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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF A DISCRETE LOGISTIC MODEL

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ABSTRACT. We consider the discrete logistic model with or without delay

$$x_{n+1} = \frac{\alpha_n x_n}{1 + \beta_n x_{n-j}}, \qquad n = 0, 1, 2, \dots, j \geq 0$$

where α_n, β_n are positive bounded sequences. A complete discussion on the oscillatory and asymptotic behavior is given for the case that $j = 0$. For the case that $j > 0$, some results on oscillation are also obtained.

1. Introduction. In 1969, Pielou posed the difference equation model (see [**8**])

$$x_{n+1} = \frac{\alpha x_n}{1 + \beta x_{n-j}}, \qquad n = 0, 1, 2, \dots, j \geq 0 \tag{1.0}$$

(where $\alpha > 1$, $\beta > 0$ are constants) as the discrete analog of the delay logistic equation

$$\dot{N}(t) = rN(t)\left[1 - \frac{N(t-\tau)}{p}\right].$$

Recently, Kuruklis and Ladas have obtained oscillation criteria for Equation (1.0) with $j > 0$ and asymptotic stability results for (1.0) with $j = 0, 1$, see [**4**].

However, from the derivation of the model (1.1) we see that α and β are related to the growth rate r and the carrying capacity p as follows:

$$\alpha = e^r \quad \text{and} \quad \beta = (e^r - 1)p,$$

and hence are not constants, and not even periodic in general.

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Our aim in this paper is to study revised models where α and β in (1.0) are replaced by bounded sequences α_n and β_n. We will consider both the difference equation without delay

$$x_{n+1} = \frac{\alpha_n x_n}{1+\beta_n x_n}, \qquad n = 0, 1, 2, \dots, x_0 > 0 \tag{1.1}$$

and the difference equation with a delay

$$x_{n+1} = \frac{\alpha_n x_n}{1+\beta_n x_{n-j}}, \qquad n = 0, 1, 2, \dots, j \geq 1 \tag{1.2}$$

with $x_i = a_i$ for $i = -j, \dots, 0$, $a_i \geq 0$ for $i = -j, \dots, -1$, $a_0 > 0$, $1 < \alpha_* \leq \alpha_n \leq \alpha^* < \infty$, and $0 \leq \beta_* \leq \beta_n \leq \beta^* < \infty$. We will give a complete discussion on the behavior of (1.1) and obtain some results on the oscillation of (1.2).

Definition 1. A sequence $\{x_n\}$ is said to be *oscillatory* if x_n is not eventually positive or eventually negative. A sequence $\{x_n\}$ is said to be oscillatory about a sequence $\{y_n\}$ if $\{x_n - y_n\}$ is oscillatory. A sequence $\{x_n\}$ is said to be k-oscillatory if $\{x_n\}$ is oscillatory about $\{k\}$. If a sequence $\{x_n\}$ is k-oscillatory for some k, then we refer to $A(x_n) = \limsup x_n - \liminf x_n$ as the amplitude of $\{x_n\}$.

Definition 2. Let $\{x_n\}, \{y_n\}$ be two sequences. $\{x_n\}$ is said to *approach* $\{y_n\}$ *asymptotically*, denoted by $x_n \sim y_n$, if $x_n - y_n \to 0$ as $n \to \infty$. Furthermore, we say that $x_n \sim y_n$ with an exponential speed if $|x_n - y_n| \leq kc^n$ for some $k > 0$, $0 < c < 1$.

The following assumptions will be used in our discussion.

(H1) $K_n = (\alpha_n - 1)/\beta_n$ is eventually monotonic and $\lim K_n = k$;

(H2) K_n is not eventually monotonic and $\liminf K_n = k_*$ and $\limsup K_n = k^*$.

2. Behavior of equation (1.1). In this section we obtain results for oscillatory and asymptotic behavior of (1.1) which are parallel to those of the continuous logistic model given in [**3**].

