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SYMMETRIC PERIODIC SOLUTIONS OF RATIONAL RECURSIVE SEQUENCES

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ABSTRACT. We consider the rational recursive sequence

$$(*)\qquad x_{n+1} = \frac{a + \sum_{i=0}^{k-1} b_i x_{n-i}}{x_{n-k}}, \qquad n = 0, \pm 1, \pm 2, \dots$$

where

$$a \in (0, \infty) \quad \text{and} \quad b_0, \dots, b_{k-1} \in [0, \infty)$$

and show that, under appropriate hypotheses, when the linearized equation

$$Ey_{n+1} + Ey_{n-k} = \sum_{i=0}^{k-1} b_i y_{n-i}, \qquad n = 0, \pm 1, \pm 2 \dots$$

about the positive equilibrium E of $(*)$ has a periodic solution with minimal period $2(k+1)$, then $(*)$ also has a periodic solution with the same minimal period.

1. Introduction. Consider the rational recursive sequence

$$(1)\qquad x_{n+1} = \frac{a + \sum_{i=0}^{k-1} b_i x_{n-i}}{x_{n-k}}, \qquad n = 0, \pm 1, \pm 2, \dots ,$$

where

$$(2)\qquad a \in (0, \infty) \quad \text{and} \quad b_0, \dots, b_{k-1} \in [0, \infty).$$

Our aim in this paper is to show that, under appropriate hypotheses, when the linearized equation

$$(3)\qquad Ey_{n+1} + Ey_{n-k} = \sum_{i=0}^{k-1} b_i y_{n-i}, \qquad n = 0, \pm 1, \pm 2, \dots$$

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about the positive equilibrium E of (1) has a periodic solution with minimal period $2(k+1)$, then (1) also has a periodic solution of the same minimal period.

A finite sequence of real numbers $\{c_l, c_{l+1}, \dots, c_{m-1}, c_m\}$ is called *symmetric* if

$$c_i = c_{l+m-i} \quad \text{for } i = l, \dots, m.$$

Throughout this paper we will assume, without further mention, that the coefficients $\{b_0, b_1, \dots, b_{k-1}\}$ form a symmetric sequence of non-negative numbers, that is

$$0 \le b_i = b_{k-1-i} \quad \text{for } i = 0, \dots, k-1 \tag{4}$$

and that the initial conditions for a solution of Equation (1) are of the form

$$x_n = \varphi_n \quad \text{for } n = 1, \dots, k+1 \tag{5}$$

where the numbers φ_n are positive and the sequence $\{\varphi_1, \dots, \varphi_{k+1}\}$ is symmetric, that is,

$$0 < \varphi_i = \varphi_{k+2-i} \quad \text{for } i = 1, \dots, k+1. \tag{6}$$

One can now show that, with such initial conditions given, (1) has a unique solution $\{x_n\}_{n=-\infty}^{\infty}$ which is positive and *symmetric* in the sense that

$$0 < x_n = x_{k+2-n} \quad \text{for } n = 0, \pm 1, \pm 2, \dots . \tag{7}$$

A sequence $\{x_n\}_{n=-\infty}^{\infty}$ is called *periodic of period* p if

$$x_{n+p} = x_n \quad \text{for } n = 0, \pm 1, \pm 2, \dots . \tag{8}$$

The least positive number p for which (8) holds is called the *minimal period* of the sequence. Equation (1) has a unique positive equilibrium, E. The equilibrium, E, satisfies the quadratic equation

$$E^2 - \left(\sum_{i=0}^{k-1} b_i\right) E - a = 0,$$

and is given by

$$E = \frac{\left(\sum_{i=0}^{k-1} b_i\right) + \sqrt{\left(\sum_{i=0}^{k-1} b_i\right)^2 + 4a}}{2}. \tag{9}$$

Equation (3) is the linearized equation of (1) about E. The main result in this paper is the following:

Theorem 1. *Assume k is odd. Suppose that the linearized equation* (3) *has a periodic solution with* (*minimal*) *period* $2(k+1)$. *Then Equation* (1) *has infinitely many symmetric periodic solutions, each with* (*minimal*) *period* $2(k+1)$ *and arbitrarily near the equilibrium* E.

