely as

$$R_{ij}{}^k{}_l = C_{ij}{}^k{}_l + 2\delta_{[i}{}^k P_{j]l} + \beta_{ij} \delta^k{}_l$$

where the *Weyl tensor* $C_{ij}{}^k{}_l$ is trace free and β_{ij} is skew. The Bianchi symmetry $R_{[ij}{}^k{}_{l]} = 0$ dictates that $2P_{[ij]} = -\beta_{ij}$. The tensor P_{ij} is a modification of the (not necessarily symmetric) Ricci tensor $R_{jl} = R_{ij}{}^i{}_l$:

$$(n-1) P_{ij} = R_{ij} + \beta_{ij}$$

and thus $(n+1)\beta_{ij} = -2R_{[ij]}$.

Under change of connection, $C_{ij}{}^k{}_l$ is invariant and the other parts transform according to

$$\widehat{P}_{ij} = P_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j \quad \text{and} \quad \hat{\beta}_{ij} = \beta_{ij} + 2\nabla_{[i} \Upsilon_{j]}. \tag{15}$$

The Bianchi identity $\nabla_{[i} R_{jk]}{}^l{}_m = 0$ yields two results that we will need later:

$$\nabla_k C_{ij}{}^k{}_l = 2(n-2) \nabla_{[i} P_{j]l} \quad \text{and} \quad \nabla_{[i} \beta_{jk]} = 0. \tag{16}$$

We note that β_{ij} considered as a 2-form is closed, and therefore given (15), we can always cause it to vanish locally by a change of connection generated by an appropriately chosen Υ_i. *We will assume henceforth that we are always using a connection with* $\beta_{ij} = 0$. The remaining freedom is to transform by Υ_i satisfying $\nabla_{[i} \Upsilon_{j]} = 0$ just as in the conformal case. A helpful consequence of this is that P_{ij} is always symmetric.

Choose a line bundle $\mathcal{E}(1)$ such that its $(-n-1)^{\text{st}}$ power $\mathcal{E}(-n-1)$ is identified with the canonical bundle of $\mathcal{M}$. We can now consider tensor fields with *projective weights*, e.g., sections of $\mathcal{E}^i(w) = \mathcal{E}^i \otimes \mathcal{E}(w)$. The bundles $\mathcal{E}(w)$ have an induced connection which behaves under transformation according to

$$\hat{\nabla}_i f = \nabla_i f + w\Upsilon_i f.$$

The connection induced on $\mathcal{E}(w)$ is flat (since for a general connection in the projective class its curvature would be $w\beta_{ij}$). Just as for conformal structures, one can associate to every nowhere vanishing section τ of $\mathcal{E}(1)$ a connection in the projective class in which that section is constant. We refer to this as a *choice of projective scale*. Using $\Omega^{-1}\tau$ as projective scale corresponds to a change of connection generated by $\Upsilon_i = \Omega^{-1}\nabla_i\Omega$ again paralleling the conformal case.

3.2. The tractor bundle and connection. One can proceed by analogy with the conformal case, starting with the projectively invariant equation $(\nabla_i\nabla_j + P_{ij})\sigma = 0$ on σ, a section of $\mathcal{E}(1)$. In the interests of brevity, we will not take that approach here. We will begin by defining the dual of the tractor bundle.

Definition 3.1. The *co-tractor* vector bundle $\mathcal{E}_I$ is $\mathcal{J}^1\mathcal{E}(1)$, the first jet bundle of $\mathcal{E}(1)$.

This immediately leads to the description:

Proposition 3.2. *For any choice of projetive scale, the* co-tractor *vector bundle $\mathcal{E}_I \to \mathcal{M}$ is identified with the direct sum $\mathcal{E}_i(1)\oplus\mathcal{E}(1)$, and, under change of scale, pairs (μ_i, σ) are identified with their counterparts $(\hat{\mu}_i, \hat{\sigma})$ in the new scale according to*

$$(\,\hat{\mu}_i \quad \hat{\sigma}\,) = (\,\mu_i + \Upsilon_i\sigma \quad \sigma\,)\,. \tag{17}$$

We see that $\mathcal{E}_I$ has composition series

$$\mathcal{E}_I = \mathcal{E}(1) + \mathcal{E}_i(1).$$

Definition 3.3. The *tractor connection* ∇ on $\mathcal{E}_I$ is given by

$$\nabla_j \begin{pmatrix} \mu_i & \sigma \end{pmatrix} = \begin{pmatrix} \nabla_j \mu_i + P_{ij}\sigma & \nabla_i \sigma - \mu_i \end{pmatrix}. \tag{18}$$

Since there is no natural inner product on the tractors in the projective case, $\mathcal{E}_I$ is not isomorphic to its dual. Instead we have the *tractor bundle*

$$\mathcal{E}^I = \mathcal{E}^i(-1) + \mathcal{E}(-1),$$

where the splitting changes under change of scale according to

$$\begin{pmatrix} \hat{\nu}^i \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \nu^i \\ \rho - \Upsilon_j \nu^j \end{pmatrix}.$$

