ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 1, Winter 1995

# ULTIMATE BOUNDS AND GLOBAL ASYMPTOTIC STABILITY FOR DIFFERENTIAL DELAY EQUATIONS

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ABSTRACT. We use an interval mapping method to produce a sequence of improved ultimate bounds for positive solutions of differential delay equation models for population growth. We obtain a general result for global asymptotic stability of a positive equilibrium as a consequence.

**1. Introduction.** We consider the population dynamics model with (possibly varying) time delay

$$\dot{x}(t) = x(t)f(x(t-\tau(t)),t). \tag{1.1}$$

Since $x(t)$ in (1.1) represents a population density, we restrict our attention to positive solutions of (1.1). (See Lemma 1 below.) Although (1.1) does not contain all biologically relevant differential delay equation models of population growth (Cushing [**2**], Freedman and Gopalsamy [**3**], Gurney, Blythe and Nisbet [**5**]), it is sufficiently general to include, for example, the modified logistic delay equation (1.1) with

$$f(x(t-\tau(t)),t) = a + bx(t-\tau) - cx^2(t-\tau), \tag{1.2}$$

treated recently by Gopalsamy and Ladas [**4**]. In this paper our main goal is to provide new checkable conditions for global asymptotic stability of the positive equilibrium of (1.1). To achieve this goal we first extend slightly one of our recent results [**1**, Proposition 1] which gives permanence for (1.1); this extension is Theorem 2 below. We then establish a refinement (Theorems 3 and 6) of Theorem 2 in which sharper estimates for the attracting set for positive solutions are obtained. This is accomplished by producing a sequence of improved estimates for the ultimate bounds for such solutions using an interval

Received by the editors on September 28, 1992, and in revised form on February 1, 1993.
*Key words and phrases.* Differential delay equations, global asymptotic stability, population dynamics.

mapping technique. As a consequence we provide, at least in the autonomous case, checkable conditions under which the attracting set shrinks to the positive equilibrium of (1.1), and we obtain global asymptotic stability (Corollary 8). We conclude with an example that indicates that our result yields an improvement of a global asymptotic stability condition given recently by Gopalsamy and Ladas [**4**, Theorem 3.1] for (1.1), (1.2).

**2. A permanence result.** Consider the non-autonomous differential delay equation

$$\dot{x}(t) = x(t)f(x(t-\tau(t)),t), \tag{2.1}$$

where $f(x,t)$ is a continuous function and $\tau = \tau(t) > 0$ is a continuous function satisfying

$$\tau_m = \liminf_{t\to+\infty} \tau(t) \le \limsup_{t\to+\infty} \tau(t) = \tau_0. \tag{2.2}$$

for some constants $\tau_m \ge 0$ and $\tau_0 \ge 0$. Let

$$\tau_* = \sup\{\tau(t) : t \in [0,+\infty)\}. \tag{2.3}$$

Of course, $\tau_* \ge 0$ is finite. For any $t_0 \ge 0$ and $\varphi \in C([-\tau_*,0],\mathbf{R})$, there exists a $t_1 > t_0$ and a unique function $x(t)$ for $t \in [t_0-\tau_*,t_1)$ such that

$$x(t_0+\theta) = \varphi(\theta), \quad \theta \in [-\tau_*,0] \tag{2.4}$$

and $x(t)$ satisfies (2.1) for $t \in [t_0,t_1)$. It is easy to see that if $\varphi(0) > 0$ then $x(t) > 0$ for all $t \ge t_0$ for which $x(t)$ is defined. In our next result we will prove that, given any initial condition (2.4), the solution of (2.1) and (2.4) globally exists to the right.