The oscillation and stability of (1) was investigated in [**2**]. The periodic character of solutions of some special cases of (1) were investigated by Lyness [**3**]. See also [**1**] and [**2**].

2. A system of algebraic equations. In this section we will establish a system of algebraic equations which yields symmetric periodic solution of (1). Suppose $k = 2m-1$ is an odd number. If $\{x_n\}_{n=-\infty}^{+\infty}$ is a symmetric periodic solution of (1) with period $2(k+1) = 4m$, then

$$x_n = x_{2m+1-n} \quad \text{and} \quad x_{n+4m} = x_n \quad \text{for all } n. \tag{10}$$

Let

$$D = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ & \vdots & & & \\ 1 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

be the $m \times m$-antidiagonal matrix. Define

$$X_i = (x_{im+1}, x_{im+2}, \dots, x_{im+m})^T \in \Re^m \quad \text{for all } i, \tag{12}$$

and set

$$W = X_2 - X_0 = (x_{2m+1} - x_1, \dots, x_{3m} - x_m)^T \in \Re^m. \tag{13}$$

Then, (10) yields

$$X_1 = DX_0, \qquad X_2 = X_0 + W, \qquad X_3 = DX_2 = D(X_0 + W), \tag{14}$$

and

$$X_{4+i} = X_i \quad \text{for } i = 0, \pm 1, \pm 2, \dots,$$

where the equality $X_3 = DX_2$ is because $x_{3m+n} = x_{2m+1-3m-n} = x_{2m+m+1-n}$ for $n = 1, 2, \dots, m$. On the other hand, if X_i satisfies (14) for some $W \in \Re^m$ and $\{x_n\}_{n=-\infty}^{+\infty}$ is determined by (12), then it is easy to verify that $\{x_n\}$ satisfies (10). Now let

$$B_0 = \begin{pmatrix} 0 & b_0 & b_1 & \cdots & b_{m-3} & b_{m-2} \\ 0 & 0 & b_0 & \cdots & b_{m-4} & b_{m-3} \\ 0 & 0 & 0 & \cdots & b_{m-5} & b_{m-4} \\ & \cdots & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & b_0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_{2m-2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_{2m-3} & b_{2m-2} & 0 & \cdots & 0 & 0 & 0 \\ & \cdots & & & & \cdots & \\ b_{m+1} & b_{m+2} & b_{m+3} & \cdots & b_{2m-2} & 0 & 0 \\ b_m & b_{m+1} & b_{m+2} & \cdots & b_{2m-3} & b_{2m-2} & 0 \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} b_{m-1} & b_m & b_{m+1} & \cdots & b_{2m-3} & b_{2m-2} \\ b_{m-2} & b_{m-1} & b_m & \cdots & b_{2m-4} & b_{2m-3} \\ b_{m-3} & b_{m-2} & b_{m-1} & \cdots & b_{2m-5} & b_{2m-4} \\ & \cdots & & & \cdots & \\ b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ b_0 & b_1 & b_2 & \cdots & b_{m-2} & b_{m-1} \end{pmatrix}$$

and let $A = (a, a, \dots, a)^T \in \Re^m$. Clearly,

$$DB_0D = B_2, \qquad DB_1D = B_1 \quad \text{and} \quad DB_2D = B_0 \tag{15}$$

or, equivalently,

$$DB_0 = B_2D, \qquad DB_1 = B_1D \quad \text{and} \quad DB_2 = B_0D.$$

For any $U = (u_1, u_2, \dots, u_m)^T$ and $V = (v_1, v_2, \dots, v_m)^T$ in $\Re^m$, define

$$U * V = (u_1 v_1, u_2 v_2, \dots, u_m v_m)^T \in \Re^m. \tag{16}$$

With this notation, (1) can be rewritten in the form

$$\begin{gathered} X_{i+2} * X_i = A + B_0 X_i + B_1 X_{i+1} + B_2 X_{i+2}, \\ i = 0, \pm 1, \pm 2, \dots . \end{gathered} \tag{17}$$

