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ON THE DISCRETE RICCATI EQUATION AND ITS APPLICATIONS TO DISCRETE HAMILTONIAN SYSTEMS

LYNN H. ERBE AND PENGXIANG YAN

ABSTRACT. We obtain some comparison theorems for the discrete Riccati equation

$$\Delta W(t) + A(t) + B^*(t)W(t) + W(t)B(t) - B^*(t)W(t)B(t) + (I - B(t)))^*W(t)(C^{-1}(t) + W(t))^{-1}W(t)(I - B(t)) = 0$$

and some applications to the discrete Hamiltonian difference system

$$\Delta y(t) = B(t)y(t+1) + C(t)z(t)$$
$$\Delta z(t) = -A(t)y(t+1) - B^*(t)z(t)$$

where A, B, and C are $d \times d$ matrix functions, $y(t)$, $z(t)$ are $d \times 1$ vectors and t takes on integer values in $[M-1, N+1]$.

1. Introduction. In [**5**–**8**] the present authors introduced the Hamiltonian vector difference system:

$$\begin{aligned} \Delta y(t) &= B(t)y(t+1) + C(t)z(t) \\ \Delta z(t) &= -A(t)y(t+1) - B^*(t)z(t) \end{aligned} \tag{1.1}$$

the corresponding matrix system

$$\begin{aligned} \Delta Y(t) &= B(t)Y(t+1) + C(t)Z(t) \\ \Delta Z(t) &= -A(t)Y(t+1) - B^*(t)Z(t) \end{aligned} \tag{1.2}$$

where $A(t), B(t), C(t), W(t), Y(t)$, and $Z(t)$ are $d \times d$ matrices with $A(t)$ and $C(t)$ Hermitian. We assume further that $C(t) > 0$ (positive definite) and $I - B(t)$ is invertible. Also, Δ denotes the forward

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difference operator $\Delta y(t) = y(t+1) - y(t)$ and B^* denotes the conjugate transpose. Here $y(t)$ and $z(t)$ are $d \times 1$ vectors and t takes on integer values in $[M-1, N+1]$, where M and N are two integers. We examined the disconjugacy for these systems by introducing the discrete Riccati equation:

(1.3)
$$\begin{aligned} R[W] := \Delta W(t) + A(t) + B^*(t)W(t) + W(t)B(t) - B^*(t)W(t)B(t) \\ + \big(I - B(t)\big)^* W(t)\big(C^{-1}(t) + W(t)\big)^{-1} W(t)\big(I - B(t)\big) = 0 \end{aligned}$$

and the quadratic forms

$$q[u] = \sum_{t=M}^{N+1} \big(z^*(t-1)C(t-1)z(t-1) - y^*(t)A(t-1)y(t)\big),$$

where

$$\begin{aligned} u &= \{y(t), z(t)\} \in \Omega \\ &= \{y, z \in C^d : y(M-1) = 0 = y(N+1), \\ &\qquad \Delta y(t) = B(t)y(t+1) + C(t)z(t)\}; \end{aligned}$$

and

$$Q[U] = \sum_{t=M}^{N+1} \big(Z^*(t-1)C(t-1)Z(t-1) - Y^*(t)A(t-1)Y(t)\big)$$

where

$$\begin{aligned} U &= \{Y(t), Z(t)\} \in \Lambda \\ &= \{Y, Z \in C^{d\times d} : Y(M-1) = 0 = Y(N+1), \\ &\qquad \Delta Y(t) = B(t)Y(t+1) + C(t)Z(t)\}. \end{aligned}$$

We introduce the further notation: $\Lambda^+ := \{U \in \Lambda$: there is a $M-1 \le t_0 \le N-1$ such that $Y(t_0) = 0$ and $Y(t_0+1)$ is nonsingular or there is $M+1 \le t_0 \le N+1$ such that $Y(t_0) = 0$ and $Y(t_0-1)$ is nonsingular$\}$.