**Lemma 1.** *Given any initial condition* (2.4)*, the solution* $x(t)$ *of* (2.1) *and* (2.4) *exists on the whole interval* $[-\tau_*+t_0,+\infty)$*. Moreover, if* $\varphi(0) > 0$*, then* $x(t) > 0$ *for all* $t \ge t_0$*.*

*Proof.* By the discussion above, we know that the solution of (2.1) and (2.4) is unique and locally exists to the right of $t_0$. Let $[-\tau_*+t_0,t_1)$

be the largest interval on which $x(t)$ exists. If $t_1 < +\infty$, then by the condition $\tau(t) > 0$ for all $t \geq 0$ there exists $t_1^* < t_1$ such that

$$t - \tau(t) \leq t_1^* \quad \text{for all } t \in [t_0, t_1]. \tag{2.5}$$

Consequently, $f(x(t-\tau(t)), t)$ is a continuous function of $t$ on the closed interval $[t_0, t_1]$. Integrating (1.1), we have

$$x(t) = \varphi(0) \exp\left\{ \int_{t_0}^{t} f(x(s - \tau(s)), s)\, ds \right\}. \tag{2.6}$$

Thus, $\lim_{t \to t_1} x(t)$ exists and is finite. By local existence, $x(t)$ can be extended to the interval $[t_1, t_2)$ for some $t_2 > t_1$. This contradicts that $[-\tau_* + t_0, t_1)$ is the maximal interval of existence for $x(t)$, and so $t_1$ cannot be finite. Thus, $x(t)$ exists on $[t_0 - \tau_*, +\infty)$. It is easy to see from (2.6) that $x(t) > 0$ for all $t \geq t_0$ if $\varphi(0) > 0$. The proof is completed. □

In the discussion of equation (2.1), the function $f(x, t)$ will be approximated by a pair of continuous functions $d(x), D(x)$ which satisfy the following conditions.

(H1) Suppose that $d(x)$ and $D(x)$ are continuous functions from $[0, +\infty)$ to $\mathbf{R}$. Assume that there exist two positive constants $\delta$ and $M$ such that

$$d(x) > 0 \quad \text{for } x \in [0, \delta), \qquad D(x) < 0 \quad \text{for } x \in (M, +\infty) \tag{2.7}$$

and

$$\liminf_{x \to +\infty} D(x) = \overline{D} < 0. \tag{2.8}$$

**Theorem 2.** *Suppose that there are two continuous functions $d(x)$ and $D(x)$ and two positive numbers $\delta$ and $M$ such that* (H1) *is satisfied. If the inequalities*

$$d(x) \leq \liminf_{t \to +\infty} f(x, t) \leq \limsup_{t \to +\infty} f(x, t) \leq D(x) \tag{2.9}$$

*are satisfied uniformly for* $x \in [0, +\infty)$*, then any positive solution* $x$ *of* (2.1) *satisfies*

$$\delta_0 \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M_0, \tag{2.10}$$

*where* $M_0 = Me^{\widetilde{D}\tau_0}$, $\widetilde{D} = \sup\{D(x) : x \in [0, M]\}$ *and* $\delta_0 = \delta e^{\tilde{d}\tau_0}$ *with* $\tilde{d} = \inf\{d(x) : x \in [0, M_0]\}$.

*Proof.* Fix $\varepsilon$ with $0 < \varepsilon < (1/2)\min\{|\overline{D}|, d(0)\}$ where $\overline{D}$ is given by (2.8). There exists $t_1^* > 0$ such that

$$\tau_m - \varepsilon \le \tau(t) \le \tau_0 + \varepsilon \quad \text{for all } t \ge t_1^* \tag{2.11}$$

and

$$\begin{gathered} d(x) - \varepsilon \le f(x,t) \le D(x) + \varepsilon \\ \text{for all } t \ge t_1^* \quad \text{and} \quad x \ge 0. \end{gathered} \tag{2.12}$$

Define

$$\begin{gathered} \tilde{\delta}^\varepsilon = \inf\{x \ge 0 : d(x) - \varepsilon < 0\}, \\ \widetilde{M}^\varepsilon = \sup\{x \ge 0 : D(x) + \varepsilon > 0\} \end{gathered} \tag{2.13}$$

It follows from the choice of $\varepsilon$ that

$$\delta \le \liminf_{\varepsilon\to0^+} \tilde{\delta}^\varepsilon \le \limsup_{\varepsilon\to0^+} \widetilde{M}^\varepsilon \le M. \tag{2.14}$$