It is obvious that equation (1.1) is equivalent to the equation

$$x_{n+1} - x_n = \frac{K_n - x_n}{1+\beta_n x_n}\beta_n x_n. \tag{2.1}$$

Theorem 2.1. *Assume that* (H1) *holds. Then every solution* $\{x_n\}$ *of* (1.1) *eventually satisfies* $x_n \geq K_n$ *or* $x_n \leq K_n$ *and* $x_n \sim k$.

Proof. Without loss of generality, we assume that K_n is increasing for $n \geq n_0$.

i) If $x_n > K_n$ for $n \geq n_0$, then by (2.1) x_n is decreasing, and hence $x_n \to x \geq k$. From (1.1),

$$\frac{\alpha_n}{1+\beta_n x_n} = \frac{x_{n+1}}{x_n} \to 1, \qquad n \to \infty.$$

Then $\alpha_n \sim 1 + \beta_n x_n$, and

$$x_n \sim \frac{\alpha_n - 1}{\beta_n} = K_n \to k.$$

ii) If there exists an i such that $x_i \leq K_i$, since the function $\varphi(x) = \alpha x/(1+\beta x)$ is increasing for any $\alpha > 0$ and $\beta > 0$, we have

$$x_{i+1} = \frac{\alpha_i x_i}{1+\beta_i x_i} \leq \frac{\alpha_i K_i}{1+\beta_i K_i} = K_i \leq K_{i+1}.$$

By induction $x_n \leq K_n$ for $n \geq i$. From (2.1), x_n is increasing, and as in i), $x_n \to k$. □

Theorem 2.2. *Assume that* (H1) *holds and* K_n *is increasing* (*decreasing*) *with*

$$K_n \sum_{i=0}^{n-1} \frac{K_i - K_n}{1+\beta_i K_n}\beta_i \to -\infty(+\infty) \quad \text{as } n \to \infty.$$

Then all solutions of (1.1) *eventually satisfy that* $x_n \leq K_n$ ($x_n \geq K_n$) *and* $x_n \sim k$.

Proof. Without loss of generality, assume that K_n is increasing. By Theorem 2.1, it suffices to show that all solutions eventually satisfy $x_n \leq K_n$.

Let $\{x_n\}, \{y_n\}$ be two solutions of (1.1). Clearly, $x_n > y_n$ implies that $x_{n+1} > y_{n+1}$. Denote

$$E = \{x_0 : \text{ solution } \{x_n\} \text{ starting at } x_0 \text{ satisfies } x_n > K_n \text{ for all } n\}.$$

Assume the conclusion is not true; then E is nonempty, connected and bounded below by K_0. Let $x^* = \inf E$. If $x^* \notin E$, then there exists an n_1 such that $x^*_{n_1} < K_{n_1}$. By the continuous dependence of solutions on their initial values, we see that there exists an $\hat{x} \in E$ such that $\hat{x}_{n_1} < K_{n_1}$, contradicting $\hat{x} \in E$.

If $x^* \in E$, then there exist $x(n) \to x^*$ as $n \to \infty$ such that the solution $x_i(n)$ of (1.1) starting from $x(n)$ satisfies that $x_n(n) = K_n$ and $x_i(n) > K_n$ for $i = 0, \dots, n-1$ since $x_i(n)$ is decreasing in i. Since $\psi(x) = (Kx - x^2)/(1+\beta x)$ is decreasing for any $K > 0$, $\beta > 0$ and $x \geq K$, noting that $K_n \geq K_i$, $i = 0, \dots, n-1$, we have

$$\begin{aligned} x_n(n) - x_0(n) &= \sum_{i=0}^{n-1} (x_{i+1}(n) - x_i(n)) \\ &= \sum_{i=0}^{n-1} \frac{K_i - x_i(n)}{1 + \beta_i x_i(n)} \beta_i x_i(n) \\ &\leq K_n \sum_{i=0}^{n-1} \frac{K_i - K_n}{1 + \beta_i K_n} \beta_i \to -\infty, \qquad n \to \infty \end{aligned}$$

by assumption. Thus, $x_n(n) = K_n \to -\infty$ since $x_0(n)$ are bounded, and this contradicts $K_n \to k$. □

