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SOME GLOBAL QUALITATIVE ANALYSES OF A SINGLE SPECIES NEUTRAL DELAY DIFFERENTIAL POPULATION MODEL

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ABSTRACT. In this paper a class of nonlinear nonautonomous single species neutral delay differential population models are introduced and analyzed. Sufficient conditions for positivity and boundedness of solutions, local and global stability of positive steady state, are established.

1. Introduction. The most frequently adopted single species population model takes the form

$$\dot{x}(t) = rx(t)[1 - x(t)/K], \qquad x(0) > 0, \tag{1.1}$$

where r is the so-called intrinsic growth rate of the species x, and K is often referred to as the environment carrying capacity for x. $r[1 - x(t)/K]$ is called the per capita growth rate of x at time t, which asserts that the growth rate is inhibited due to the self crowdedness effect.

If we take into account the fact that species x may have a gestation period of length τ, then a more suitable expression of the per capita growth rate should be

$$r[1 - ax(t) + bx(t - \tau)] \tag{1.2}$$

with $a - b = 1/K$. The term $rbx(t - \tau)$ reflects the birth rate due to the part of the population of age τ. Equation (1.1) thus becomes

$$\dot{x}(t) = rx(t)[1 - ax(t) + bx(t - \tau)]. \tag{1.3}$$

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Assume that species x gives births seasonally, say twice in a time period of τ, and the total growth rate in that period is governed by the per capita growth rate (1.2). Assume further that the rate of newborns surviving a time period of τ is $\rho(< 1)$, then a proper model for its growth takes the form of

$$\dot{x}(t) + \rho\dot{x}(t-\tau) = rx(t)[1 - ax(t) + bx(t-\tau)]. \tag{1.4}$$

In this paper we consider the following more general neutral delay model

$$\dot{x}(t) + \rho\dot{x}(t-\tau) = x(t)G(x(t), x(t-\tau)), \tag{1.5}$$

where G is decreasing with respect to $x(t)$ and nondecreasing with respect to $x(t-\tau)$. The main advantage of (1.5) is that we allow $G(\cdot,\cdot)$ to be nonlinear. Of course, an even more realistic model may take the form

$$\dot{x}(t) + \rho\int_{-\tau_2}^{-\tau_1} \dot{x}(t)\, d\mu(s) = x(t)G(t, x(t-\tau_3), x_t) \tag{1.6}$$

where $\tau_1 \leq \tau_2$, $\int_{-\tau_2}^{-\tau_1} |\, d\mu(s)| = 1$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$, $\tau \geq \max\{\tau_2, \tau_3\}$; G is decreasing with respect to $x(t-\tau_3)$, and increasing with respect to x_t. Indeed, most of our results in this paper can be extended to model (1.6) with proper modifications. For more details on this, see Kuang [**16**]. We chose model (1.5) simply to avoid the technical complexity.

Other kinds of neutral delay population models are studied in Gopalsamy and Zhang [**8**], Gopalsamy, et al. [**7**], Kuang [**13**–**16**], Kuang and Feldstein [**17**], Giyori and Wu [**9**], Wu and Freedman [**21**].

In the following section we present some local stability results. Hopf bifurcation results for equation (1.5) can be obtained by applying the Hopf bifurcation theorem for neutral delay equations established in de Oliveira [**2**]. In Section 3 we obtain conditions for solutions of (1.5) to be positive and bounded. In Section 4 we establish nonlocal (which we call global) stability results for positive steady states in a properly defined region.

2. Preliminaries. We consider the following nonlinear neutral delay differential equation as a single species growth model

$$\frac{d}{dt}(x(t) + \rho x(t-\tau)) = x(t)G(x(t), x(t-\tau)), \tag{2.1}$$

where $\tau > 0$, $0 \leq \rho < 1$, $G(x, y)$ is continuously differentiable and satisfies

(H1) $(\partial/\partial x)G(x,y) < 0$, $(\partial/\partial y)G(x,y) \geq 0$ for $x \geq 0$, $y \geq 0$.

(H2) There is a nondecreasing function $g(y)$ such that $G(g(y), y) = 0$, $g(0) > 0$. Moreover, there is a $K > 0$ such that $G(K, K) = 0$.