Further define

$$\widetilde{D}^\varepsilon = \sup\{D(x) + \varepsilon : \ x \in (0, \widetilde{M}^\varepsilon]\} \tag{2.15}$$

and

$$\tilde{d}^\varepsilon = \inf\{d(x) - \varepsilon : \ x \in (0, \widetilde{M}^\varepsilon e^{\widetilde{D}^\varepsilon(\tau_0+\varepsilon)} + \varepsilon]\}.$$

Then

$$\lim_{\varepsilon\to0^+} \tilde{d}^\varepsilon = \tilde{d} \quad \text{and} \quad \lim_{\varepsilon\to0^+} \widetilde{D}^\varepsilon = \widetilde{D}. \tag{2.16}$$

With these notations established we are ready to begin the heart of the proof. Let $x$ be an arbitrary positive solution of (2.1) on the interval $[t_0 - \tau_*, +\infty)$. We divide the proof into two cases.

**Case 1.** $x(t)$ is eventually monotone. With no loss of generality $x(t)$ is monotone increasing for $t \geq t_0^* \geq \max\{t_0, t_1^*\}$. Then (2.12) and (2.13) imply

$$x(t - \tau(t)) \leq \widetilde{M}^{\varepsilon} \quad \text{for } t \geq t_0^*.$$

Thus $\lim_{t\to+\infty} x(t) > 0$ exists and $\lim_{t\to+\infty} \dot{x}(t) = 0$. From the last equality, (2.7), and (2.9) it follows that

$$\delta \leq \lim_{t\to+\infty} x(t) \leq M. \tag{2.17}$$

Notice that from (2.7) one can show that $\widetilde{D} > 0$ and $\tilde{d} < 0$. Thus (2.10) follows from (2.17) for this case.

**Case 2.** $x(t)$ is eventually oscillating. There exist sequences $t_n$, $t_n' \to +\infty$ as $n \to \infty$ such that $x(t_n)$ and $x(t_n')$ are local maxima and minima of $x(t)$, respectively, and

$$\lim_{n\to\infty} x(t_n) = \limsup_{t\to+\infty} x(t) \tag{2.18}$$

and

$$\lim_{n\to\infty} x(t_n') = \liminf_{t\to+\infty} x(t). \tag{2.19}$$

Therefore, $\dot{x}(t_n) = \dot{x}(t_n') = 0$, or equivalently,

$$f(x(t_n - \tau_n), t_n) = f(x(t_n' - \tau_n'), t_n') = 0, \tag{2.20}$$

where $\tau_n = \tau(t_n)$ and $\tau_n' = \tau(t_n')$. Without loss of generality, we can assume

$$t_n \geq t_1^* + \tau^* \quad \text{and} \quad t_n' \geq t_1^* + \tau_*, \quad \text{for all } n.$$

Thus (2.12) and (2.20) imply

$$\begin{aligned} \tilde{\delta}^{\varepsilon} &\leq x(t_n - \tau_n) \leq \widetilde{M}^{\varepsilon}, \\ \tilde{\delta}^{\varepsilon} &\leq x(t_n' - \tau_n') \leq \widetilde{M}^{\varepsilon}, \quad \text{for all } n. \end{aligned} \tag{2.21}$$

Integrating (2.1), we have, for all $n$,

$$x(t_n) = x(t_n - \tau_n) \exp \int_{t_n - \tau_n}^{t_n} f(x(s - \tau(s)), s)\, ds \leq \widetilde{M}^\varepsilon \exp(\widetilde{D}^\varepsilon(\tau_0 + \varepsilon)). \tag{2.22}$$

Thus

$$\limsup_{t \to +\infty} x(t) = \lim_{n \to +\infty} x(t_n) \leq \widetilde{M}^\varepsilon \exp(\widetilde{D}^\varepsilon(\tau_0 + \varepsilon)). \tag{2.23}$$

It follows that there exists $t_2^* \geq t_1^*$ such that

$$x(t) \leq \widetilde{M}^\varepsilon \exp(\widetilde{D}^\varepsilon(\tau_0 + \varepsilon)) + \varepsilon \quad \text{for all } t \geq t_2^*. \tag{2.24}$$