Lemma 2.3. *Assume that* (H2) *holds. Then, for any* $\varepsilon > 0$*, all solutions of* (1.1) *eventually satisfy*

$$k_* - \varepsilon < x_n < k^* + \varepsilon. \tag{2.2}$$

Proof. Assume that there exists a solution $\{x_n\}$ which does not satisfy (2.2) eventually. Consider the following four cases:

i) $x_n \geq k^* + \varepsilon$ eventually,

ii) $x_n \leq k_* - \varepsilon$ eventually,

iii) there exist $n_i \to \infty$ such that (2.2) holds for $n = n_i$,

iv) there exist sequences $n_i \to \infty$ and $m_i \to \infty$ such that i) holds for $n = n_i$ and ii) holds for $n = m_i$.

For case i), from (2.1) eventually we have

$$\begin{aligned} x_{n+1} - x_n &= \frac{K_n - x_n}{1 + \beta_n x_n}\beta_n x_n \\ &\leq -\frac{\varepsilon}{2}\frac{\beta_n(k^* + \varepsilon)}{1 + \beta_n(k^* + \varepsilon)} \\ &\leq -\frac{\varepsilon}{2}\frac{\beta_* k^*}{1 + \beta_* k^*}, \end{aligned}$$

so $x_n \to -\infty$ as $n \to \infty$, contradicting that $x_n \geq k^* + \varepsilon$.

For case ii), the discussion is similar.

For case iii), if $x_n > K_n$ eventually, then there exists an n_i such that (2.2) holds for n_i and $x_n > K_n$ for $n \geq n_i$. Hence, $\{x_n\}$ is decreasing for $n \geq n_i$. As a result, $x_n \leq x_{n_i} < k^* + \varepsilon$ for $n \geq n_i$. At the same time, $x_n > K_n > k_* - \varepsilon$. Thus, (2.2) holds for $n \geq n_i$. A similar argument holds for the case that $x_n < K_n$ eventually. If $\{x_n\}$ is not eventually monotonic, from (2.1), we see that $\{x_n\}$ assumes its local maximum at some $n = i + 1$ if $x_i \leq K_i$ and $x_{i+1} > K_{i+1}$. So, from (1.1), for large i,

$$x_{i+1} = \frac{\alpha_i x_i}{1 + \beta_i x_i} \leq \frac{\alpha_i K_i}{1 + \beta_i K_i} = K_i < k^* + \varepsilon.$$

Similarly, if $\{x_n\}$ assumes its local minimum at some $n = i + 1$, then $x_{i+1} > k_* - \varepsilon$ for large i. Hence, for sufficiently large n, we have that (2.2) holds, contradicting the assumption.

For case iv), the proof is exactly the same as the second half of case iii). □

Theorem 2.4. *Assume that* (H2) *holds with* $k_* = k^* := k$. *Then all solutions of* (1.1) *satisfy* $x_n \sim k$.

Proof. This is an immediate corollary of Lemma 2.3. □

Lemma 2.5. *Assume that* (H2) *holds with* $k_* < k^*$. *Then, for every solution* $\{x_n\}$ *of* (1.1), *there exists an interval* $(a, b) \subset (k_*, k^*)$ *with*

$$b - a \geq \frac{b_* k_*}{2 + 3\beta_* k_*}(k^* - k_*) \tag{2.3}$$

such that $\{x_n\}$ *is* k*-oscillatory for every* $k \in (a, b)$, *and hence* $A(x_n) \geq b - a > 0$.

Proof. Let $a = \liminf x_n$, $b = \limsup x_n$. First we show that $k_* \leq a < b \leq k^*$, and hence $\{x_n\}$ is k-oscillatory for any $k \in (a, b)$.