For $i = 0$ and 1, Equation (17) becomes

$$\begin{aligned} (X_0 + W) * X_0 &= A + (B_0 + B_1 D + B_2) X_0 + B_2 W \\ (D(X_0 + W)) * (DX_0) &= A + B_0 D X_0 + B_1 (X_0 + W) \\ &\quad + B_2 D (X_0 + W), \end{aligned} \tag{18}$$

where the relations in (14) were used. Notice that $D^2 = I$ (identity matrix), $DA = A$, and $(DU) * (DV) = D(U * V)$ for any U, V in $\Re^m$. By multiplying by D and by using (15), the second equation in (18) becomes

$$(X_0 + W) * X_0 = A + (B_2 + B_1 D + B_0) X_0 + B_1 D W + B_0 W.$$

By subtracting this equation from the first equation in (18), Equation (18) is equivalent to the following equations

$$\begin{gathered} (X_0 + W) * X_0 = A + (B_0 + B_1 D + B_2) X_0 + B_2 W \\ \widetilde{B}_1 W = 0, \end{gathered} \tag{19}$$

where $\widetilde{B}_1 = B_0 + B_1 D - B_2$. Therefore, a symmetric periodic sequence $\{x_n\}_{n=-\infty}^{+\infty}$ with period $4m$ (that is, satisfying (10)) is a solution of (1) if and only if (17) is satisfied for $i = 0, 1, 2, 3$. When $i = 2, 3$, (17) becomes

$$\begin{aligned} X_4 * X_2 &= A + B_0 X_2 + B_1 X_3 + B_2 X_4 \\ X_5 * X_3 &= A + B_0 X_3 + B_1 X_4 + B_2 X_5. \end{aligned} \tag{20}$$

Since

$$X_4 * X_2 = X_0 * X_2 = X_2 * X_0, \qquad X_5 * X_3 = X_3 * X_1,$$

$$\begin{aligned} B_0X_2 + B_1X_3 + B_2X_4 &= B_0(X_0 + W) + B_1D(X_0 + W) + B_2X_0 \\ &= (B_0 + B_1D + B_2)X_0 + B_0W + B_1DW \end{aligned}$$

and

$$\begin{aligned} B_0X_3 + B_1X_4 + BX_5 &= B_0D(X_0 + W) + B_1X_0 + B_2DX_0 \\ &= B_0DX_0 + B_1X_0 + B_2DX_0 + B_0DW, \end{aligned}$$

we see that (20) is also equivalent to (19). In summary, we have established the following lemma.

Lemma 1. *Suppose $k = 2m - 1$ is an odd number. If $\{x_n\}_{n=-\infty}^{+\infty}$ is a symmetric periodic solution of* (1) *with period $4m$, then (X_0, W) as given by* (12) *and* (13) *is a solution of* (19). *On the other hand, if $(X_0, W) \in \Re^m \times \Re^m$ is a solution of* (19) *and X_i for $i = \pm 1, \pm 2, \dots$, and $\{x_n\}_{n=-\infty}^{+\infty}$ are given by* (14) *and* (12), *respectively, then $\{x_n\}_{n=-\infty}^{+\infty}$ is a symmetric periodic solution of* (1) *with period $4m$.*

In a similar way, consider (3), and suppose that $\{y_n\}_{n=-\infty}^{+\infty}$ is a symmetric periodic solution of (3) with period $2(k + 1) = 4m$. Define

$$Y_i = (y_{im+1}, y_{im+2}, \dots, y_{im+m})^T \in \Re^m, \qquad i = 0, \pm 1, \pm 2, \dots \tag{21}$$

and

$$V = Y_2 - Y_0. \tag{22}$$

Then (14) becomes

$$\begin{gathered} Y_1 = DY_0, \qquad Y_2 = Y_0 + V, \qquad Y_3 = D(Y_0 + V) \\ \text{and} \quad Y_{4+i} = Y_i \quad \text{for all } i. \end{gathered} \tag{23}$$

As in the above discussion with (1), (3) is reduced to the following equations:

$$\begin{aligned} E(Y_0 + V) + EY_0 &= (B_0 + B_1D + B_2)Y_0 + B_2V \\ \widetilde{B}_1V &= 0. \end{aligned}$$

That is,

$$\begin{aligned} (2EI - (B_0 + B_1D + B_2))Y_0 &= (B_2 - EI)V \\ \widetilde{B}_1V &= 0, \end{aligned} \tag{24}$$

where $\widetilde{B}_1 = B_0 + B_1 D - B_2$.

