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A NEW ANGLE ON STURM-LIOUVILLE PROBLEMS

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Introduction. The problem under study takes the form

$$-(py')' + qy = \lambda ry, \tag{1}$$

where

$$p > 0, \quad r > 0, \quad 1/p, q, r \in L_1([0,1]), \mathbf{R}),$$

subject to boundary-conditions of the form

$$y(0)\cos\beta_0 = (py')(0)\sin\beta_0, \qquad 0 \le \beta_0 < \pi \tag{2}$$

and

$$(a\lambda + b)y(0) = (c\lambda + d)(py')(0), \tag{3}$$

where

$$0 \ne (a,b,c,d) \in \mathbf{R}^4 \quad \text{and} \quad e = ad - bc. \tag{4}$$

Extensive bibliographies for this problem can be found in Walter [**8**] and Fulton [**5**]. Most of the cited work deals with completeness and expansion theory in $L_2[0,1] \oplus \mathbf{C}$. Here we consider Sturm (oscillation, comparison, etc.) theory for three cases:

(I) $c = 0 \ne d$, $e \ge 0$ (also discussed by Reid [**7**] via different methods);

(II) $c \ne 0$, $e > 0$ (joint work with P.J. Browne and K. Seddighi [**4**]);

(III) $c \ne 0$, $e < 0$ (joint work with P.J. Browne).

Further details for these and other cases (e.g., with both end conditions λ dependent, indefinite r, etc.) will appear elsewhere.

(I) *The simplest case.* We remark that this case includes the Sturm-Liouville (λ-independent end condition) one where $a = 0$. We define θ by means of the differential equation

$$\theta' = (1/p)\cos^2\theta + (\lambda r - q)\sin^2\theta$$

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with initial condition $\theta = \beta_0$, see (2). Thus, $\cot\theta = py'/y$, which is essentially the definition given by Prüfer [**6**]. Let $f(\lambda) = \cot\theta(\lambda, 1)$, and define $\lambda_{-1}^D = -\infty$, with λ_n^D, $n = 0, 1, 2, \dots$, as the eigenvalues of the Sturm-Liouville problem (1), (2) and the Dirichlet condition $y(1) = 0$.

Theorem 1. *The graph of f has countably many branches $B_0, B_1, \dots$, on each of which f decreases strictly and continuously. Indeed, B_n corresponds to $\lambda_{n-1}^D < \lambda < \lambda_n^D$ and $\lim_{\lambda\downarrow\lambda_{n-1}^D} f(\lambda) = -\infty$, $\lim_{\lambda\uparrow\lambda_n^D} f(\lambda) = +\infty$.*

This follows from standard results (see Atkinson [**1**, Section 8.4] for the L_1 coefficient assumptions used here).

Since (3) may be written $\cot\theta(\lambda, 1) = g(\lambda) := (a\lambda/d) + (b/d)$ and $e \geq 0$ ensures $a/d \geq 0$, we have the following result for the eigenvalues λ_n, $n = 0, 1, 2, \dots$, of our problem (1)–(3).

Theorem 2. (i) *Interlacing.* $\lambda_{n-1}^D < \lambda_n < \lambda_n^D$, $n = 0, 1, 2, \dots$.

(ii) *Oscillation. λ_n corresponds to a unique (up to scaling) eigenfunction y_n with n zeros in $]0, 1[$.*

(iii) *Asymptotics. $\lambda_n = (n\pi/\sigma)^2 + o(n^2)$ as $n \to \infty$, where $\sigma = \int_0^1 (r/p)^{1/2}$.*

(iv) *Dependence. If a and d are constant, while $-\beta_0$, $-bd$, p and q are nondecreasing (respectively continuous) in a parameter t, then each λ_n is nondecreasing (respectively continuous) in t.*

Sketch Proof. (i) The graph of g meets each branch B_n precisely once.

(ii) Each point on B_n corresponds to $n\pi < \theta(\lambda, 1) < (n+1)\pi$.

(iii) Use (i) and $\lambda_n^D = (n\pi/\sigma)^2 + o(n^2)$, established in [**2**].

(iv) θ is nonincreasing, so $f(\lambda)$ is nondecreasing, while $g(\lambda)$ is nonincreasing, in t. □

Remark. If $pr \in AC([0, 1])$, then (iii) may be improved to give an estimate as far as the constant term. For $p = r \equiv 1$, this result is due to Fulton [**5**].

(II) *A less simple case.* Recall that now $c \neq 0$. We introduce the new angle

$$\theta^- = \theta - \gamma \quad \text{where} \quad \gamma = \cot^{-1}(a/c) \in \,]0, \pi[\,.$$

Qualitatively, θ^- behaves like θ except that $\lim_{\lambda \to -\infty} \theta^-(\lambda, 1) = -\gamma$. Thus, the graph of $f^- : \lambda \to \cot \theta^-(\lambda, 1)$ resembles that of f except that the left hand branch has a horizontal asymptote. To consider the vertical asymptotes, we define λ_n^A as the eigenvalues of (1), (2) and the asymptotic condition $ay(1) = c(py')(1)$ which is obtained from (3) by formally dividing by λ and then setting $\lambda = \infty$.

Theorem 3. *Theorem* 1 *holds for* f^- *except that the left hand branch has a horizontal asymptote at* $-a/c$*, and the vertical asymptotes are at* λ_n^A*,* $n = 0, 1, \ldots$.

This follows from the fact that $\theta = \gamma + n\pi \Leftrightarrow \cot\theta = \cot\gamma = a/c$.