The connection on $\mathcal{E}^I$ is easily computed to give

$$\nabla_j \begin{pmatrix} \nu^i \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_j \nu^i + \rho \delta_j{}^i \\ \nabla_j \rho - P_{kj}\nu^k \end{pmatrix}.$$

3.3. Primary and secondary parts. We use the terminology of primary and secondary parts exactly as for the conformal case. The projection $\mathcal{E}_I \to \mathcal{E}(1)$ and injection $\mathcal{E}(-1) \to \mathcal{E}^I$ can be expressed in terms of a preferred section X^I of $\mathcal{E}^I(1)$, according to

$$\sigma = X^I U_I \quad \text{and} \quad \rho \mapsto \rho X^I,$$

respectively. In a chosen scale, X^I is represented by

$$X^I = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

There is another preferred object Y_I^j, a section of $\mathcal{E}_I^j(-1)$, which defines the projection $\mathcal{E}^I \to \mathcal{E}^i(-1)$ and the injection $\mathcal{E}_i(1) \to \mathcal{E}_I$ in the obvious way. In a chosen scale it is represented as $(\delta_i{}^j \ \ 0)$ and $X^I Y_I^j = 0$.

3.4. Curvature and that flat model. The flat model of an n-dimensional projective structure is the projective space $\mathbf{P}(\mathbf{R}^{n+1})$, where $\mathbf{R}^{n+1}$ comes equipped with a fixed volume form. In that case, $\mathcal{E}^I$ is just the product bundle with fiber $\mathbf{R}^{n+1}$ and the obvious flat connection. (This explains our decision as to which bundle is $\mathcal{E}^I$ and which its dual!)

The *tractor curvature* is defined by

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)U^K = \Omega_{ij}{}^K{}_L U^L$$

and a calculation shows that

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\begin{pmatrix}\nu^k\\ \rho\end{pmatrix} = \begin{pmatrix}C_{ij}{}^k{}_l & 0\\ -2\nabla_{[i}P_{j]l} & 0\end{pmatrix}\begin{pmatrix}\nu^l\\ \rho\end{pmatrix},$$

thus identifying the square matrix as the curvature. The projecting part is the Weyl curvature. This necessarily vanishes for $n = 2$ in which case we see that $\nabla_{[i}P_{j]l}$ is an invariant. Given (16), we immediately obtain

Proposition 3.4. *The tractor connection is flat if and only if* $C_{ij}{}^k{}_l = 0$ *for* $n \geq 3$ *or* $\nabla_{[i}P_{j]k} = 0$ *for* $n = 2$.

An analogous argument to the conformal case shows that if the tractor connection is flat then $\mathcal{M}$ is locally equivalent to the flat model.

3.5. The D operator. Thomas also defined an analogue, for projetive structures, of the operator D_I for conformal structures.

Definition 3.5. On a section f of $\mathcal{E}(w)$, the operator $D_I : \mathcal{E}(w) \to \mathcal{E}_I(w-1)$ is defined by

$$D_I f = (\,\nabla_i f \quad wf\,).$$

It is easy to check that this definition is invariant and that it remains so even if f has tractor indices.

One can use D to construct invariant operators, etc., much as in the conformal case. We give a single example here.

Proposition 3.6. *For each integer $k \geq 0$ there exists a projectively invariant linear differential operator*

$$\mathcal{E}(k) \to \underbrace{\mathcal{E}_{i\ldots m}}_{k+1}(k)$$

with symbol $\nabla_{(i} \cdots \nabla_{m)}$.

The $(k+1)^{\text{st}}$ order operator $\nabla_{(i} \cdots \nabla_{m)}$ on $\mathcal{E}(k)$ is invariant in the flat model, and one can thus regard the proposition as a proof of the existence of curved analogues of these operators.

Proof. If ϕ is a quantity of weight w, possibly with tractor indices, then $X^I D_I \phi = w\phi$ and it is also easy to check that

$$D_I(X^J \phi) - X^J D_I \phi = \phi \delta^J{}_I.$$

From this it follows that

$$X^I \underbrace{D_{(I} \cdots D_{M)}}_{k+1} f = 0$$

for any section f of $\mathcal{E}(k)$. Thus, the projecting part of $D_{(I} \cdots D_{M)} f$ is the "last possible" piece, and this is clearly a differential operator with the required symbol. □

There is also an analogue for projective structures of the Fefferman-Graham construction for conformal structures. It was also found by Thomas [**29**]. Stated briefly, on the total space of $\mathcal{E}(-1)$ (with the zero section removed) there is defined a canonical affine connection which preserves a naturally defined volume form. Tractors are essentially tangents to this space, and D_I is the connection. In the flat case, this constructed space *is* just the vector space of which the flat model is the projective space, and D_I is essentially differentiation with respect to homogeneous coordinates.

