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# OSCILLATORY PROPERTIES OF SOLUTIONS AND NONLINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

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To the memory of Geoffrey J. Butler
and to the dedication of his colleagues in Edmonton

**1. Introduction.** On September 3, 1979, returning from a meeting in Szeged, Geoffrey J. Butler visited Louvain-la-Neuve and delivered a lecture on *The Poincaré-Birkhoff theorem and periodic solutions of second order nonlinear differential equations.* In his talk, Butler described some of his recent work on the use of the Poincaré-Birkhoff fixed point theorem for the obtention of nontrivial $T$-periodic solutions of 'unforced' second order differential equations of the form

$$x'' + f(t,x) = 0,$$

when $f(t,0) = 0$ for all $t$ and $f$ is superlinear with respect to $x$, i.e.,

$$\frac{f(t,x)}{x} \longrightarrow +\infty \quad \text{as } |x| \longrightarrow \infty.$$

He conjectured in his lecture that this approach could be modified to be applicable to the 'forced' case, i.e., when $f(t,0) \not\equiv 0$, but never published any further paper about this conjecture (which is explicitly mentioned in [**6**]). His results on the unforced case [**4, 5**] make an extensive use of the *oscillatory* properties of the solutions of the differential equation, a topic to which Butler has masterly contributed in several directions (see, e.g., the analysis in [**13**]).

It may therefore be appropriate to show in these Butler lectures how the use of oscillatory properties of the solutions of ordinary differential equations may help in proving the existence of periodic solutions for ordinary differential equations. The recent results described here will, hopefully, convince the reader how fruitful and vivid are the ideas that

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Butler already used in the seventies, especially when combined with some recent techniques and some seminal ideas of Opial and Fučik, two other mathematicians whose untimely death is still in our memories.

The direct study of the oscillatory properties of ordinary differential equations appears at the very beginning of the qualitative approach to differential equations. Indeed, when the mathematicians realized that the explicit solution of ordinary differential equations was either impossible or would lead to the creation and study of new classes of transcendental functions, they turned their efforts to another way, which consisted in getting qualitative properties of the solutions from the differential equation itself, without any explicit knowledge of the solutions. Sturm in 1836 was the first to use this methodology for second order linear differential equations, and he obtained information about the oscillatory properties of the nontrivial solutions of those equations (e.g., the number and position of their zeros, the maxima and the minima, etc.) *by the sole consideration of the differential equation itself, without needing to integrate it.* After the pioneering work of Dirichlet in 1846 on the stability of motion, Poincaré extended systematically this approach around 1880 to nonlinear differential equations and the qualitative method has since been of fundamental importance in the theory of ordinary differential equations.

In what follows, we shall show how the combination of the use of *oscillatory* properties of solutions with *topological, symplectic* and *variational* techniques provides sharp results about the existence and the number of periodic solutions of some forced second order ordinary differential equations.

**2. The Leray-Schauder continuation theorem revisited.** Let $X$ and $Z$ be real vector normed spaces, $L : D(L) \subset X \to Z$ a linear Fredholm mapping of index zero and $N : X \times [0,1] \to Z$ a mapping which is $L$-completely continuous (see [**24**] or [**25**] for the corresponding definitions). A version of the Leray-Schauder continuation theorem states that if there exists a bounded open set $\Omega \subset X$ such that the following conditions hold:

$$Lu \neq N(u,\lambda) \quad \text{for each } (u,\lambda) \in \partial\Omega \times [0,1],$$
$$D_L(L - N(.,0), \Omega) \neq 0,$$

where $D_L(L - N(.,0), \Omega)$ denotes the coincidence degree of $L - N(.,0)$

with respect to $\Omega$ (see [**24**] or [**25**] for the corresponding definition in terms of the Leray-Schauder topological degree), then the equation

$$Lu = N(u, \lambda)$$

has at least one solution in $\Omega$ for each $\lambda \in [0, 1]$.