We always assume that the initial condition for (2.1) satisfies

$$x(\theta) = \phi(\theta) \geq 0, \qquad \theta \in [-\tau, 0], \tag{2.2}$$

$\phi(\theta)$ is continuously differentiable on $[-\tau, 0]$, and $\phi(0) > 0$. The existence and uniqueness of solution of (2.1) and (2.2) (denoted as $x(t) = x(t, \phi)$) are thus assured [**11**]. However, without further assumptions on (2.1) and (2.2), solutions may not stay positive and/or bounded. For example, when $\phi(0)$ is very small and ρ is a positive constant and $\dot{\phi}(-\tau) > 0$ is large enough, we see that $\dot{x} \approx -\rho\dot{\phi}(-\tau) < 0$. Therefore, for small t, $x(t)$ may become negative. Even when $\rho(t) \equiv 0$, solutions of (2.1) and (2.2) may not be bounded. For example, assume that $G(x(t), x(t-\tau)) = 1 - (e+1)x(t) + e^{x(t-\tau)}$, then we can show that the solution of

$$\dot{x}(t) = x(t)G(x(t), x(t-\tau)), \tag{2.3}$$
$$x(\theta) = 2, \qquad \theta \in [-\tau, 0] \tag{2.4}$$

tends to $+\infty$ as $t \to +\infty$. Indeed, we can show that the solution of (2.3) and (2.4) is strictly increasing for $t > 0$. Note that (2.3) has two positive steady states.

We consider first the local stability of steady state $x(t) \equiv K$. For this purpose, we linearize it to obtain

$$\dot{x}(t) + \rho\dot{x}(t-\tau) + \beta x(t) + \gamma x(t-\tau) = 0, \tag{2.5}$$

where

$$B = -K\frac{\partial}{\partial x}G(K, K) > 0, \qquad \gamma = -K\frac{\partial}{\partial y}G(K, K) < 0. \tag{2.6}$$

A direct application of Theorem 3.1 in Freedman and Kuang [**5**] yields

Theorem 2.1. *For equation* (2.5), *the following statements are true*

(i) *If* $(\partial/\partial y)G(K,K) < -(\partial/\partial x)G(K,K)$*, then* $x(t) \equiv K$ *is locally asymptotically stable.*

(ii) *If* $(\partial/\partial y)G(K,K) > -(\partial/\partial x)G(K,K)$*, then* $x(t) \equiv K$ *is unstable.*

Roughly, the above theorem asserts that, near the steady state $x(t) \equiv K$, if the nondelayed self-competition effect is stronger than the growth effect due to the past population, then $x(t) \equiv K$ is locally stable; otherwise, it is unstable. However, we note that the length of delay is irrelevant.

If equation (2.1) has a unique positive steady state $x(t) \equiv K$, then it is necessary that $(\partial/\partial y)G(K,K) \leq |(\partial/\partial x)G(K,K)|$.

If $\rho = 0$, then by similar arguments as the proof of the main result in Haddock and Kuang [**10**], we have

Theorem 2.2. *In equation* (2.1)*, assume* $\rho = 0$ *and* (H1) *and* (H2) *and*

$$\text{(H3)} \quad \max\left\{\frac{\partial}{\partial y}G(x,y) : x \geq 0, y \geq 0\right\} < \min\left\{\left|\frac{\partial}{\partial x}G(x,y)\right| : x \geq 0, y \geq 0\right\}$$

are satisfied. Then it has a unique positive steady state $x(t) \equiv K$*, and this steady state is globally asymptotically stable with respect to nonnegative continuous initial function* $\phi(\theta)$ *such that* $\phi(0) > 0$.

The proof of the above theorem is a simple application of the well-known Razumikhin-type theorems (e.g., see Hale [**11**]).

For continuous function ϕ on $[-\tau, 0]$, we define its norm as

$$||\phi|| = \max_{-\tau \leq \theta \leq 0}\{|\phi(\theta)|\}. \tag{2.7}$$

3. Positivity and boundedness. Our objective in this section is to derive sufficient conditions for the solutions of (2.1) and (2.2) to be positive and bounded.