Lemma 2. *Suppose $k = 2m-1$ is an odd number. If $\{y_n\}_{n=-\infty}^{+\infty}$ is a symmetric periodic solution of* (3) *with period $4m$, then (Y_0, V) as defined by* (21) *and* (22) *is a solution of* (24). *On the other hand, if $(Y_0, V) \in \Re^m \times \Re^m$ is a solution of* (24) *and Y_i for $i = \pm 1, \pm 2, \dots,$ and $\{y_n\}_{n=-\infty}^{+\infty}$ are defined by* (23) *and* (21), *respectively, then $\{y_n\}_{n=-\infty}^{+\infty}$ is a symmetric periodic solution of* (3) *with period $4m$.*

3. The proof of Theorem 1. In view of Lemma 1, finding a symmetric periodic solution of (1) with period $2(k+1) = 4m$ is equivalent to finding a solution of (19). Since $x_n \equiv E$ is an equilibrium of (1), if $\widehat{X}_0 = (E, E, \dots, E)^T \in \Re^m$, then $(X_0, W) = (\widehat{X}_0, 0)$ is a solution of (19). It is clear that the linearization of (19) for (X_0, W) around $(\widehat{X}, 0)$ is just (24). Since (24) is equivalent to (3), by Lemma 2 we have the following result about the symmetric periodic solutions of (3).

Lemma 3. *Suppose that $a > 0$ and $b_i \geq 0$ for $i = 0, 1, 2, \dots, k-1$. Then the following statements are true*:

(i) *Equation* (3) *has a nontrivial periodic solution with period p if and only if, for some integer q, $\lambda = e^{2q\pi i/p}$, $i^2 = -1$, is a solution of the equation*

$$E(\lambda^{k+1} + 1) = \sum_{j=0}^{k-1} b_j \lambda^{k-j}. \tag{25}$$

(ii) *Equation* (3) *has no nontrivial periodic solution with period* $(k+1)$.

(iii) *If $k = 2m-1$ is odd and* (3) *has a nontrivial periodic solution with* (*minimal*) *period $4m$, then* (3) *has a nontrivial symmetric periodic solution with* (*minimal*) *period $4m$.*

Proof. (i) is obviously true because (25) is the eigen-equation of (3).

Since

$$\left|\sum_{j=0}^{k-1} b_j \lambda^{k-j}\right| \le \sum_{j=0}^{k-1} b_j \quad \text{for any } |\lambda| = 1$$

and

$$E = \frac{\sum_{j=0}^{k-1} b_j + \sqrt{(\sum_{j=0}^{k-1} b_j)^2 + 4a}}{2} > \sum_{j=0}^{k-1} b_j,$$

there is no solution of (25) satisfying $\lambda^{k+1} - 1 = 0$. Therefore, it follows from (i) that there is no nontrivial periodic solution of (3) with period $(k+1)$. Finally, if (3) has a nontrivial periodic solution of period $2(k+1) = 4m$, then it follows from (i) that there exists a solution $\lambda = e^{2q\pi i/4m}$ of (25), for some integer q. According to (ii), the integer q is odd. Since $\lambda = e^{-2q\pi i/4m}$ is also a solution of (25), $\{c\sin(nq\pi/(2m) + \theta)\}_{n=-\infty}^{+\infty}$ is a periodic solution of (3) for any fixed $c, \theta \in \Re$. In particular, if $c = 1$ and $\theta = -q\pi/2$, then $\{\sin[(2n-1)q\pi/4m]\}_{n=-\infty}^{+\infty}$ is a symmetric periodic solution of (3) with period $4m$. If the nontrivial periodic solution is of minimal period $4m$, then q is relatively prime to $4m$. Therefore, the symmetric periodic solution is of minimal period $4m$ also. The proof is complete. □

Now we are ready to establish Theorem 1.