The difficulty in applying such a theorem lies, of course, in finding an open bounded set $\Omega$ satisfying the two conditions above, and any alternative formulation of those conditions may be of interest in applications. To this effect, let us define the function $\varphi : X \times [0, 1] \to \mathbf{R}$ by

$$\varphi(u, \lambda) = -\text{dist}\,(u, \partial\Omega) \quad \text{if } u \in \Omega,$$
$$\varphi(u, \lambda) = \text{dist}\,(u, \partial\Omega) \quad \text{if } u \notin \Omega,$$

so that $\varphi(u, \lambda) = 0$ if and only if $u \in \partial\Omega$. If we set

$$\Sigma = \{(u, \lambda) \in D(L) \times [0, 1] : Lu = N(u, \lambda)\},$$

and, for each $\lambda \in [0, 1]$,

$$\Sigma_\lambda = \{u \in D(L) : (u, \lambda) \in \Sigma\},$$

then $\varphi$ is continuous on $X \times [0, 1]$, proper on $\Sigma$ and bounded below by $-\text{diam}\,\Omega$. If we assume that $\Sigma_0 \subset \Omega$, it is easy to check that, under the assumptions of the Leray-Schauder continuation theorem above, we have

$$-\text{diam}\,\Omega \le \varphi_- := \inf_{\Sigma_0} \varphi \le \varphi_+ := \sup_{\Sigma_0} \varphi < 0,$$

and hence, by taking $c_- < -\text{diam}\,\Omega$ and $c_+ = 0$, we see that, when $\Sigma_0 \subset \Omega$, the assumptions of the Leray-Schauder theorem imply the existence of a continuous mapping $\varphi : X \times [0, 1] \to \mathbf{R}$ proper on $\Sigma$ and of numbers

$$c_- < \varphi_- \le \varphi_+ < c_+,$$

such that $\varphi(u, \lambda) \notin \{c_-, c_+\}$ for $(u, \lambda) \in \Sigma$. It has been shown in [**7**] that, in some sense, the converse is also true, in that the existence of such a functional $\varphi$ and of numbers $c_-$, $c_+$ verifying the above conditions implies the solvability of the equation when the nonzero degree condition holds.

**Lemma 1.** *Assume that* $\Sigma_0$ *is bounded, that* $D_L(L-N(.,0),\Omega_0) \neq 0$ *for some open bounded set* $\Omega_0 \supset \Sigma_0$ *and that there exist a continuous functional* $\varphi : X \times [0,1] \to \mathbf{R}$ *which is proper on* $\Sigma$ *and real numbers*

$$c_- < \varphi_- \leq \varphi_+ < c_+,$$

*such that* $\varphi(u,\lambda) \notin \{c_-, c_+\}$ *for all* $(u,\lambda) \in \Sigma$. *Then the equation*

$$Lu = N(u,\lambda)$$

*has at least one solution for each* $\lambda \in [0,1]$.

*Proof.* If

$$\mathcal{A} = \varphi^{-1}(]c_-, c_+[) \subset X \times I,$$

and

$$\Sigma^* = \varphi^{-1}([c_-, c_+]) \cap \Sigma = \varphi^{-1}(]c_-, c_+[) \cap \Sigma,$$

then $\Sigma^*$ is compact, $\mathcal{A}$ is open and $\Sigma^* \subset \mathcal{A}$. Consequently, there exists an open bounded set $\mathcal{B}$ such that

$$\Sigma^* \subset \mathcal{B} \subset \overline{\mathcal{B}} \subset \mathcal{A},$$

which easily implies that, for every $\lambda \in [0,1]$, $Lu \neq N(u,\lambda)$ for each

$$u \in (\partial\mathcal{B})_\lambda = \{u \in X : (u,\lambda) \in \partial\mathcal{B}\}.$$

Therefore, for each $\lambda \in [0,1]$, one has

$$\begin{aligned} D_L(L-N(.,\lambda),\mathcal{B}_\lambda) &= D_L(L-N(.,\lambda),\mathcal{B}_0) \\ &= D_L(L-N(.,0),\Omega_0) \neq 0, \end{aligned}$$

where $\mathcal{B}_\lambda = \{u \in X : (u,\lambda) \in \mathcal{B}\}$, and the result follows from the existence property of the degree. □

A special case of this result which is well suited to deal with applications to differential equations whose solutions have some oscillatory properties is the following one, whose proof is given in [**7**].