Theorem 3.1. *Consider equation* (2.1) *with initial function* ϕ *satisfying* (2.2). *Assume that* (H1) *and* (H2) *hold, and*

(H3) *there is a* $\bar{\phi} > ||\phi||$ *such that*

$$\phi(0) + \rho\phi(-\tau) < g(\bar{\phi}) + \rho\bar{\phi} < \bar{\phi}, \tag{3.1}$$

$$\phi(0) + \rho\phi(-\tau) > g(0) > \rho\bar{\phi}. \tag{3.2}$$

Then the solution $x(t) \equiv x(t, \phi)$ *satisfies* $0 < g(0) - \rho\bar{\phi} < x(t) < \bar{\phi}$, $t \geq 0$.

Remark 3.1. Clearly, $\bar{\phi}$ depends on $||\phi||$. For too large or too small initial function ϕ (in terms of $||\phi||$), (3.1) or (3.2) cannot be satisfied. Also, in order to satisfy (3.1), we must have $g(y) < (1-\rho)y$ for large value y, which restricts $G(x, y)$. However, Theorem 3.1 does not depend on the value τ.

Proof of Theorem 3.1. Denote

$$z(t) = x(t) + \rho x(t - \tau).$$

Then (2.1) is equivalent to

$$\dot{z}(t) = x(t)G(z(t) - \rho x(t - \tau), x(t - \tau)). \tag{3.3}$$

From (3.1) and (3.2), we see that

$$g(0) < z(0) = \phi(0) + \rho\phi(-\tau) < g(\bar{\phi}) + \rho\bar{\phi}. \tag{3.4}$$

We claim that, for $t \geq 0$,

$$g(0) \leq z(t) < g(\bar{\phi}) + \rho\bar{\phi}. \tag{3.5}$$

Otherwise, there are two possibilities:

(i) there is a $t_1 > 0$, such that

$$\begin{gathered} z(t_1) = g(\bar{\phi}) + \rho\bar{\phi}, \qquad \dot{z}(t_1) \geq 0, \\ g(0) < z(t) < g(\bar{\phi}) + \rho\bar{\phi} \end{gathered} \tag{3.6}$$

for $t \in [0, t_1)$.

(ii) There exist $t_2 > t_1 > 0$, $\varepsilon > 0$, such that

$$\begin{aligned} \varepsilon < g(0) - \rho\bar{\phi}, \qquad z(t_1) = g(0), \qquad \dot{z}(t_1) \le 0, \\ z(t_2) = g(0) - \varepsilon, \qquad \dot{z}(t_2) < 0, \end{aligned} \tag{3.7}$$

and $g(0) - \varepsilon < z(t) < g(\phi) + \rho\bar{\phi}$ for $t \in [0, t_2)$.

We divide the rest of the proof into two parts.

Part A. We show first that as long as $g(0) - \varepsilon \le z(t) \le g(\bar{\phi}) + \rho\bar{\phi}$, for $t \in [0, t_1]$, where $g(0) > \varepsilon + \rho\bar{\phi}$. Then

$$0 < g(0) - \varepsilon - \rho\bar{\phi} < x(t) < \bar{\phi}, \qquad t \in [0, t_1]. \tag{3.8}$$

Otherwise, there are two possibilities:

(a) there is a $t_0 \in (0, t_1]$ such that $x(t_0) = \bar{\phi}$, $g(0) - \varepsilon - \rho\bar{\phi} < x(t) < \bar{\phi}$ for $t \in (0, t_0)$.

(b) There is a $t_0 \in (0, t_1]$ such that $x(t_0) = g(0) - \varepsilon - \rho\phi$, $g(0) - \varepsilon - \rho\bar{\phi} < x(t) < \bar{\phi}$ for $t \in (0, t_0)$.

Assume first that case (a) is true. Then, by (3.1),

$$x(t_0) = z(t_0) - \rho x(t_0 - \tau) \le z(t_0) \le g(\bar{\phi}) + \rho\bar{\phi} < \bar{\phi},$$

a contradiction. In case that (b) is true, then

$$x(t_0) = z(t_0) - \rho x(t_0 - \tau) > g(0) - \varepsilon - \rho\bar{\phi},$$

also a contradiction.