Similarly to (2.22) and (2.23) we get, for all large $n$

$$x(t_n') \geq \tilde{\delta}^\varepsilon \exp(\tilde{d}^\varepsilon(\tau_0 + \varepsilon)) \tag{2.25}$$

and

$$\liminf_{t \to +\infty} x(t) = \lim_{n \to \infty} x(t_n') \geq \tilde{\delta}^\varepsilon \exp(\tilde{d}^\varepsilon(\tau_0 + \varepsilon)). \tag{2.26}$$

Taking limits in (2.23) and (2.26) as $\varepsilon \to 0^+$, noting (2.16), we complete the proof. □

**3. Ultimate bounds and asymptotic stability.** In this section we will present improved estimates on the upper and lower bounds of positive solutions of (2.1). The estimates come from an interval mapping defined by the functions $d(x)$ and $D(x)$ satisfying (2.7)–(2.9). We now define this mapping $\mathcal{F}$. First, for any interval $[a, b] \subseteq [0, +\infty)$, let

$$d[a, b] = \min\{d(x) : a \leq x \leq b\}$$
$$D[a, b] = \max\{D(x) : a \leq x \leq b\};$$

then we define the mapping $\mathcal{F}$ by

$$\mathcal{F}([a, b]) = [\tilde{a}, \tilde{b}] \tag{3.1}$$

where

$$\tilde{a} = \delta \exp(\tau_0 d[a,b]) \tag{3.2}$$

and

$$\tilde{b} = M \exp(\tau_0 D[a,b]). \tag{3.3}$$

Here $\delta$, $M$, and $\tau_0$ are given in (2.7) and (2.2). It is clear that $\mathcal{F}$ is well-defined on the set of closed subintervals of $[0,+\infty)$, and that $\mathcal{F}$ is monotone nondecreasing: if $[a^1,b^1] \subseteq [a,b] \subseteq [0,+\infty)$,

$$\mathcal{F}([a^1,b^1]) \subseteq \mathcal{F}([a,b]). \tag{3.4}$$

We can state our main result now.

**Theorem 3.** *Assume the same conditions as in Theorem* 2*, and let* $\mathcal{F}$ *be the interval mapping defined by* (3.1)–(3.3)*. Then, there is an interval* $[\overline{\delta},\overline{M}] \subseteq (0,M_0]$ *such that*

$$\lim_{n\to\infty} \mathcal{F}^n([0,M_0]) = [\overline{\delta},\overline{M}] \tag{3.5}$$

*where* $\mathcal{F}^n$ *is the nth iteration of the mapping* $\mathcal{F}$ *and* $M_0$ *is defined in Theorem* 2*. Furthermore any positive solution* $x$ *of* (2.1) *satisfies*

$$\overline{\delta} \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to\infty} x(t) \le \overline{M}. \tag{3.6}$$

*Remark.* To see that Theorem 3 indeed improves Theorem 2, we compute $\mathcal{F}([0,M_0])$. Comparing (3.2) and (3.3) with $a=0$ and $b=M_0$ to the definitions of $\delta_0$ and $M_0$ given in Theorem 2 we have that

$$\mathcal{F}([0,M_0]) = [\delta_0, Me^{\tau_0 D[0,M_0]}] = [\delta_0, M_0]. \tag{3.7}$$

The last equality in (3.7) follows because

$$\max\{D(x) : 0 \le x \le M_0\} = \max\{D(x) : 0 \le x \le M\}$$

since by (2.9)

$$D(x) > 0, \quad \text{for } x \in [0,\delta),$$

and

$$D(x) < 0, \quad \text{for } x \in (M, M_0].$$

By the monotonicity of $\mathcal{F}$,

$$\mathcal{F}^2([0, M_0]) = \mathcal{F}([\delta_0, M_0]) \subseteq \mathcal{F}([0, M_0]) = [\delta_0, M_0]. \tag{3.8}$$

Iterating (3.8) gives the existence of the interval $[\overline{\delta}, \overline{M}]$ satisfying (3.5) and

$$[\overline{\delta}, \overline{M}] \subseteq [\delta_0, M_0]. \tag{3.9}$$

So (3.6) is at least as good as (2.10).