By Lemma 2.3, it is obvious that $k_* \leq a \leq b \leq k^*$. Now we show that $a < b$. If not, $\lim x_n = x^*$ exists and $x^* \in [k_*, k^*]$. Without loss of generality, assume that $x^* < k^*$. Let $k^* - x^* = 2l$. Then there exists an i such that $K_n - x_n \geq l$ and (2.2) holds for some $0 < \varepsilon < k_*$ and $n \geq i$. Therefore, by (2.1)

$$x_{n+1} - x_n = \frac{K_n - x_n}{1 + \beta_n x_n}\beta_n x_n \geq \frac{l\beta_*(k_* - \varepsilon)}{1 + \beta^*(k^* + \varepsilon)} > 0,$$

contradicting that $\lim x_n = x^*$.

Next we show that (2.3) holds. Assume the contrary. We have

$$A(x_n) < \frac{\beta_* k_*}{2 + 3\beta_* k_*}(k^* - k_*). \tag{2.4}$$

Then

$$\begin{aligned} \limsup\{|K_n - x_n|\} &\geq [(k^* - k_*) - A(x_n)]/2 \\ &> \frac{1 + \beta_* k_*}{2 + 3\beta_* k_*}(k^* - k_*) \end{aligned}$$

and

$$\begin{aligned} \limsup\{|x_{n+1} - x_n|\} &\geq \limsup\{|K_n - x_n|\} \liminf \frac{\beta_n x_n}{1 + \beta_n x_n} \\ &> \frac{1 + \beta_* k_*}{2 + 3\beta_* k_*}(k^* - k_*)\frac{\beta_* k_*}{1 + \beta_* k_*} \\ &= \frac{\beta_* k_*}{2 + 3\beta_* k_*}(k^* - k_*). \end{aligned}$$

This implies that

$$\limsup x_n - \liminf x_n \geq \frac{\beta_* k_*}{2 + 3\beta_* k_*}(k^* - k_*)$$

and hence contradicts (2.4). □

Corollary 2.6. *Assume that* (H2) *holds. Then every solution of* (1.1) *is oscillatory about* $\{K_n\}$.

Proof. Assume the contrary and, without loss of generality, assume that there exists a solution $\{x_n\}$ of (1.1) satisfying $x_n > K_n$ eventually. By (2.1), $\{x_n\}$ is eventually decreasing, contradicting that $\{x_n\}$ is k-oscillatory for $k \in (a, b) \subset (k_*, k^*)$. □

Lemma 2.7. *Assume that* (H2) *holds, and* $\{x_n\}$ *and* $\{x_n^*\}$ *are two solutions of* (1.1). *Then* $x_n \sim x_n^*$ *with an exponential speed.*

Proof. Assume that $x_n \not\equiv x_n^*$. Then $x_n \neq x_n^*$. Without loss of generality, assume that $x_n > x_n^*$. Make a change of variables $x_n = e^{y_n}$ in (1.1). Then we have

$$y_{n+1} = y_n - \ln\left[\frac{1}{\alpha_n}(1 + \beta_n e^{y_n})\right] := f(n, y_n). \tag{2.5}$$

Noting that

$$\frac{d}{dz} f(n, z) = 1 - \frac{\beta_n e^z}{1 + \beta_n e^z} = \frac{1}{1 + \beta_n e^z}.$$

It is easy to see that $0 < (d/dz) f(n, z) \leq c < 1$ for $e^z \in (k_* - \varepsilon, k^* + \varepsilon)$ where $\varepsilon > 0$. Hence, for the solutions $x_n = e^{y_n}$, $x_n^* = e^{y_n^*}$, we have

$$\begin{aligned} 0 < y_{n+1} - y_{n+1}^* &= f(n, y_n) - f(n, y_n^*) \\ &= \frac{d}{dz} f(n, \xi_n)(y_n - y_n^*) \leq c(y_n - y_n^*) \end{aligned} \tag{2.6}$$

where $\xi_n \in (y_n, y_n^*)$, and hence $e^{\xi_n} \in (k_* - \varepsilon, k^* + \varepsilon)$. Now (2.6) implies that $y_n \sim y_n^*$ with an exponential speed, and so $x_n \sim x_n^*$ with an exponential speed. □

Theorem 2.8. *Assume that* (H2) *holds with* $k_* < k^*$. *Then there exists an interval* $(a, b) \subset (k_*, k^*)$ *satisfying* (2.3) *such that all solutions of* (1.1) *are* k*-oscillatory for any* $k \in (a, b)$.