4. Other geometries. The constructions above can be carried out on other structures with appropriate geometry. Our main example is paraconformal structure and we outline the proof below of an assertion about the equivalence of such structures to their flat model which was made in [**1**].

4.1. Paraconformal (PCF) manifolds. The analogue of Thomas's structure bundle for PCF manifolds will be referred to as the "local twistor" bundle. This is because in the special case $p = 2$, $q = 2$ (see below), a PCF manifold is a 4-dimensional conformal spin manifold, and the bundle we define is precisely the local twistor bundle of Penrose. It is not hard to verify that in that case, the tractor bundle can be identified with the second exterior power of the local twistor bundle.

Paraconformal manifolds are manifolds locally modelled on Grassmannians. We consider only the holomorphic category here, although real forms do exist. A complete definition and a description of their differential geometry and twistor theory can be found in [**1**]. Roughly speaking, a PCF is a manifold whose tangent bundle $\mathcal{E}^a$ splits as a tensor product of two vector bundles of ranks $q, p : \mathcal{E}^a = \mathcal{E}^{AA'} = \mathcal{E}^A \otimes \mathcal{E}^{A'}$. So as to avoid repetition, we will adopt the conventions and notation of [**1**] without explanation. Thus, in this section, we use lower case letters a, b, c, d for tangent and cotangent bundle indices, A, B, C, D for indices pertaining to the defining bundles of the PCF structure, and lower case Greek for the local twistor indices.

We will consider only the special case of *torsion-free quaternionic conformal* (QCF) manifolds here, although the arguments can be generalized to PCFs with some extra work. A torsion-free QCF has $p = 2$, $q = 2k$ (with $k \geq 2$) for the ranks of the vector bundles and they also satisfy a certain condition of vanishing torsion. They are essentially a complexified version of quaternionic manifolds [**22, 23**].

In [**1**], the local twistor bundle of a PCF is defined and necessary and sufficient conditions for its flatness (and hence for equivalence with the flat model by the usual type of argument) are stated. Our aim here is to outline the proof of this statement, in the simple case of torsion-free QCFs.

The curvature of the local twistor connection is given by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{pmatrix} \omega^C \\ \pi_{C'} \end{pmatrix} = \begin{pmatrix} \Psi_{ABD}{}^C \varepsilon_{A'B'} & 0 \\ -2\nabla_{[a} P_{b]DC'} & 0 \end{pmatrix} \begin{pmatrix} \omega^D \\ \pi_{D'} \end{pmatrix},$$

thus identifying the square matrix as the *local twistor curvature* $\Omega_{ab}{}^{\gamma}{}_{\delta}$.

Several appeals to the Bianchi identities are made in the ensuing argument. These are rather difficult to derive directly for QCFs but everything that is needed is a consequence of, and can most easily be derived from, the *local twistor Bianchi identities* $\nabla_{[a}\Omega_{bc]}{}^{\delta}{}_{\varepsilon} = 0$.

The local twistor curvature splits according to

$$\Omega_{ab}{}^{\gamma}{}_{\delta} = \Omega_{A'B'AB}{}^{\gamma}{}_{\delta} + \Omega_{AB}{}^{\gamma}{}_{\delta}\varepsilon_{A'B'}$$

where the summands have the symmetries $\Omega_{A'B'AB}{}^{\gamma}{}_{\delta} = \Omega_{(A'B')[AB]}{}^{\gamma}{}_{\delta}$ and $\Omega_{AB}{}^{\gamma}{}_{\delta} = \Omega_{(AB)}{}^{\gamma}{}_{\delta}$. It is a consequence of the Bianchi identities that $\nabla_{A(A'}P_{B')BCC'} - \nabla_{B(A'}P_{B')ACC'} = 0$ and hence that $\Omega_{A'B'AB}{}^{\gamma}{}_{\delta} = 0$. Another consequence of the Bianchi identities is that

$$\nabla_{AA'}\Psi_{BCD}{}^{A} + (2k-1)\nabla^{B'}_{(B}P_{C)B'DA'},$$

which implies that the local twistor curvature vanishes if and only if $\Psi_{ABC}{}^{D} = 0$, as asserted in [**1**] for this special case. As we remarked, the general case is similar but more involved.

4.2. Other structures modelled on Hermitian symmetric spaces. Baston [**5**] has shown the existence of a local twistor construction for other structures locally modelled on Hermitian symmetric spaces. It is possible to proceed in these cases in an analogous way to that used in the PCF theory and construct an explicit local twistor vector bundle and connection. From this point of view, the relevant geometry is that the manifolds have a tangent bundle that can be written as $\bigwedge^2$ or $\bigodot^2$ of some other vector bundle.

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