Part B. We now prove that (3.5) is true for all $t \ge 0$.

Assume first that case (i) is true. By Part A, we know that for $t \in [0, t_1]$,

$$0 < g(0) - \rho\bar{\phi} < x(t) < \bar{\phi}.$$

Hence, by (H1) and the fact that $x(t - \tau) < \bar{\phi}$, we have

$$\begin{aligned} \dot{z}(t_1) &= x(t_1)G(z(t_1) - \rho x(t_1 - \tau), x(t_1 - \tau)) \\ &< x(t_1)G(z(t_1) - \rho\bar{\phi}, x(t_1 - \tau)) \le x(t_1)G(z(t_1) - \rho\bar{\phi}, \bar{\phi}) \\ &= x(t_1)G(g(\bar{\phi}), \bar{\phi}) = 0, \end{aligned}$$

a contradiction.

Assume now that (ii) is true. Then, by Part A, we know that for $t \in [0, t_2]$,

$$0 < g(0) - \varepsilon - \rho\bar{\phi} < x(t) < \bar{\phi}.$$

Therefore,

$$\begin{aligned}\dot{z}(t_2) &= x(t_2)G(z(t_2) - \rho x(t_2 - \tau), x(t_2 - \tau)) \\ &\geq x(t_2)G(z(t_2), 0) \\ &= x(t_2)G(g(0) - \varepsilon, 0) > 0,\end{aligned}$$

also a contradiction. This proves the claim of (3.5).

Now it is clear from Part A that $g(0) - \rho\bar{\phi} < x(t) < \bar{\phi}$, proving the theorem. □

Next we apply Theorem 3.1 to the following equation

$$\dot{x}(t) + \rho\dot{x}(t - \tau) = rx(t)[1 - ax(t) + bx(t - \tau)], \tag{3.9}$$

where r, a and b are positive constants. For this equation, we have

$$G(x(t), x(t - \tau)) = r[1 - ax(t) + bx(t - \tau)].$$

Without loss of generality, we assume $a - b = 1$. Then $G(1, 1) = 0$ $G(g(y), y) = 0$ with $g(y) = a^{-1}(1 + by)$ strictly increasing. Clearly, (H1) and (H2) are satisfied. By Theorem 3.1, we thus have

Corollary 3.1. *Consider equation* (3.9) *with initial function* ϕ *satisfying* (2.2). *Assume that* $a - b = 1$, *and there exists* $\phi > 0$ *such that* $||\phi|| \leq \phi$, *and*

(A) $\phi(0) + \rho\phi(-\tau) < a^{-1}(1 + b\bar{\phi}) + \rho\bar{\phi} < \bar{\phi}$,

(B) $\phi(0) + \rho\phi(-\tau) > a^{-1} > \rho\bar{\phi}$.

Then the solution $x(t) \equiv x(t, \phi)$ *satisfies*

$$0 < a^{-1} - \rho\bar{\phi} < x(t) < \bar{\phi}, \qquad t \geq 0.$$

For example, if we apply Corollary 3.1 to the following equation,

$$\dot{x}(t) + 0.2\dot{x}(t - 1) = 2[1 - 2x(t) + x(t - 1)] \tag{3.10}$$

we have $\rho = 0.2$, $\tau = 1$, $r = 2$, $a = 2$ and $b = 1$. We can choose $\bar{\phi} = 2.49$. We conclude that if ϕ satisfies (2.2) and

$$(0.498 <)0.5 < \phi(0) + 0.2\phi(-1) < 2.243(< 2.49), \tag{3.11}$$

then $x(t) \equiv x(t, \phi)$ satisfies $0.002 < x(t) < 2.49$, $t \geq 0$. It is easy to see that condition (3.11) is not very restrictive in view of the fact that the positive steady state is 1.