Proof. Let $\{x_n\}$ be a solution of (1.1). By Lemma 2.5 there exists an interval (a, b) satisfying (2.3) such that $\{x_n\}$ is k-oscillatory for $k \in (a, b)$. We show that this interval is suitable for all solutions. Since $A(x_n) > 0$, there exist $\varepsilon > 0$ and $n_i \to \infty$ such that $x_{n_i} \le k - \varepsilon$. Let $\{x_n^*\}$ be any other solution of (1.1). By Lemma 2.7, $x_n \sim x_n^*$. Without loss of generality, assume that $x_n^* > x_n$. We claim that $\{x_n^*\}$ is also k-oscillatory. Otherwise, $x_n^* > k$ for $n \ge n_0 \ge 0$. Then $x_{n_i}^* > k$ for $n_i \ge n_0$ and thus $x_{n_i}^* - x_{n_i} > \varepsilon$, contradicting the fact that $x_n^* \sim x_n$. □

For equations with periodic coefficients, we have the following result.

Corollary 2.9. *Assume that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are positive periodic functions with period* $j > 0$. *Then there exists a unique* j*-periodic solution of* (1.1) *which is globally asymptotically stable.*

Proof. By induction it is easy to see that (1.1) is equivalent to the following equation

$$x_{n+j} = \frac{\alpha_n \cdots \alpha_{n+j-1} x_n}{1 + (\beta_n + \alpha_n \beta_{n+1} + \cdots + \alpha_n \cdots \alpha_{n+j-2} \beta_{n+j-1}) x_n}. \tag{2.7}$$

$\{x_n^*\}$ is a j-periodic solution of (1.1) if and only if $x_{n+j}^* = x_n^*$. Solving this equation we find

$$x_n^* = \frac{\alpha_n \cdots \alpha_{n+j-1} - 1}{\beta_n + \alpha_n \beta_{n+1} + \cdots + \alpha_n \cdots \alpha_{n+j-2} \beta_{n+j-1}}, \qquad n = 1, 2, \dots .$$

Therefore (1.1) has a unique j-periodic solution $\{x_n^*\}$ if $\{\alpha_n\}$ and $\{\beta_n\}$ are j-periodic. By Lemma 2.7, all solutions of (1.1) approach $\{x_n^*\}$ as $n \to \infty$, i.e., $\{x_n^*\}$ is globally asymptotically stable. □

3. Oscillation of equation (1.2). In this section we present some oscillation results for the delay difference equation (1.2). (1.2) is equivalent to the following equation

$$x_{n+1} - x_n = \frac{K_n - x_{n-j}}{1 + \beta_n x_{n-j}} \beta_n x_n. \tag{3.1}$$

Denote

$$\begin{aligned} N(i) &= \{n \in N, n \geq i\}, \\ N_1 &= \{n \in N : K_n \text{ assumes its local minimum at } n\}, \\ N_2 &= \{n \in N : K_n \text{ assumes its local maximum at } n\}, \\ N_1(i) &= N_1 \cap N(i), \qquad N_2(i) = N_2 \cap N(i), \end{aligned}$$

and

$$i_* = \min\{N_1(i)\}, \qquad i^* = \min\{N_2(i)\}.$$

Theorem 3.1. *Assume that* (H2) *holds with* $k_* < k^*$. *Then every solution of* (1.2) *is oscillatory about* $\{K_{n+j}\}$.