We say q is positive on Ω provided $q[u] \ge 0$ for all $u \in \Omega$ and $q = 0$ if and only if $u \equiv 0$; Q is positive definite on Λ provided for all $U \in \Lambda$, $Q[U] \ge 0$ and $Q = 0$ if and only if $U \equiv 0$; Q is strictly positive on Λ^+ if $Q[U] > 0$ for all $U \in \Lambda^+$. The relation among (1.1)-(1.3), q, Q is given in the "Reid Roundabout Theorem" [**6**, Theorem 2.5], which will be stated below for completeness.

We say (1.1) is disconjugate on $[M-1, N+1]$ if and only if for any nontrivial solution $\{y(t), z(t)\}$ of (1.1) there exists at most one integer $p \in [M-1, N]$ such that $y^*(p)C^{-1}(p)\big(I - B(p)\big)y(p+1) \le 0$. A solution $\{Y(t), Z(t)\}$ of (1.2) is said to be prepared if $Y^*(t)Z(t)$ is Hermitian. We say a prepared solution of (1.2) is a conjoined basis if Rank $\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} \equiv d$, and it is said to be recessive at ∞ if there exists an integer M_0 for which

$$Y^*(t)C^{-1}(t)\big(I - B(t)\big)Y(t+1) > 0, \qquad t \ge M_0 \tag{1.4}$$

and

$$\lim_{n\to\infty} \sum_{s=M_0}^{n} u^* \Big(Y^*(s)C^{-1}(s)\big(I - B(s)\big)Y(s+1)\Big)^{-1} u = \infty \tag{1.5}$$

for every unit vector u. A prepared solution of (1.2) is said to be dominant at ∞ if (1.5) holds for some integer M_0 and

$$\sum_{s=M_0}^{\infty} u^* \Big(Y^*(s)C^{-1}(s)\big(I - B(s)\big)Y(s+1)\Big)^{-1} u \tag{1.6}$$

converges for every unit vector u.

Equation (1.1) is said to be eventually disconjugate in case there exists an integer M_0 such that (1.1) is disconjugate on $[M_0-1, N_1+1]$ for all integers $N_1 > M_0$. A "Reid Roundabout Theorem," (cf. Ahlbrandt [**2**]) and many results of disconjugacy for (1.1) and (1.2) analogous to the continuous case of Coppel [**4**] are given in [**6**–**8**]. Peterson [**11**] gives the corresponding disfocality criteria for (1.1). For completeness we state: (cf. [**6**, Theorem 2.5])

Theorem A. *The following are equivalent*:

(i) *Equation* (1.1) *is disconjugate on* $[M-1, N+1]$;

(ii) $q[u]$ *is positive definite on* Ω;

(iii) $Q[U]$ *is positive definite on* Λ *and strictly positive on* Λ^+

(iv) *There exists a Hermitian solution of the Riccati equation (3) such that* $C^{-1}(t) + W(t) > 0,\ t \in [M-1, N]$;

(v) *There exists a solution of equation* (1.2) *such that*

$$Y^*(t)C^{-1}(t)\big(I - B(t)\big)Y(t+1) > 0, \qquad t \in [M-1, N].$$

Ahlbrandt [**3**] recently showed that under certain implicit solvability hypotheses, (1.1) is equivalent to the discrete Euler equation, and also proved that discrete linear Hamiltonian systems have a symplectic transition matrix. Ahlbrandt's results show that there is a possibility to apply the results of discrete Hamiltonian systems to the control theory.

In control theory (cf. [**12**]), for the linear discrete-time system

$$\begin{aligned} x(k+1) &= \tilde{A}x(k) + \tilde{B}v(k), \qquad k > 0 \\ \lim_{k\to\infty} x(k) &= 0, \end{aligned} \tag{1.7}$$

one wishes to minimize the cost functional

$$J[v] = \sum_{k=0}^{\infty} \begin{pmatrix} x(k) \\ v(k) \end{pmatrix}^* \begin{pmatrix} \tilde{Q} & \tilde{C}^* \\ \tilde{C} & \tilde{R} \end{pmatrix} \begin{pmatrix} x(k) \\ v(k) \end{pmatrix}. \tag{1.8}$$