4. Global stability. Given a set Q of initial functions satisfying (2.2) and containing steady state $x(t) \equiv K$ of equation (2.1), we say $x(t) \equiv K$ is *globally asymptotically stable* with respect to Q for equation (2.1) if

$$\lim_{t\to\infty} x(t, \phi) = K, \quad \text{for all } \phi \in Q. \tag{4.1}$$

We denote for $v \in (0, K)$,

$$\begin{aligned} g_1 &= g_1(v) \\ &= \min\left\{\left|\frac{\partial G}{\partial x}(x, y)\right| : |x - K| \leq v, |y - K| \leq v\right\} \end{aligned} \tag{4.2}$$

$$\begin{aligned} g_2 &= g_2(v) \\ &= \max\left\{\left|\frac{\partial G}{\partial y}(x, y)\right| : |x - K| \leq v, |y - K| \leq v\right\} \end{aligned} \tag{4.3}$$

$$\begin{aligned} Q(v) = \Big\{\phi : \phi \text{ satisfies (2.2)}, ||\phi - K|| < v, \\ ||\dot{\phi}|| < \frac{g_1 + g_2}{1 - \rho}(K + v)v\Big\}. \end{aligned} \tag{4.4}$$

The following theorem is the main result of this section.

Theorem 4.1. *If there exists a $v \in (0, K)$ such that*

$$g_1 - g_2 > \frac{\rho}{1 - \rho}\frac{K + v}{K - v}(g_1 + g_2), \tag{4.5}$$

then the steady state $x(t) \equiv K$ of equation (2.1) *is globally asymptotically stable with respect to $Q(v)$.*

Note that $Q(v)$ places conditions on the derivatives of its members, and condition (4.5) implies that $\rho < 1/2$, a requirement that also appeared in Kuang [**16**] for a different class of neutral delay population models.

We prove first the following useful lemma, which amounts to saying that $Q(v)$ is positive invariant with respect to equation (2.1).

Lemma 4.1. *If all conditions of Theorem* 4.1 *are satisfied and $\phi \in Q(v)$, then for all $t \geq 0$, $|x(t,\phi) - K| < v$, $|\dot{x}(t)| < [(g_1 + g_2)/(1 - \rho)](K + v)v$.*

Proof. If the above lemma is not true, then there are two cases to consider:

(i) there is a $t_0 \geq 0$, $|x(t_0) - K| = v$, $|x(t) - K| < v$ for $t < t_0$ and $|\dot{x}(t)| \leq ((g_1 + g_2)/(1 - \rho))(K + v)v$ for $t \leq t_0$;

(ii) there is a $t_0 \geq 0$, $|\dot{x}(t_0)| = ((g_1 + g_2)/(1 - \rho))(K + v)v$, $|x(t) - K| < v$ for $t < t_0$ and $|\dot{x}(t)| < ((g_1 + g_2)/(1 - \rho))(K + v)v$ for $t < t_0$.

Consider first case (i). If $x(t_0) = K + v$, then $\dot{x}(t_0) \geq 0$, while if $x(t_0) = K - v$, then we must have $\dot{x}(t_0) \leq 0$. We consider first the case $x(t_0) = K + v$, the case of $x(t_0) = K - v$ will be dealt with similarly.

Note that, for $x(t_0) = K + v$, we have

$$G(x(t_0), x(t_0 - \tau)) = G(x(t_0), x(t_0 - \tau)) - G(K, K).$$

Using the above equation and the fact that $2\rho < 1$ from condition (4.5), we obtain

$$\begin{aligned} \dot{x}(t_0) &= x(t_0)G(x(t_0), x(t_0 - \tau)) - \rho\dot{x}(t_0 - \tau) \\ &\leq (K + v)(-g_1 v + g_2 v) + \rho\frac{g_1 + g_2}{1 - \rho}(K + v)v \\ &= \left[\frac{\rho}{1 - \rho}(g_1 + g_2) - (g_1 - g_2)\right](K + v)v < 0, \end{aligned} \tag{4.6}$$

a contradiction.

If $x(t_0) = K - v$, then we have

$$G(x(t_0), x(t_0 - \tau)) \geq g_1 v - g_2 v = (g_1 - g_2)v.$$

Hence, by (ii), we have

$$\begin{aligned}\dot{x}(t_0) &\geq (K - v)(g_1 - g_2)v - \rho\frac{g_1 + g_2}{1 - \rho}(K + v)v \\ &= (K - v)\left[g_1 - g_2 - \frac{\rho}{1 - \rho}\frac{K + v}{K - v}(g_1 + g_2)\right]v > 0,\end{aligned}$$

also a contradiction. This disproves case (i).