**Corollary 1.** *Assume that* $\Sigma_0$ *is bounded, that* $D_L(L-N(.,0),\Omega_0) \neq 0$ *for some open bounded set* $\Omega_0 \supset \Sigma_0$ *and that there exist a continuous* *functional* $\varphi : X \times [0,1] \to \mathbf{R}$ *and a number* $R > 0$ *such that* $\varphi(\Sigma \backslash (B(R) \times [0,1])) \subset \mathbf{N}$ *and such that* $\varphi^{-1}(n) \cap \Sigma$ *is bounded for each* $n \in \mathbf{N}$*. Then the equation*

$$Lu = N(u, \lambda)$$

*has at least one solution for each* $\lambda \in [0,1]$*.*

**3. A functional for planar systems.** In this section we shall introduce a functional $\varphi$ which is particularly appropriate for the study of periodic solutions of planar systems of the form

$$u'(t) = f(t, u(t), \lambda), \tag{1}$$

where

$$f : \mathbf{R} \times \mathbf{R}^2 \times [0,1] \to \mathbf{R}^2, (t, u, \lambda) \mapsto f(t, u, \lambda)$$

is $T$-periodic with respect to $t$ and continuous. We set

$$\omega = \frac{2\pi}{T}, \qquad \delta(u) = \min\left(1, \frac{1}{|u|}\right), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

(symplectic matrix),

$$C_T = \{u : \mathbf{R} \to \mathbf{R}^2 : u \text{ is continuous and } T\text{-periodic}\},$$

with the uniform norm $||u|| = \max_{t \in \mathbf{R}} |u(t)|$, where $|v|$ denotes the Euclidean norm of $v \in \mathbf{R}^2$ and $(v|w)$ the corresponding inner product. We define $\varphi : C_T \times [0,1] \to \mathbf{R}$ by

$$\varphi(u, \lambda) = \left| \frac{1}{2\pi} \int_0^T (f(t, u(t), \lambda) | Ju(t)) \delta(u(t))\, dt \right|.$$

It is easy to show that $\varphi$ is continuous and, if

$$\Sigma = \{(u, \lambda) \in C_T \times [0,1] : u \text{ is a } T\text{-periodic solution of (1)}\},$$

then, for each $T$-periodic solution $u$ of (1) such that $\min_{t \in \mathbf{R}} |u(t)| \geq 1$, $\varphi(u, \lambda)$ denotes the absolute value of the winding number of the closed curve

$$(u(t), u'(t)), \;\; (T \in [0, T])$$

around the origin and is therefore a nonnegative integer.

The following estimates about the oscillatory character of the periodic solutions of (1) are proved in [**7**].

**Lemma 2.** *If there exists $\gamma \geq 0$ and a positive definite and positive homogeneous mapping of degree two $S : \mathbf{R}^2 \to \mathbf{R}$ such that either*

$$(f(t,u,\lambda) \mid Ju) \geq S(u) - \gamma|u|, \tag{2}$$

*or*

$$(f(t,u,\lambda) \mid Ju) \leq -S(u) + \gamma|u|, \tag{3}$$

*for all $u \in \mathbf{R}^2$ and $\lambda \in [0,1]$, then there exists $R_1 \geq 1$ such that, for each $(u,\lambda) \in \Sigma$ with $\min_{\mathbf{R}} |u(t)| \geq R_1$, one has*

$$\varphi(u,\lambda) \geq \frac{1}{\omega\langle 1/S\rangle},$$

*where*

$$\left\langle \frac{1}{S} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{S(\cos\theta, \sin\theta)}.$$

**Lemma 3.** *If there exists $\gamma \geq 0$ and a positive definite and positively homogeneous mapping of degree two $S : \mathbf{R}^2 \to \mathbf{R}$ such that either*

$$-\gamma|u| \leq (f(t,u,\lambda) \mid Ju) \leq S(u) + \gamma|u|, \tag{4}$$

*or*

$$\gamma|u| \geq (f(t,u,\lambda) \mid Ju) \geq -S(u) - \gamma|u|, \tag{5}$$

*for all $u \in \mathbf{R}^2$ and $\lambda \in [0,1]$, then there exists $R_1 \geq 1$ such that for each $(u,\lambda) \in \Sigma$ with $\min_{\mathbf{R}} |u(t)| \geq R_1$, one has*

$$\varphi(u,\lambda) \leq \frac{1}{\omega\langle 1/S\rangle}.$$

The following variant of Lemma 3 is useful in problems with jumping nonlinearities and one-sided growth restrictions.