Remark 4. In case (I), B_n corresponds to n internal zeros for the corresponding eigenfunctions. Now, however, the n^{th} branch B_n^- corresponds to $n\pi < \theta^- < (n+1)\pi$. B_n^- intersects the horizontal asymptote for B_0^- where $\cot\theta^- = -a/c$, i.e., where $\theta = n\pi$, so $y(1) = 0$. Thus, above (respectively, below) this asymptote, B_n^- corresponds to eigenfunctions with $n - 1$ (respectively, n) internal zeros.

We are now in a position to give the analogue of Theorem 2.

Theorem 5. *Theorem* 2 *holds with the following modifications*:

(i) $\lambda_n \in I_n := \left]\lambda_{n-1}^A, \lambda_n^A\right[$, $n = 0, 1, 2, \ldots,$

(ii) y_n *has* n *zeros if* $n \leq N$ *and* $n - 1$ *zeros if* $n > N$*, where* N *is defined by* $\lambda_{n-1}^D < -d/c \leq \lambda_n^D$,

(iii) *is unchanged*,

(iv) *replace "a and d" by "a, c and e" and "$-bd$" by "$-(ab + cd)$."*

Sketch Proof. (i) The graph of

$$g^- : \lambda \to \cot\theta^-(\lambda, 1) = e^{-1}[(a^2 + c^2)\lambda + ab + cd] \tag{5}$$

intersects that of f^- precisely once on each branch.

(ii) This follows from Remark 4 and the fact that the graph of g^- meets the horizontal asymptote of the graph of f^- at $\lambda = -d/c$, i.e., $g^-(-d/c) = -a/c$.

(iv) Again $f^-(\lambda)$ is nondecreasing and $g^-(\lambda)$ (5) is nonincreasing in t. □

Remark 6. As in (I), (iii) may be improved when $pr \in AC[0,1]$. In fact then one obtains the extremely accurate estimate $\lambda_n = \lambda_{n-1}^A + O(n^{-2})$. Again the expansion up to the constant term is due to Fulton [**5**] in case $p = r \equiv 1$.

(III) *A more difficult case.* By scaling, if necessary, we can assume that $e = -1$. We then define inner product spaces $H_\pm = L_2[0,1] \oplus \mathbf{C}$ where L_2 is weighted by r. Specifically, if $Y = (y, y_1) \in H_\pm$ where $y \in L_2[0,1]$ and $y_1 \in \mathbf{C}$, then we have

$$||Y||_\pm^2 = \int_0^1 r|y|^2 \pm |y_1|^2. \tag{6}$$

It follows that H_+ (respectively H_-) is a Hilbert (respectively Pontryagin) space. On H_+ we define the bounded symmetric involution R by $R : (y, y_1) \to (y, -y_1)$.

Following ideas of Walter [**8**] and Fulton [**5**], we can define an operator $A : (y, y_1) \to (r^{-1}(-(py')' + qy),\ d(py')(1) - by(1))$ on a domain guaranteeing that the equation $AY = \lambda Y$ is equivalent to our problem (1)–(3). Specifically, $D(A)$ consists of those (y, y_1) for which the above expression for $A(y, y_1)$ makes sense as an element of H_+, and for which $y_1 = ay(1) - c(py')(1)$. In the case $e = +1$, A is self-adjoint, bounded below with compact resolvent on H_+, and this is the chief tool in [**8**] and [**5**].

Here (with $e = -1$) we have a similar behavior in H_-. Actually, it is more convenient to rewrite $AY = \lambda Y$ in the equivalent form $RAY = \lambda RY$ and then to prove (essentially as in [**5**]) that RA is self-adjoint, bounded below with compact resolvent in H_+. We are now in a position to study the (λ, μ) eigenvalues (cf. [**3**]) for

$$RAY + \mu Y = \lambda RY. \tag{7}$$

Splitting (7) into components, we obtain

$$-(py')' + qy = (\lambda - \mu)ry, \qquad by(0) = d(py')(0)$$

and

$$(a(\lambda + \mu) + b)y(1) = (c(\lambda + \mu) + d)(py')(1).$$

Thus, λ has been replaced by $\lambda - \mu$ in (1) (and (2)) and by $\lambda + \mu$ in (3). Comparing this with Π, we see that the eigenvalues λ_n correspond to the intersections of the (translated) graphs of $f^-(\lambda - \mu)$ and $g^-(\lambda + \mu)$.

More specifically, a continuous dependence argument (based on varying μ) shows that if (λ, μ) is on the nth (variational) eigencurve for (7) then $f_n^-(\lambda - \mu) = g^-(\lambda + \mu)$ (where f_n^- is the restriction of f^- to the nth branch) and vice-versa. We may now apply the results of [**3**] to conclude the following modifications of Theorem 5.

(i) All but two of the eigenvalues may be indexed λ_n, where $\lambda_n \in I_n$, $n = 1, 2, \dots$. Both the other two, say $\tilde{\lambda}_j$, $j = 1, 2$, are either in the same I_M where $M \geq 0$ or else form a nonreal conjugate pair.

(ii) y_n has $n - 1$ zeros if $n < N$ and n zeros if $n > N$. y_N has $N - 1$ (respectively N) zeros if $\lambda_N \leq$ (respectively greater than) $-d/c$, and $\tilde{y}_j$ (corresponding to $\tilde{\lambda}_j$) have $M - 1$ or M zeros depending on whether $\tilde{\lambda}_j \leq$ or $> -d/c$.

(iii) The asymptotics (including Remark 6) remain unchanged.

(iv) If we change $-(ab + cd)$ to $ab + cd$ then Theorem 5(iv) holds locally (i.e., for a sufficiently small t interval) for all λ_n, provided we index λ_M so that $||y_M||_-^2 \geq 0$, see (6). (It can be shown that this is automatic if $\tilde{\lambda}_j$ are nonreal, and if they are real then at most one of λ_M, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ has an eigenfunction y satisfying $||y||_-^2 < 0$).

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