The control v which minimizes this cost functional is given by the formula

$$v(k) = -(\tilde{R} + \tilde{B}^* P_+ \tilde{B})^{-1}(\tilde{C} + \tilde{B}^* P_+ \tilde{B})^{-1}(\tilde{C} + \tilde{B}^* P_+ \tilde{A}), \tag{1.9}$$

where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}$ and $\tilde{R}$ have the dimensions $d \times d$, $d \times l$, $l \times d$, $d \times l$, $d \times d$, $l \times l$. The pair $(\tilde{A}, \tilde{B})$ is assumed to be stabilized, i.e., there exists a $d \times d$ matrix K such that all eigenvalues of $\tilde{A} - \tilde{B}K$ are inside the unit circle, and P_+ is the maximal Hermitian solution of the following algebraic Riccati equation

$$P = \tilde{A}^* P \tilde{A} + \tilde{Q} - \big(\tilde{C} + \tilde{B}^* P \tilde{A}\big)^* \big(\tilde{R} + \tilde{B}^* P \tilde{B}\big)^{-1} \big(\tilde{C} + \tilde{B}^* P \tilde{A}\big), \tag{1.10}$$

In this paper, we are going to derive some comparison theorems for (1.3), and derive sufficient conditions of disconjugacy for autonomous systems, i.e., the systems with all the coefficient matrices in (1.1) constant.

2. Main results.

The following theorem is an extension of Proposition 2.4 [**7**].

Theorem 1. *If $V(t)$ is a Hermitian matrix solution of the inequality $R[V] \leq 0$ for $t \in [k_0, \infty), W(t)$ is a Hermitian solution of* (1.3) *with $W(k_0) > V(k_0)$, $W(k_0) \geq V(k_0)$, and $V(t) + C^{-1}(t) > 0$, then $W(t) > V(t)$, $W(t) \geq V(t)$, for $t \in [k_0, \infty)$.*

Proof. Suppose $U = \{Y(t), Z(t)\}$ is a prepared solution of (1.2) corresponding to $W(t)$, as in Proposition 1.4 [**6**]. Then

$$\begin{aligned} Q[U] &= Y^*(N+1)Z(N+1) - Y^*(k_0)Z(k_0) \\ &= Y^*(N+1)W(N+1)Y(N+1) - Y^*(K_0)W(k_0)Y(k_0) \end{aligned} \tag{2.1}$$

At the same time, since $R[V] \leq 0$, we have

$$\begin{aligned} \Delta V(t) \leq -A(t) &- B^*(t)V(t) - V(t)B(t) + B^*(t)V(t)B(t) \\ &- \big(I - B(t)\big)^* V(t)\big(C^{-1}(t) + V(t)\big)^{-1} V(t)\big(I - B(t)\big). \end{aligned}$$

By using the proof of Proposition 1.4 [**6**], we have

$$\begin{aligned} Q[U] \geq Y^*(N+1)&V(N+1)Y(N+1) - Y^*(k_0)V(k_0)Y(k_0) \\ &+ \sum_{t=k_0-1}^{N} \big(V(t)Y(t) - Z(t)\big)^* \big(C^{-1}(t) \\ &\qquad + V(t)\big)^{-1}\big(V(t)Y(t) - Z(t)\big) \end{aligned} \tag{2.2}$$

Now combining (2.1) and (2.2), we get

$$\begin{aligned} Y^*(N+1)\big(W(N+1) &- V(N+1)\big)Y(N+1) \\ &\geq Y^*(k_0)\big(W(k_0) - V(k_0)\big)Y(k_0) > 0. \end{aligned}$$

where N can be any integer in $[k_0 + 1, \infty)$. Therefore the proof is completed. □

The next theorem concerns the minimal solution of (1.3). For $B(t) \equiv 0$ Ahlbrandt [**1**] gave the explicit constructions for "maximal" and "minimal" solutions of (1.3).