Further, we notice that

$$\mathcal{F}^2([0, M_0]) = \mathcal{F}([\delta_0, M_0]) = [\tilde{\delta}_0, \widetilde{M}_0]$$

where

$$\tilde{\delta}_0 = \delta \exp(\tau_0 d[\delta_0, M_0])$$

and

$$\widetilde{M}_0 = M \exp(\tau_0 D[\delta_0, M_0]).$$

Since $d[\delta_0, M_0] = d[0, M_0]$, we see that $\tilde{\delta}_0 = \delta_0$. If $D[\delta_0, M_0] = D[0, M_0]$, then we also have $\widetilde{M}_0 = M_0$, and consequently iteration of $\mathcal{F}([0, M_0])$ implies $[\overline{\delta}, \overline{M}] = [\delta_0, M_0]$. Hence, Theorem 3 does *not* yield an improvement of Theorem 2 in this case. So we have

**Corollary 4.** *The estimate* (3.6) *is sharper than* (2.10) *if and only if*

$$D[\delta_0, M_0] < D[0, M_0]. \tag{3.10}$$

Obviously (3.10) holds if $D$ is monotone decreasing. We now proceed with the proof of Theorem 3.

*Proof.* From the remark above, the interval $[\overline{\delta}, \overline{M}]$ satisfying (3.5) exists, $[\overline{\delta}, \overline{M}] \subseteq [\delta_0, M_0]$, and

$$[\overline{\delta}, \overline{M}] = \bigcap_{n=1}^{\infty} \mathcal{F}^n([0, M_0]). \tag{3.11}$$

It remains to show (3.6).

Suppose $x$ is any positive solution of (2.1), and let

$$\delta_1 = \liminf_{t\to+\infty} x(t), \qquad M_1 = \limsup_{t\to+\infty} x(t). \tag{3.12}$$

Then, by Theorem 2,

$$[\delta_1, M_1] \subseteq [\delta_0, M_0] \subseteq [0, M_0]. \tag{3.13}$$

Consider

$$\mathcal{F}([\delta_1, M_1]) = [\tilde{\delta}_1, \widetilde{M}_1];$$

by definition of $\mathcal{F}$,

$$\begin{aligned} \tilde{\delta}_1 &= \delta \exp(\tau_0 d[\delta_1, M_1]), \\ \widetilde{M}_1 &= M \exp(\tau_0 D[\delta_1, M_1]). \end{aligned}$$

Similarly to the proof of Theorem 2 (take $\overline{D}^\varepsilon = \sup\{D(x) + \varepsilon : x \in [\delta_1 - \varepsilon, M_1 + \varepsilon]\}$ and $\overline{d}^\varepsilon = \inf\{d(x) - \varepsilon : x \in [\delta_1 - \varepsilon, M_1^* + \varepsilon]\}$ where $M_1^* = M_1 \exp(\overline{D}^\varepsilon(\tau_0 + \varepsilon)))$, one can show that

$$\tilde{\delta}_1 \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le \widetilde{M}_1,$$

i.e.,

$$[\delta_1, M_1] \subseteq [\tilde{\delta}_1, \widetilde{M}_1] = \mathcal{F}([\delta_1, M_1]). \tag{3.14}$$

Since $[\delta_1, M_1] \subseteq [0, M_0]$, it follows by monotonicity that

$$\mathcal{F}([\delta_1, M_1]) \subseteq \mathcal{F}([0, M_0]). \tag{3.15}$$

By iterating (3.14) and (3.15) we have, then,

$$[\delta_1, M_1] \subseteq [\overline{\delta}, \overline{M}].$$

This completes the proof. $\square$

Theorem 3 immediately yields the following result.

**Corollary 5.** *Suppose that the same conditions as in Theorem* 2 *are satisfied. If there exists* $\bar{x}_0 \in [0, M_0]$ *such that*

$$\mathcal{F}^n([0, M_0]) \to \{\bar{x}_0\},$$

*then every positive solution* $x(t)$ *of* (2.1) *satisfies* $\lim_{t\to+\infty} x(t) = \bar{x}_0$.