**Lemma 4.** *If there exists $R > 0$, $\gamma \geq 0$ and a positive definite and positively homogeneous mapping of degree two $S : \mathbf{R}^2 \to \mathbf{R}$ such that either*

$$0 < (f(t,u,\lambda) \mid Ju) \leq S(u) + \gamma|u|, \tag{6}$$

*or*

$$0 > (f(t,u,\lambda) \mid Ju) \geq -S(u) - \gamma|u|, \tag{7}$$

*for all $u \in \mathbf{R}^2$ with $|u| \geq R$ and $u_1 \geq 0$ and all $\lambda \in [0,1]$, then there exists $R_1 \geq 1$ such that, for each $(u,\lambda) \in \Sigma$ with $\min_{\mathbf{R}} |u(t)| \geq R_1$, one has*

$$\varphi(u,\lambda) \leq \frac{2}{\omega\{1/S\}},$$

*where*

$$\left\{\frac{1}{S}\right\} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{S(\cos\theta, \sin\theta)}.$$

*Proof.* If we write $u_1 = \rho\cos\theta$, $u_2 = \rho\sin\theta$, and if $|u(t)| \geq \max(A,R)$ with $A \geq 1$, then

$$\rho^2(t)\theta'(t) = (f(t,u(t),\lambda) \mid Ju(t)) = (f(t,u(t),\lambda) \mid Ju(t))\delta(u(t)),$$

and hence, on $\{t \in [0,T] : u_1(t) \geq 0\}$, one has if we assume, say, that (6) holds,

$$0 < \theta'(t) \leq S(\cos\theta(t), \sin\theta(t)) + \gamma/A,$$

and hence

$$0 < \int_{\{t\in[0,T]:u_1(t)\geq 0\}} \frac{\theta'(t)}{S(\cos\theta(t),\sin\theta(t))}\,dt \leq \left(1 + \frac{\gamma}{\sigma A}\right)T,$$

where $\sigma = \min_{S^1} S$. Now, if

$$\{t \in [0,T] : u_1(t) \geq 0\} = \bigcup_{j=1}^{k} [\tau_j, t_j],$$

we have necessarily $k = \varphi(u,\lambda)$ and

$$\int_{\tau_j}^{t_j} \frac{\theta'(t)}{S(\cos\theta(t),\sin\theta(t))} = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{S(\cos\theta,\sin\theta)} = \pi\left\{\frac{1}{S}\right\},$$

for each $1 \leq j \leq k$. Consequently, one has

$$0 < \sum_{j=1}^{\varphi(u,\lambda)} \int_{\tau_j}^{t_j} \frac{\theta'(t)}{S(\cos\theta(t),\sin\theta(t))}\,dt = \varphi(u,\lambda)\pi\left\{\frac{1}{S}\right\} \leq \left(1+\frac{\gamma}{\sigma A}\right)T.$$

This implies immediately that

$$0 < \varphi(u,\lambda) \leq \frac{2}{\omega\{1/S\}}\left(1+\frac{\gamma}{\sigma A}\right),$$

and hence, if we take $A = R_1$ so large that

$$\frac{2}{\omega\{1/S\}}\left(\frac{\gamma}{\sigma A}\right) < \frac{1}{2},$$

and use the fact that $\varphi(u,\lambda)$ is an integer, we obtain the result. □

The continuation theorem of the previous section can be applied to the study of $T$-periodic solutions of the planar system (1). Setting

$$X = Z = C_T = \{u : \mathbf{R} \to \mathbf{R}^2 : u \text{ is continuous and } T\text{-periodic}\},$$

$D(L) = C_T \cap C^1(\mathbf{R},\mathbf{R}^2)$, $Lu = u'$, $N(u,\lambda) = f(.,u(.),\lambda)$, it is standard to check that $L$ is Fredholm of index zero and $N$ is $L$-completely continuous. We therefore obtain the following continuation theorem whose proof can be found in [**7**], where $\varphi$ is the functional defined above.

**Lemma 5.** *Assume that the following conditions are satisfied.*

(H1) $f(t,u,0) = f_0(u)$ *and there exists an* $r_0 > 0$ *such that every possible* $T$*-periodic solution of*

$$u'(t) = f_0(u(t)) \tag{8}$$

*satisfies the inequality* $||u|| < r_0$.