Proof of Theorem 1. Let $k = 2m - 1$, and suppose that (3) has a nontrivial periodic solution $\{y_n\}_{n=-\infty}^{+\infty}$ with period $4m$. According to (iii) of Lemma 3, we may assume that $\{y_n\}_{n=-\infty}^{+\infty}$ is symmetric. By Lemma 2, there is a nonzero solution $(\widehat{Y}_0, \widehat{V})$ of (24) corresponding to $\{y_n\}_{n=-\infty}^{+\infty}$. Notice that if $(Y_0, 0)$ is a nonzero solution of (24), then $\{y_n\}_{n=-\infty}^{+\infty}$ as defined by (23) and (21), it is a periodic solution of (3) with period $k + 1 = 2m$. Therefore, it follows from (ii) of Lemma 3 that

$$\det(2EI - (B_0 + B_1 D + B_2)) \ne 0.$$

Consequently, the existence of $(\widehat{Y}_0, \widehat{V})$ implies that

$$\widehat{V} \ne 0 \quad \text{and} \quad \det(\widehat{B}_1) = 0.$$

By the discussion about the linearization of (19) at the beginning of this section, it follows from the implicit function theorem that there exists $\alpha_0 > 0$ and a continuous function $X_0 = X_0(\alpha)$ from $(-\alpha_0, \alpha_0)$ into $\Re^m$ such that $(X_0(\alpha), \alpha\widehat{V})$ satisfies (19) for all $\alpha \in (-\alpha_0, \alpha_0)$ and $X_0(0) = (E, E, \dots, E)^T \in \Re^m$. Moreover, one can write

$$X_0(\alpha) = (E, E, \dots, E)^T + \alpha\widehat{Y}_0 + \alpha^2\widehat{X}_0(\alpha), \tag{26}$$

where $\widehat{X}_0(\alpha)$ is a continuous function from $(-\alpha_0, \alpha_0)$ to $\Re^m$. By Lemma 1, $(X_0, W) = (X_0(\alpha), \alpha\widehat{V})$ yields a symmetric periodic solution $\{x_n(\alpha)\}_{n=-\infty}^{+\infty}$ of (1) with period $4m$ for each $\alpha \in (-\alpha_0, \alpha_0)$. Moreover, it follows from (26) that one can write

$$x_n(\alpha) = E + \alpha y_n + \alpha^2\tilde{x}_n(\alpha) \quad \text{for } n = 0, \pm 1, \pm 2, \dots$$

and

$$\alpha \in (-\alpha_0, \alpha_0)$$

where $\tilde{x}_n(\alpha)$ for $n = 0, \pm 1, \pm 2, \dots,$ are continuous functions in $\alpha \in (-\alpha_0, \alpha_0)$. Consequently, if $\{y_n\}_{n=-\infty}^{+\infty}$ is of minimal period $4m$, then $\{x_n\}_{n=-\infty}^{+\infty}$ is also of minimal period $4m$ for α near zero. The proof is complete. □

4. Examples. Before we present some examples, we obtain the following consequence of Lemma 3.

Lemma 4. *Assume that* $k = 2m - 1$. *Then* (3) *has a nontrivial* (*symmetric*) *periodic solution of period* $4m$ *if and only if the polynomials* $(\sum_{j=0}^{2m-2} b_j\lambda^{2m-1-j})$ *and* $(\lambda^{2m} + 1)$ *have a common factor. If there is a solution* $\lambda = e^{q\pi i/2m}$ *of the equation*

$$\sum_{j=0}^{2m-2} b_j\lambda^{2m-1-j} = 0$$

with q *and* $2m$ *being relatively prime, then this* λ *is a solution of* (25), *and the corresponding symmetric periodic solution of* (3) *in* (iii) *of Lemma* 3 *is of minimal period* $4m$.