Proof. Assume the contrary. Then there exists a solution $\{x_n\}$ which is not oscillatory about $\{K_{n+j}\}$. Without loss of generality, assume that $x_n > K_{n+j}$ for $n \geq i \geq 0$. By (3.1), $\{x_n\}$ is decreasing for $n \geq i$ and hence $\lim x_n = x^*$ exists. Noting that $N_1(i)$ is an infinite set, by (3.1) we have

$$\begin{aligned} x^* - x_i &= \sum_{n=i}^{\infty} (x_{n+1} - x_n) < \sum_{n \in N_1(i)} (x_{n+1} - x_n) \\ &= \sum_{n \in N_1(i)} \frac{K_n - x_{n-j}}{1 + \beta_n x_{n-j}} \beta_n x_n \\ &\leq \sum_{n \in N_1(i)} \frac{K_n - x^*_{n-j}}{1 + \beta_n x_{n-j}} \beta_n x_n \\ &\leq \sum_{n \in N_1(i)} \frac{K_n - K_{n^*}}{1 + \beta_n x_{n-k}} \beta_n x_n \\ &\leq \sum_{n \in N_1(i)} \frac{M}{2} (k_* - k^*) \end{aligned} \tag{3.2}$$

where M is a constant, $0 < M < \beta_* k_*/(1+\beta^* k^*)$. This implies that $x^* = -\infty$, which is impossible. □

Theorem 3.2. *Assume* (H2) *holds with* $k_* < k^*$. *Then, for every solution* $\{x_n\}$ *of* (1.2)*, there exists an interval* (a,b) *such that* $[a,b] \cap [k_*, k^*] \neq \varnothing$*,* (2.3) *holds and* $\{x_n\}$ *is* k*-oscillatory for any* $k \in (a,b)$.

The proof is similar to that of Lemma 2.5. But here we may not have $(a,b) \subset (k_*, k^*)$. Instead, since $\{x_n\}$ is oscillatory about $\{K_{n+j}\}$, we have

$$\limsup x_n \geq k_* \quad \text{or} \quad \liminf x_n \leq k^*.$$

Therefore, $[a,b] \cap [k_*, k^*] \neq \varnothing$.

Theorem 3.3. *Assume that* (H2) *holds, with* $k_* < k^*$. *Let* $\{x_n\}$, $\{y_n\}$ *be two solutions of* (1.2). *Then either* $\{x_n\}$ *is oscillatory about* $\{y_n\}$ *or* $x_n \sim y_n$.

Proof. Suppose that $\{x_n\}$ is not oscillatory about $\{y_n\}$. Without loss of generality, assume that $x_n > y_n$, $n \geq i-j$ for some i. By induction from (1.2), we get

$$\frac{x_{n+1}}{y_{n+1}} = \frac{x_i}{y_i} \frac{1+\beta_i y_{i-j}}{1+\beta_i x_{i-j}} \cdots \frac{1+\beta_n y_{n-j}}{1+\beta_n x_{n-j}}.$$

If $x_n \not\sim y_n$, then there exists $n_i \to \infty$ such that

$$\frac{1+\beta_{n_i} y_{n_i-j}}{1+\beta_{n_i} x_{n_i-j}} \leq \delta < 1.$$

Hence, $x_{n+1}/y_{n+1} \to 0$ as $n \to \infty$, contradicting that $x_n/y_n > 1$. □

Theorem 3.4. *Assume that* (H2) *holds with* $k_* = k^* := k$*, and*

$$\sum_{n \in N_1} (K_n - K_{n^*}) = -\infty \quad \text{and} \quad \sum_{n \in N_2} (K_n - K_{n^*}) = \infty.$$

Then every solution of (3.1) *is oscillatory about* $\{K_{n+j}\}$.

Proof. The proof is similar to that of Theorem 3.1. Note that, from (3.2), we get

$$x^* - x_i \leq \frac{M}{2} \sum_{n \in N_1(i)} (K_n - K_{n^*}) = -\infty$$

contradicting that $x^* > 0$. □

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