(H2) *The Brouwer degree* $d_B(f_0, B(r_0), 0)$ *is different from zero.*

(H3) *For each* $r_1 > 0$ *there exists* $r_2 \geq r_1$ *such that for each possible* $T$*-periodic solution* $u$ *of* (1) *with* $\min_{t\in\mathbf{R}} |u(t)| \leq r_1$, *one has* $||u|| \leq r_2$.

(H4) *For each* $n \in \mathbf{N}$, *there exists* $K_n \geq 0$ *such that for each possible* $T$*-periodic solution* $u$ *of* (1) *with* $\varphi(u, \lambda) = n$, *one has* $\min_{t\in\mathbf{R}} |u(t)| \leq K_n$.

*Then* (1) *has at least one solution for each* $\lambda \in [0, 1]$.

We shall study the $T$-periodic solutions of second order differential equations of the form

$$x''(t) + g(x(t)) = e(t) \tag{9}$$

where $g : \mathbf{R} \to \mathbf{R}$ is continuous and $e : \mathbf{R} \to \mathbf{R}$ is continuous and $T$-periodic. We shall apply Lemma 5 to the equivalent planar system

$$u_1' = u_2, \qquad u_2' = -g(u_1) + e(t), \tag{10}$$

so that

$$f(t, u, \lambda) = (u_2, -g(u_1) + \lambda e(t)),$$

and $f_0(u) = f(t, u, 0) = (u_2, -g(u_1))$. We also have

$$(f(t, u, \lambda) \mid Ju) = -[g(u_1)u_1 + u_2^2] + \lambda e(t)u_1.$$

We set $G(u) = \int_0^u g(s)\, ds$.

**Lemma 6.** *Assume that* $G(x) \to +\infty$ *as* $|x| \to \infty$. *Then condition* (H3) *of Lemma* 5 *holds.*

*Proof.* Let $G_0 = \min_{\mathbf{R}^2} G$ and define $V : \mathbf{R}^2 \to \mathbf{R}$ by $V(u) = G(u_1) - G_0 + (u_2^2 + 1)/2$, so that, by assumption, $V(u) \to +\infty$ as $|u| \to \infty$ and

$$(V'(u) \mid f(t, u, \lambda)) = \lambda e(t)u_2 \leq \max_{t\in\mathbf{R}} |e(t)| V(u)|.$$

Hence property (H3) follows from a classical differential inequality argument (see, e.g., [**24**, Proof of Theorem VI.2]). □

**4. Second order equations with jumping nonlinearities.** We shall first consider the case where $g$ has at most a linear growth and may have a different asymptotic behavior at $+\infty$ and $-\infty$ (*jumping* or *asymmetric* nonlinearities).

**Theorem 1.** *Assume that there exist $K \geq 0$ and positive constants $A_+ \leq B_+$ and $A_- \leq B_-$ such that*

$$A_+ x^2 - K|x| \leq x g(x) \leq B_+ x^2 + K|x| \quad \textit{for } x \geq 0, \tag{11}$$

*and*

$$A_- x^2 - K|x| \leq x g(x) \leq B_- x^2 + K|x| \quad \textit{for } x \leq 0, \tag{12}$$

*for all $x \in \mathbf{R}$. If*

$$\left[\frac{2}{1/\sqrt{A_+} + 1/\sqrt{A_-}}, \frac{2}{1/\sqrt{B_+} + 1/\sqrt{B_-}}\right] \cap \omega \mathbf{N} = \varnothing, \tag{13}$$

*then equation* (9) *has at least one $T$-periodic solution.*

*Proof.* We apply Lemma 5 to the equivalent planar system (10). By our assumptions (11) or (12) we have easily that $G(u) \to +\infty$ as $|u| \to \infty$ and hence condition (H3) of Lemma 5 follows from Lemma 6. If we define the positive definite and positively homogeneous of degree two mappings $S_1$ and $S_2$ by

$$S_1(u) = A_+(u_1^+)^2 + A_-(u_1^-)^2 + u_2^2,$$
$$S_2(u) = B_+(u_1^+)^2 + B_-(u_1^-)^2 + u_2^2,$$

we easily check that

$$-S_2(u) - M|u| \leq (f(t,u,\lambda) \mid Ju) \leq -S_1(u) + M|u|, \tag{14}$$

for $M = L + \max_{t \in \mathbf{R}} |e(t)|$, and hence Lemmas 2 and 3 imply the existence of $R_1 > 0$ such that for each possible $T$-periodic solution of (10) such that $\min_{t \in \mathbf{R}} |u(t)| \geq R_1$, one has

$$\frac{1}{\omega \langle 1/S_1 \rangle} \leq \varphi(u, \lambda) \leq \frac{1}{\omega \langle 1/S_2 \rangle}.$$