**4. Global asymptotic stability for autonomous equations.** Consider the autonomous equation

$$\dot{x}(t) = g(x(t - \tau(t))) \tag{4.1}$$

where $g(x)$ and $\tau(t)$ are continuous functions for $x \geq 0$ and $t \geq 0$, respectively, and $\tau(t)$ satisfies

$$\tau_m = \liminf_{t\to+\infty} \tau(t) \leq \limsup_{t\to+\infty} \tau(t) = \tau_0 \tag{4.2}$$

for some constants $\tau_m$ and $\tau_0$. We assume

(H1′) There exist positive constants $\delta$ and $M$ such that

$$g(x) > 0 \quad \text{for } x \in [0, \delta)$$

and

$$g(x) < 0 \quad \text{for } x \in (M, +\infty).$$

For the choice $d(x) = D(x) = g(x)$, we have the conditions of Theorem 3 satisfied except for the condition $\liminf_{x\to+\infty} g(x) < 0$. One can see from the proof that Theorem 2 is valid for (4.1) if (H1) is replaced by (H1′) (choose $\varepsilon < d(0)/2$; there is no need for (2.12)–(2.14) in this case: replace $\tilde{\delta}^\varepsilon$ and $\widetilde{D}^\varepsilon$ by just $\tilde{\delta}$ and $\widetilde{D}$, respectively. See also Proposition 1 or Theorem 3 in [**1**]). The latter conditions were needed in Theorem 2 but not Theorem 3. So we have

**Theorem 6.** *Suppose* (H1′) *and the interval mapping* $\mathcal{G}$ *is defined by*

$$\mathcal{G}([a, b]) = [\tilde{a}, \tilde{b}]$$

*where*

$$\tilde{a} = \min\{\delta e^{g(x)\tau_0} : a \leq x \leq b\}$$

*and*

$$\tilde{b} = \max\{Me^{g(x)\tau_0} : a \le x \le b\}.$$

*Let* $\widetilde{G} = \sup\{g(x) : 0 \le x \le M\}$, $M_0 = Me^{\widetilde{G}\tau_0}$, $\tilde{g} = \inf\{g(x) : 0 \le x \le M_0\}$, *and* $\delta_0 = \delta e^{\tilde{g}\tau_0}$. *Then, as* $n \to \infty$, $\mathcal{G}^n([0, M_0]) \to [\overline{\delta}, \overline{M}]$ *for some interval* $[\overline{\delta}, \overline{M}] \subseteq [\delta_0, M_0]$, *and further, for any positive solution* $x$ *of* (4.1),

$$\overline{\delta} \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le \overline{M}.$$

We observe, of course, that if $\overline{\delta} = \overline{M}$ Theorem 6 yields global attractivity of the positive equilibrium $\bar{x}_0 = \overline{\delta} = \overline{M}$. Suppose now that in (H1′) $\delta = M$, so that this hypothesis can be rewritten as

(H1″) $\qquad g(x)(x - M) < 0 \quad$ for all $x \ne M$ in $(0, +\infty)$.

We consider the point mapping $G : (0, M_0] \to (0, +\infty)$ defined by $G(x) = Me^{g(x)\tau_0}$. Theorem 6, then, yields the following

**Corollary 7.** *Suppose that* (H1″) *holds, and that* $\bar{x} = M$ *is the globally asymptotically stable fixed point of the mapping* $G$. *Then* $\bar{x} = M$ *is the globally attractive equilibrium for positive solutions of* (4.1).

*Proof.* $\bar{x} = M$ is the globally asymptotically stable fixed point of $\mathcal{G}$ implies that $\mathcal{G}^n([0, M_0]) \to \{M\}$ as $n \to +\infty$, and so the corollary follows by Corollary 5. $\square$

Finally we note that the condition in Corollary 7 holds if $|G'(x)| < 1$. So we have

**Corollary 8.** *Suppose that* (H1″) *holds,* $\tau(t) \equiv \tau_0$, *for all* $t \ge 0$, *and*

$$|Mg'(x)\tau_0 e^{g(x)\tau_0}| < 1 \tag{4.3}$$

*for all* $x \in (0, M_0]$, *where* $M_0$ *is given in Theorem* 6. *Then* $\bar{x} = M$ *is the globally asymptotically stable equilibrium for positive solutions of* (4.1).