Theorem 2. *Suppose that $\{Y_0(t), Z_0(t)\}$ is the recessive solution of (1.2) on $[M, \infty)$, and set $W_0(t) = Z_0(t)Y_0^{-1}(t)$. Then for any Hermitian solution $W(t)$ of (1.3), there exists an integer $M_0 \geq M$ such that $W(t) \geq W_0(t)$ for all $t \geq M_0$.*

Proof. From the proof of Proposition 2.4 of [**7**], we have

$$\begin{aligned} W(t) = W_0(t) + \big(\big(S + H_0(t)T\big)^{-1} Y_0^{-1}\big) \\ \big(S^*T + T^*H_0(t)T\big)\big(\big(S + H_0(t)T\big)^{-1} Y_0^{-1}\big) \end{aligned} \tag{2.3}$$

since $\{Y_0(t), Z_0(t)\}$ is a recessive solution of (1.1), it follows that for any unit vector u, we have

$$u^* H_0(t) u \to \infty \quad \text{as } t \to \infty.$$

Therefore, there exists an integer $M_0 \geq M$ such that if $t \geq M_0$, we have

$$S^*T + T^*H_0(t)T \geq 0,$$

i.e., $W(t) \geq W_0(t)$. This completes the proof. □

Note. $W_0(t)$ is evidently a minimal solution of (1.3) in a neighborhood of ∞.

We say that (1.3) is solvable on the interval $[M-1, N+1]$ if there is a Hermitian solution $W(t)$ of (1.3) with $C^{-1}(t) + W(t) > 0$, and is strongly solvable if this property is preserved under a sufficiently small variation of the coefficients $A(t)$ and $B(t)$.

Theorem 3. *Suppose that there exists an $\alpha > 0$ such that $C(t) > \alpha I$ for $t \in [M-1, N+1]$. Then (1.3) is strongly solvable if and only if there exists a $\delta > 0$ such that*

$$q[u] \geq \delta(||y||^2 + ||w||^2) \tag{2.4}$$

where $u \in \Omega$, $||y||^2 = \sum_{t=M-1}^{N} y^*(t)y(t)$, $\Delta y(t) = B(t)y(t+1) + w(t)$.

Proof. If (1.3) is strongly solvable, then there exists a $\delta > 0$ such that if we replace $A(t)$, $C(t)$ by $A(t) + \delta I$, $C(t) - \delta I$, in (1.3) we still may find a Hermitian solution $W(t)$ of (1.3) with $W(t) + C^{-1}(t) > 0$. Therefore the corresponding quadratic form $\tilde{q}[u] \geq 0$. Denote

$$D(t) = \begin{pmatrix} I & 0 \\ B^*(t) & I \end{pmatrix} \begin{pmatrix} C^{-1}(t) & 0 \\ 0 & -A(t) \end{pmatrix} \begin{pmatrix} I & B(t) \\ 0 & I \end{pmatrix}.$$

Then

$$\begin{aligned} \tilde{q}[u] &= \sum_{t=M-1}^{N} \big(\Delta y^*(t), y^*(t+1)\big) \tilde{D}(t) \begin{pmatrix} \Delta y(t) \\ y(t+1) \end{pmatrix} \\ &= q[u] - \sum_{t=M-1}^{N} \big(\Delta y^*(t), y^*(t+1)\big) G(t) \begin{pmatrix} \Delta y(t) \\ y(t+1) \end{pmatrix} \end{aligned}$$

where

$$G(t) = \begin{pmatrix} I & 0 \\ B^*(t) & I \end{pmatrix} \begin{pmatrix} \delta I & 0 \\ 0 & \delta I \end{pmatrix} \begin{pmatrix} I & B(t) \\ 0 & I \end{pmatrix}.$$

Since $\tilde{q} \geq 0$, we obtain (2.4). On the other hand, if (2.4) holds, by Theorem 2.5 of [**6**], we know (1.3) is strongly solvable. □