Consider now case (ii). We have

$$\begin{aligned}|\dot{x}(t_0)| &< \rho|\dot{x}(t_0 - \tau)| + (K + v)(g_1 + g_2)v \\ &\leq \frac{\rho}{1 - \rho}(g_1 + g_2)(K + v)v + (K + v)(g_1 + g_2)v \\ &\leq \frac{g_1 + g_2}{1 - \rho}(K + v)v,\end{aligned} \tag{4.7}$$

which shows that case (ii) is impossible, proving the lemma. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. From Lemma 4.1, we see that there exists a constant v^*, $0 \leq v^* \leq v$, such that

$$\limsup_{t \to \infty} |x(t) - K| = v^* \tag{4.8}$$

where $x(t) = x(t, \phi)$ for some $\phi \in Q(v)$. We need to show $v^* = 0$.

If $v^* > 0$, then (4.5) implies that there exists a small $\varepsilon \in (0, v^*)$ such that $v^* + \varepsilon < K$ and

$$g_1\frac{v^* - \varepsilon}{v^* + \varepsilon} - g_2 > \frac{\rho}{1 - \rho}\frac{K + v^* + \varepsilon}{K - v^* - \varepsilon}(g_1 + g_2) + \rho\varepsilon. \tag{4.9}$$

The definition of v^* implies that there exists a $t_1 \geq \tau$ such that, for $t \geq t_1$,

$$|x(t) - K| < v^* + \varepsilon.$$

Let N be a positive integer such that

$$\rho^N \frac{g_1 + g_2}{1-\rho}(K+v)v < \varepsilon(K - v^* - \varepsilon)(v^* + \varepsilon) \tag{4.10}$$

and denote

$$\alpha = (K + v^* + \varepsilon)(g_1 + g_2)(v^* + \varepsilon). \tag{4.11}$$

Then, for $t \geq t_1 + N\tau$, we have

$$\begin{aligned} |\dot{x}(t)| &\leq (K + v^* + \varepsilon)(g_1 + g_2)(v^* + \varepsilon) + \rho|\dot{x}(t-\tau)| \\ &= \alpha + \rho|\dot{x}(t-\tau)|. \end{aligned}$$

Similarly, we have

$$\begin{gathered} |\dot{x}(t - i\tau)| \leq \alpha + \rho|\dot{x}(t - (i+1)\tau)|, \\ i = 1, 2, \ldots, N-1. \end{gathered}$$

From Lemma 4.1, we know that $|\dot{x}(t - N\tau)| < (1-\rho)^{-1}(g_1 + g_2)(K+v)v$ for $t \geq t_1 + N\tau$. Hence, for $t \geq t_1 + N\tau$,

$$\begin{aligned} |\dot{x}(t)| &\leq \alpha(1 + \rho + \cdots + \rho^{N-1}) + \rho^N|\dot{x}(t - N\tau)| \\ &< \frac{\alpha}{1-\rho} + \rho^N \frac{g_1 + g_2}{1-\rho}(K+v)v \\ &< \frac{g_1 + g_2}{1-\rho}(K + v^* + \varepsilon)(v^* + \varepsilon) \\ &\quad + \varepsilon(K - v^* - \varepsilon)(v^* + \varepsilon). \end{aligned}$$

The definition of v^* and the assumption that $v^* > 0$ implies that at least one of the following three cases is true.

Case 1. *There is a* $t_2 > t_1 + (N+1)\tau$ *such that* $x(t_2) \geq K + v^* - \varepsilon$ *and* $\dot{x}(t_2) \geq 0$.

Case 2. *There is a* $t_2 \geq t_1 + (N+1)\tau$ *such that* $x(t_2) \leq K - v^* + \varepsilon$ *and* $\dot{x}(t_2) \leq 0$.

Case 3. *There is a* $t_2 \geq t_1 + (N+1)\tau$ *such that* $x(t)$ *is monotone for* $t \geq t_2$.