Now,

$$\left\langle \frac{1}{S_1} \right\rangle = \frac{1}{2}\left( \frac{1}{\sqrt{A_+}} + \frac{1}{\sqrt{A_-}} \right),$$
$$\left\langle \frac{1}{S_2} \right\rangle = \frac{1}{2}\left( \frac{1}{\sqrt{B_+}} + \frac{1}{\sqrt{B_-}} \right),$$

and, from assumption (13), we see that, for each $n \in \mathbf{N}$ and each possible $T$-periodic solution $u$ of (1) such that $\varphi(u,\lambda) = n$, one has $\min_{t\in\mathbf{R}} |u(t)| < R_1$, which shows that condition (H4) of Lemma 5 holds with $K_n$ independent of $n$. As a consequence, all possible $T$-periodic solutions of (10) are a priori bounded independently of $\lambda$ and assumption (H1) of Lemma 5 is also satisfied. Now the inequality (11) implies that, for sufficiently large values of $|u|$, one has

$$(f(t,u,0) \mid -Ju) = (f_0(u) \mid -Ju) > 0,$$

and hence $d_B(f_0, B(r), 0) = d_B(-J, B(r), 0) = 1$, showing that assumption (H1) of Lemma 5 holds. □

*Remark.* Theorem 1 is a result on periodic solutions of differential equations with jumping or asymmetric nonlinearities which can be traced to [**8**] and [**19**] (see [**18**] and [**7**] for more references). The set $\mathcal{F}$ of curves in the positive quadrant of the $(m_+, m_-)$ plane made by the positive axes and the curves

$$\frac{2}{1/\sqrt{m_+} + 1/\sqrt{m_-}} = \omega n,$$

for the positive integers $n$ was first introduced independently for periodic problems in [**8**] and [**19**] and called the *Fučik spectrum*, as it generalizes the usual spectrum from linear to piecewise linear situations. The set $\mathcal{F}$ is characterized by the fact that the problem

$$x'' + m_+x^+ - m_-x^- = 0,$$

has a nontrivial $T$-periodic solution if and only if $(m_-, m_+) \in \mathcal{F}$. The approach used here is a slight variant of that in [**7**] where more general planar systems are considered.

**5. Second order equations with one-sided growth restrictions.** We shall show in this section that the same approach can be applied to situations where the linear growth restriction only holds in one direction. We again consider the second order equation (9).

**Theorem 2.** *Assume that there exist $R > 0$, $K \geq 0$ and positive constants $A_+ \leq B_+$ and $A_- \leq B_-$ such that*

$$A_+x^2 - K|x| \leq xg(x) \leq B_+x^2 + K|x| \quad \text{for } x \geq 0, \tag{15}$$

*and*

$$A_-x^2 - K|x| \leq xg(x) \quad \text{for } x \leq 0, \tag{16}$$

*for all $x \in \mathbf{R}$. If*

$$\left[\frac{2}{1/\sqrt{A_+} + 1/\sqrt{A_-}}, \frac{2}{1/\sqrt{B_+}}\right] \cap \omega\mathbf{N} = \varnothing, \tag{17}$$

*then equation* (9) *has at least one $T$-periodic solution.*

*Proof.* The proof follows the same line as that of Theorem 1 except that this time we define $S_2$ by

$$S_2(u) = B_+(u_1^+)^2 + u_2^2,$$

which implies that

$$(f(t,u,\lambda) \mid Ju) \leq -S_1(u) + L|u|$$

for all $u \in \mathbf{R}^2$ and some $L \geq 0$ and

$$(ft,u,\lambda) \mid Ju) \geq -S_2(u) - L|u| > 0,$$

for all $u \in \mathbf{R}^2$ such that $u_1 \geq 0$, $|u| \geq R_1$ and some $R_1$ sufficiently large. Lemmas 2 and 4 then imply that

$$\frac{2}{\omega(1/\sqrt{A_+} + 1/\sqrt{A_-})} = \frac{1}{\omega\langle 1/S_1\rangle} \leq \varphi(u,\lambda) \leq \frac{2}{\omega\{1/S_2\}} = \frac{2\sqrt{B_+}}{\omega},$$

for all possible $T$-periodic solutions $u$ of (1) such that $\min_{t\in\mathbf{R}}|u(t)|$ is sufficiently large. The remainder of the proof is similar to that of Theorem 1. $\square$

*Remark*. This result is related to the earlier papers [**26, 9**] and [**14**] but the approach is different.