For the autonomous system, we introduce the notations as in [**12**]:

$$\Pi = \{W \mid W = W^*,\ C^{-1} + W > 0,\ \tilde{R}(W) \leq 0\}$$

where

$$\begin{aligned} \tilde{R}(W) &= W - \big(I - B^*\big) W \big(I - W\big) + A \\ &\quad + \big(I - B\big)^* W \big(C^{-1} + W\big)^{-1} W \big(I - B\big) \\ \Pi' &= \{W \mid W \in \Pi,\ \tilde{R}(W) = 0\} \\ T &= \begin{bmatrix} -\tilde{A} & 0 \\ 0 & \tilde{C}^{-1} \end{bmatrix} \end{aligned}$$

Let $W^+(t)$ denote the maximal Hermitian solution of (1.3), (here the maximal means in a neighborhood of ∞.) If $W(t)$ is a constant solution of $R(W) = 0$, it is obvious that it is also a solution of $\tilde{R}(W) = 0$. Suppose Π_i, Π_i', $\tilde{T}_i$, R_i, $\tilde{R}_i$ correspond to A_i, B_i, C_i, $i = 1, 2$. (i.e. we replace A, B, C by A_i, B_i, C_i in the previous notations).

Lemma. *If* (1.1) *is an autonomous system, then* $R(W) = 0$ *has a solution* $W(t)$ *with* $W(t) + C^{-1} > 0$ *if and only if* $\tilde{R}(W) = 0$ *has a solution* $\tilde{W} \in \Pi$.

Proof. The "if" is obvious. Now we want to prove "only if." Suppose $R[W] = 0$ has a solution $W(t)$ with $W(t) + C^{-1} > 0$. Then we can find a minimal solution $W_0(t)$ as shown in Theorem 2, and we can then show that after some time $W_0(t)$ must be a constant matrix. In fact, since $W_0(t+1)$ is also a solution of (1.3), by Theorem 2, we know that there exists an integer M_1 such that when $t > M_1$ we have

$$W_0(t+1) \geq W_0(t)$$

In the same way, we can show that there exists an integer M_2 such that when $t > M_2$ we have

$$W_0(t-1) \geq W_0(t)$$

therefore when $t > \max\{M_1, M_2\}$, we have

$$W_0(t) = W_0(t+1) = \text{Constant matrix.}$$

This completes the proof. □

Note. From Riccati equation (1.11) in [**8**], we know that if $A - B^*C^{-1}B > 0$, then $R[W] = 0$ does not have any Hermitian solutions $W(t)$ with $W(t) + C^{-1} > 0$.

Note that if W^+ exists, it must be a constant matrix after some time, and by setting $\tilde{A} = (I - B)$, $\tilde{Q} = -A$, $\tilde{R} = C^{-1}$, $\tilde{B} = I$, then (1.9) is equivalent to $\tilde{R}(W) = 0$. As a corollary of Theorem 3.2 [**12**], we have

Theorem 4. (i) *Suppose* Π_1' *is not empty and* $\Pi_1' \subseteq \Pi_2$; *then* W_1^+ *and* W_2^+ *exist and both are constant Hermitian matrices, with*

$$W_2^+ \geq W_1^+$$

(ii) *If* $B_1 = B_2$, $T_2 \geq T_1$, *then we have* (i).

(iii) *If* $T \geq 0$, *then* W^+ *exists and is a constant*; *furthermore,* $W^+ \geq 0$.

Note. From (ii), we know that if, for $i = 1$, the system (1.1) is disconjugate, then it is also disconjugate for $i = 2$. A similar comparison theorem for nonautonomous systems is given in [**6**]. Similarly, from (iii) if $T \geq 0$, then (1.1) is disconjugate.

Theorem 5. *Equation* (1.1) *is disconjugate if and only if* Π *is not empty.*

Proof. From Theorem 3.1 of [**12**] we know that $W^+ \in \Pi'$ exists if and only if Π is not empty. From the Lemma and Theorem 2.5 of [**6**] we obtain the conclusion. □

3. Discussion. In control theory, the condition $C(t) > \alpha I$, where $\alpha > 0$ is of critical importance to guarantee the existence of the optimal control for the system (cf. [**12**] and the references therein). However, in the study of disconjugacy, the case $C(t) \geq 0$ includes an important class of difference equations, as pointed out by Ahlbrandt [**3**], since any even order "self adjoint" difference equation may be represented by an equivalent Hamiltonian difference system.