If case 1 is true, then (4.9) implies that

$$\begin{aligned}\dot{x}(t_2) &= x(t_2)G(x(t_2), x(t_2-\tau)) - \rho\dot{x}(t_2-\tau)\\ &\le (K-v^*-\varepsilon)[-g_1\cdot(v^*-\varepsilon)+g_2\cdot(v^*+\varepsilon)]\\ &\quad+\frac{\rho}{1-\rho}(g_1+g_2)(K+v^*+\varepsilon)(v^*+\varepsilon)\\ &\quad+\rho\in(K-v^*-\varepsilon)(v^*+\varepsilon)\\ &< (K-v^*-\varepsilon)(v^*+\varepsilon)\bigg[-g_1\frac{v^*-\varepsilon}{v^*+\varepsilon}+g_2\\ &\qquad\qquad+\frac{\rho}{1-\rho}\frac{K+v^*+\varepsilon}{K-v^*-\varepsilon}(g_1+g_2)+\rho\varepsilon\bigg]<0,\end{aligned}$$

a contradiction. If case 2 is true, then

$$\begin{aligned}\dot{x}(t_2) &\ge (K-v^*-\varepsilon)[g_1\cdot(v^*-\varepsilon)-g_2\cdot(v^*+\varepsilon)]\\ &\quad-\frac{\rho}{1-\rho}(g_1+g_2)(K+v^*+\varepsilon)(v^*+\varepsilon)\\ &\quad-\rho\in(K-v^*-\varepsilon)(v^*+\varepsilon)>0,\end{aligned}$$

also a contraction.

Finally, if only case 3 is true, then we have either $x(t)$ is monotonely decreasing such that

$$\lim_{t\to\infty} x(t) = K + v^* \tag{4.12}$$

or $x(t)$ is monotonely increasing such that

$$\lim_{t\to\infty} x(t) = K - v^*. \tag{4.13}$$

In case of (4.12), there is a $\delta > 0$ and $t_3 \ge t_2 + \tau$ such that, for $t \ge t_3$,

$$x(t)G(x(t), x(t-\tau)) < -\delta.$$

Therefore, for $t \ge t_3$,

$$\begin{aligned}&x(t)+\rho x(t-\tau)-x(t_3)-\rho x(t_3-\tau)\\ &\qquad\qquad=\int_{t_3}^{t} x(s)G(x(s),x(s-\tau))\,ds < -\delta(t-t_3)\end{aligned}$$

which tends to $-\infty$ as $t \to +\infty$, contradicting the fact that $x(t)$ is bounded. The case of (4.13) can be dealt with similarly. This proves the theorem. □

Applying Theorem 4.1 to equation (3.9), we obtain

Corollary 4.1. *Assume that* $a - b = 1$ *in* (3.9); *then the steady state* $x(t) \equiv 1$ *is globally asymptotically stable with respect to* $Q(v)$ *provided that*

$$0 < v < \frac{1 - \rho - \rho(a+b)}{1 - \rho + \rho(a+b)}. \tag{4.14}$$

Proof. Observe that $g_1 = ra$, $g_2 = rb$, $K = 1$ and (4.5) reduces to

$$1 > \frac{\rho}{1-\rho}\frac{1+v}{1-v}(a+b),$$

which is equivalent to (4.14). □

For example, for equation (3.10), we have $\rho = 0.2$, $a = 2$, $b = 1$. Let

$$\Delta = \Delta(\rho) \equiv \frac{1 - \rho - \rho(a+b)}{1 - \rho + \rho(a+b)}; \tag{4.15}$$

then $\Delta(0.2) = 1/7$, that is, $x(t) \equiv 1$ is globally asymptotically stable with respect to $Q(v)$ provided that $v \in (0, 1/7)$. If we set $\rho = 0.1$ in (4.15), then $\Delta(0.1) = 0.5$. It is easy to see that $\Delta(\rho)$ is decreasing for $\rho < (1 + a + b)^{-1}$, and $\Delta(0) = 1 = K$.

The above discussion suggests that the smaller the neutral coefficient ρ, the larger the allowable value for v in order to have $Q(v)$ as a part of the region of attraction of steady state $x(t) \equiv K$. Again, Theorem 4.1 does not depend on the delay length τ.

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