*Proof.* Since (4.3) implies $|Mg'(M)\tau_0| < 1$, the equilibrium $x = M$ is stable. So it is globally asymptotically stable for positive solutions of (4.1) by Corollary 7. (Notice that $\mathcal{G}([0, M_0])$ is a closed interval contained in $(0, M_0]$, so (4.3) is not needed at $x = 0$. Compare with (3.7).) $\square$

*Remark.* In some cases, like our next example, it is possible to verify (4.3).

**5. An example.** Consider the differential delay equation

$$\dot{x}(t) = x(t)(a + bx(t-\tau) - cx^2(t-\tau)) \tag{5.1}$$

where $a$, $b$, and $c$ are real constants with $a > 0$ and $c > 0$. Equation (5.1) has a unique positive equilibrium given by

$$x^* = (b + \sqrt{b^2 + 4ac})/(2c). \tag{5.2}$$

Let $L = 2cx^* - b = \sqrt{b^2 + 4ac}$. In [**4**], Gopalsamy and Ladas obtain the following sufficient condition for global attractivity of $x^*$ for positive solutions of (5.1)

$$[Lx^*\tau + c(x^*)^2\tau(e^{Lx^*\tau} - 1)]e^{Lx^*\tau} < 1; \tag{5.3}$$

(see [**4**, Theorem 3.1]). Inequality (5.3) implies

$$Lx^*\tau e^{Lx^*\tau} < 1, \tag{5.4}$$

and it can be checked that (5.4) implies

$$|x^*(b - 2cx)\tau e^{(a+bx-cx^2)\tau}| < 1, \quad \text{for all } x \geq 0, \tag{5.5}$$

which guarantees (4.3). So Corollary 8 gives a sharper condition, namely (5.5), for global asymptotic stability for (5.1) than the result of Gopalsamy and Ladas. (Our result does not, however, improve the condition of Wright [**6**] for the logistic delay case of (5.1) ($a > 0, b < 0, c = 0$) which is $a\tau \leq 3/2$, or the generalization given in [**7**].)

*Remark.* Finally, consider a state-dependent differential delay equation

$$\dot{x}(t) = x(t)f(x(t - \mu(x(t))), t). \tag{5.6}$$

Equation (5.6) is of the form (1.1) if we take $\tau(t) = \mu(x(t))$. Consequently, all results above can apply to equation (5.6).

## REFERENCES

**1.** Y. Cao, J.-P. Fan and T.C. Gard, *Uniform persistence for population interaction models with time delay*, Appl. Analysis **51** (1993), 197–210.

**2.** J.M. Cushing, *Volterra integrodifferential equations in population dynamics*, Proc. Centro Internationale Matematico Estivo Summer Session on Mathematics in Biology, 1979.

**3.** H.I. Freedman and K. Gopalsamy, *Global stability in time-delayed single-species dynamics*, Bull. Math. Biol. **48** (1986), 483–492.

**4.** K. Gopalsamy and G. Ladas, *On the oscillation and asymptotic behavior of* $\dot{N}(t) = N(t)[a + bN(t-\tau) - cN^2(t-\tau)]$ , Quart. Appl. Math. **48** (1990), 433–440.

**5.** W.S.C. Gurney, S.P. Blythe and R.M. Nisbet, *Nicholson's blowflies revisited*, Nature **287** (1980), 17–21.

**6.** E.M. Wright, *A non-linear difference-differential equation*, J. Reine Angew. Math. **494** (1955), 66–87.

**7.** B.G. Zhang and K. Gopalsamy, *Global Attractivity and oscillations in a periodic delay-logistic equation*, J. Math. Anal. Appl. **150** (1990), 274–283.

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