**6. Second order superlinear equations.** Lemma 5 can also be applied to periodic solutions of second order equations which are superlinear at $+\infty$ and $-\infty$, i.e., when the function $g$ satisfies the superlinearity condition

$$g(x)/x \to +\infty \quad \text{as } |x| \to \infty. \tag{18}$$

Under such an assumption, the equation

$$x'' + g(x) = 0,$$

or the equivalent planar system

$$u_1' = u_2, \qquad u_2' = -g(u_1),$$

has infinitely many $T$-periodic solutions with arbitrary large norms, and hence the assumption (H1) of Lemma 5 will never be satisfied. A way to overcome this difficulty is to define $f(t,u,\lambda)$ by

$$f(t,u,\lambda) = \left(u_2, -(1-\lambda)\frac{u_2}{1+|u_2|} - g(u_1) + \lambda e(t)\right).$$

In this case, the system for $\lambda = 0$ is equivalent to the equation

$$x'' + \frac{x'}{1+|x'|} + g(x) = 0.$$

If $x$ is a $T$-periodic solution of this equation, then

$$\int_0^T \frac{(x'(t))^2}{1+|x'(t)|}\,dt = 0,$$

so that $x(t) = c$, $c$ constant, and $g(c) = 0$. As $g(x)x > 0$ for $|x|$ large, it immediately follows that there exists $r > 0$ such that $||(x, x')|| = |c| < r$ for any possible $T$-periodic solution and that $d_B(f_0, B(r), 0) \neq 0$. Thus, assumptions (H1) and (H2) of Lemma 5 hold. Assumption (H3) is checked as in the previous cases. The new term $-(1-\lambda)u_2/(1+|u_2|)$ in the second component of $f(t,u,\lambda)$ does not essentially affect $(f(t,u,\lambda) \mid Ju)$ as it is bounded, and the superlinear property of $g$ indeed implies that assumptions of the type of Lemma 2 hold for a sequence of quadratic forms $S_n$ of the type $S_n(u) = 2n^2u_1^2 + u_2^2/2$. This allows us to show that assumption (H4) also holds. We therefore have the following existence result, contained in [**7**], where the details of the proof and the corresponding bibliography can be found.

**Theorem 3.** *Assume that $g$ satisfies the superlinearity condition* (18). *Then equation* (9) *has at least one $T$-periodic solution.*

*Remark.* Theorem 3 indeed holds for the more general equation

$$x'' + g(x) = p(t, x, x')$$

when $p$ has at most linear growth with respect to $x$ and $x'$. In this setting, the result is the best possible because, for example, the special case

$$x'' + x' + x^3 = 0$$

has only the trivial $T$-periodic solution. On the other hand, for the autonomous equation

$$x'' + g(x) = 0$$

with $g$ superlinear, the energy integral easily shows that all the solutions $\xi(.,c)$ with initial conditions $x(0) = c > 0$, $x'(0) = 0$ have orbits $\Gamma(c)$ in the phase plane which are simple closed curves encircling the origin when $c$ is sufficiently large. If $(-l(c), 0)$ denotes the intersection of $\Gamma(c)$ with the negative $x$-axis, then the period of the corresponding periodic solution is given by