Consider the following scalar difference equation of order $2d$ (for the continuous case, see [**9**])

$$\sum_{k=0}^{d}(-1)^k \Delta^k \big(p_k(t)\Delta^k x(t-k+1)\big) = 0$$

where $p_d(t) > 0$. If we suppose

$$\begin{aligned}
y_{(k)}(t) &= \Delta^{k-1}x(t-k+1), \qquad k = 1\dots d\\
z_{(d)}(t) &= p_d(t)\Delta y_{(d)}(t)\\
z_{(d-k)}(t) &= -\Delta z_{(d-k+1)} + p_{d-k}(t)y_{(d-k+1)}(t+1),\\
&\qquad k = 1\dots d-1\\
B(t) &= (a_{ij})_{d\times d}
\end{aligned}$$

with $a_{ij} = 1$ if $i = j + 1$ and $= 0$ otherwise

$$C(t) = \operatorname{diag}\big(0, ..., (1/p_d)(t)\big)$$
$$A(t) = \operatorname{diag}\big(p_0(t), ..., p_{d-1}(t)\big)$$

then we get the equivalent system (1.1) with $C(t) \geq 0$.

For the case $C(t) \geq 0$, we may rewrite the Riccati equation (1.3) as

(3.1)
$$\begin{aligned} R[W] &:= \Delta W(t) + A(t) + B^*(t)W(t) + W(t)B(t) - B^*(t)W(t)B(t) \\ &\quad + \big(I - B(t)\big)^* W(t)\big(I + C(t)W(t)\big)^{-1} C(t)W(t)\big(I - B(t)\big) \\ &= 0. \end{aligned}$$

Similar to Proposition 1.3 [**6**], we can show

Theorem 6. *If $C(t) \geq 0$, then, $\{Y(t), Z(t)\}$ is a prepared solution of* (1.2) *with $Y(t)$ invertible if and only if there exists a Hermitian solution of* (3.1) *on* $[M, N+1]$.

Theorem 7. *If there exists a Hermitian solution of* (3.1) *with* $(\big(I + C(t)W(t)\big)^{-1} C(t) \geq 0$, *then the functional q is positive definite.*

Proof. For any $u \in \Omega$, from Proposition 1.4 [**6**], we get

$$\begin{aligned} Q[U] = \sum_{t=M-1}^{N} &\big(W(t)y(t) - z(t)\big)^* \\ &\times \big(I + C(t)W(t)\big)^{-1} C(t)\big(W(t)y(t) - z(t)\big) \geq 0 \end{aligned}$$

and $q[u] = 0$ if and only if

$$C(t)\big(W(t)y(t) - z(t)\big) = 0$$
$$\Delta y(t) = B(t)y(t+1) + C(t)z(t)$$
$$y(M-1) = 0 = y(N+1),$$

i.e.,

$$\big(I - B(t)\big)y(t+1) = \big(I + C(t)W(t)\big)y(t)y(M-1) = 0;$$

therefore, $y(t) \equiv 0$, i.e., q is positive definite. $\square$

It is natural to ask whether there exists a Hermitian solution of (3.1) with $\big(I + C(t)W(t)\big)^{-1}C(t) \geq 0$ when q is positive definite. In the case of $C(t) > 0$, we have the conclusion "Reid Roundabout theorem" [**6**, Theorem 2.5], $W(t)$ is given by finding a solution $\{Y(t), Z(t)\}$ of (1.2). We summarize this as a conjecture:

Conjecture. *If $C(t) \geq 0$ and q positive definite, then there exists a Hermitian solution $W(t)$ of* (3.1) *such that $I + C(t)W(t))^{-1}C(t) \geq 0$.*

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Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1