Proof. According to Lemma 3, Equation (3) has a nontrivial (symmetric) periodic solution of period $4m$ if and only if there exists a solution $\lambda_0 = e^{2q\pi i/4m}$ (q integer) of (25). By (ii) of Lemma 3, q is an odd integer. Therefore, $\lambda_0^{2m} + 1 = 0$. This is equivalent to the statement that the polynomials $(\sum_{j=0}^{2m-2} b_j \lambda^{2m-1-j})$ and $(\lambda^{2m} + 1)$ have a common factor. From this equivalence, the last part of Lemma 4 is a consequence of part (iii) of Lemma 3. □

Example 1. For $k = 2m - 1 = 3$, (1) becomes

$$x_{n+1} = \frac{a + b_0 x_n + b_1 x_{n-1} + b_0 x_{n-2}}{x_{n-3}}, \quad n = 0, \pm 1, \pm 2, \dots \tag{27}$$

where $b_2 = b_0$, for the symmetry. By Lemma 4 we compare the polynomial $(b_0\lambda^3 + b_1\lambda^2 + b_0\lambda)$ and $(\lambda^4 + 1)$. Since

$$\lambda^4 + 1 = (\lambda^2 + \sqrt{2}\lambda + 1)(\lambda^2 - \sqrt{2}\lambda + 1),$$

these two polynomials have a common factor if and only if $b_1 = \sqrt{2}b_0$. $\lambda^2 + \sqrt{2}\lambda + 1 = 0$ has two solutions $\lambda = e^{3\pi i/4}$ and $\lambda = e^{5\pi i/4}$, which yield symmetric periodic solutions of (3) with minimal period 8 according to Lemma 4. Therefore, by Theorem 1 we have the following:

Theorem 2. *If $b_1 = \sqrt{2}b_0 > 0$ and $a > 0$, then there exist infinitely many symmetric periodic solutions of* (27) *with minimal period* 8 *near the positive equilibrium E of* (27).

Example 2. For $k = 2m - 1 = 5$, (1) becomes

$$x_{n+1} = \frac{a + b_0 x_n + b_1 x_{n-1} + b_0 x_{n-2}}{x_{n-3}}, \quad n = 0, \pm 1, \pm 2, \dots \tag{28}$$

where $b_3 = b_1$ and $b_4 = b_0$ for the symmetry. Since

$$\lambda^6 + 1 = (\lambda^2 + 1)(\lambda^2 + \sqrt{3}\lambda + 1)(\lambda^2 - \sqrt{3}\lambda + 1),$$

one can write

$$\begin{aligned} f(\lambda) &= b_0\lambda^5 + b_1\lambda^4 + b_2\lambda^3 + b_1\lambda^2 + b_0\lambda \\ &= \lambda[(b_0\lambda^2 + b_1\lambda + b_2 - b_0)(\lambda^2 + 1) + (2b_0 - b_2)], \end{aligned}$$

or

$$f(\lambda) = \lambda[(b_0\lambda^2 + (b_1 \mp \sqrt{3}b_0))(\lambda^2 \pm \sqrt{3}\lambda + 1) + (b_2 \mp \sqrt{3}b_1 + b_0)\lambda^2].$$

By Lemma 4, if $b_2 = 2b_0$ or $b_2 - \sqrt{3}b_1 + b_0 = 0$, then (3) has nontrivial symmetric periodic solutions of period 12. Observe that $\lambda^2 + \sqrt{3}\lambda + 1 = 0$ yields solutions $\lambda = e^{11\pi i/12}$ and $e^{13\pi i/12}$. Therefore, it follows from Lemma 4 that if $b_2 - \sqrt{3}b_1 + b_0 = 0$, then there exist periodic solutions of (3) with minimal period 12. In view of the above, we have the following result:

Theorem 3. *Assume that $b_1, b_2 \in [0, \infty)$ and $a > 0$. If $b_2 = 2b_0$ or $b_2 - \sqrt{3}b_1 + b_0 = 0$, then* (28) *has infinitely many symmetric periodic solutions of period* 12 *near the positive equilibrium E of* (28)*. More precisely, if $b_2 - \sqrt{3}b_1 + b_0 = 0$, then* (28) *has infinitely many symmetric periodic solutions, each with minimal period 12 and arbitrarily near the positive equilibrium E.*

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