$$\tau(c) = \int_{-l(c)}^{c} \frac{ds}{\sqrt{G(c) - G(s)}},$$

depends continuously upon $c$ and is such that $\tau(c) \to 0$ as $c \to +\infty$. Consequently, for all the integers $k$ such that $T/k$ belongs to the range of $\tau$, if we take $c_k$ such that $\tau(c_k) = T/k$, the corresponding solution $\xi(.; c_k)$ is $(T/k)$-periodic and hence $T$-periodic. Thus, the autonomous equation has infinitely many $T$-periodic solutions with arbitrary large norms, and one can ask if the same result can be extended to equation (9). Topological methods seem unable to reach this result, but we shall see in the next section that the symplectic structure of (9) can be used together with a fixed point theorem of symplectic nature to prove the existence of infinitely many $T$-periodic solutions for (9). Notice that if we take integers $\zeta > m > 1$ such that $mT/\zeta$ belongs to the range of $\tau$, and a corresponding $c_{\zeta,m}$ such that $\tau(c_{\zeta,m}) = mT/\zeta$, then the corresponding solution will be $mT$-periodic (makes $\zeta$ revolutions), and, as checked later in more generality, will not be $kT$-periodic for any $1 \leq k \leq m-1$ when $\zeta$ is prime. One can therefore also raise the question of the existence of such $mT$-periodic solutions for (9), and the same approach will provide a positive question.

Recall that $x$ is called a *harmonic solution* of (9) if $x$ is $T$-periodic and is called a *subharmonic solution* of order $m \geq 2$ if $x$ is $mT$-periodic and is not $kT$-periodic for any integer $1 \leq k \leq m-1$.

**7. The Poincaré-Birkhoff theorem and some extensions.** In the year 1880, Poincaré observed that when the Cauchy problem for equation (9) is uniquely solvable over an interval $[0, T]$, the initial conditions $a \in \mathbf{R}^2$ leading to $T$-periodic solutions are the solutions of the equations

$$a_1 = \xi(T; a_1, a_2), \qquad a_2 = \xi'(T; a_1, a_2),$$

where $\xi(t; a_1, a_2)$ denotes the solution of the Cauchy problem

$$x'' + g(x) = e(t), \qquad x(0) = a_1, \qquad x'(0) = a_2.$$

In other words, they are the fixed points of the Poincaré operator $P$ defined in $\mathbf{R}^2$ by

$$P(a) = (\xi(T; a_1, a_2), \xi'(T; a_1, a_2)).$$

Recall that $P$ is a homeomorphism and that, because of the Hamiltonian character of equation (9), $P$ is area preserving. In 1912, Poincaré

[**30**] published the following fixed point conjecture, with some partial indications of the proof, which was given in 1913 by G.D. Birkhoff [**2**], one year after Poincaré's untimely death.

**Lemma 7.** *Let* $A = B[R]\backslash B(r) \subset \mathbf{R}^2$ *(with* $0 < r < R$*) be a closed annulus and* $H : A \to A$ *an area-preserving homeomorphism which rotates the two boundary components of* $A$ *in opposite angular directions. Then* $H$ *has at least two fixed points.*

As already observed by Poincaré, the statement about the rotations of the boundary components, usually called the *twist condition*, is ambiguous in that an anticlockwise rotation by $\theta$ is the same as a clockwise rotation by $2\pi - \theta$. This ambiguity can be resolved by going to the universal cover of $A$ associated to polar coordinates and assuming that $H$ can be lifted to a homeomorphism $\widetilde{H} : \tilde{A} \to \tilde{A}$ of the form

$$\widetilde{H}(\theta, \rho) = (\theta + g(\theta, \rho), h(\theta, \rho)),$$

where $g$ and $h$ are $2\pi$-periodic in $\theta$ and continuous, $\tilde{A} = \mathbf{R} \times [r, R]$ and

$$g(\theta, r) \cdot g(\theta, R) < 0$$

for all $\theta \in \mathbf{R}$. The proof of the Poincaré-Birkhoff theorem is still delicate and we shall refer to [**3**] for a modern treatment of Birkhoff arguments. The result can be proved easily if we assume that, for each $\theta \in \mathbf{R}$, $g(\theta, .)$ is strictly monotone. Indeed, if it is the case, it follows from the twist condition that, for each $\theta \in \mathbf{R}$, there exists a unique $\rho(\theta) \in ]r, R[$ such that $g(\theta, \rho(\theta)) = 0$. Consequently, the closed simple curve $C$ of equation $\rho = \rho(\theta)$ in $A$ is such that each of its points is moved radially under the action of $H$. Its image $H(C)$ is a simple closed curve in $A$ of equation $\rho = \rho^*(\theta)$. By the area preserving property of $H$, we have

$$\int_0^{2\pi} \int_r^{\rho(\theta)} \rho \, d\rho \, d\theta = \int_0^{2\pi} \int_r^{\rho^*(\theta)} \rho \, d\rho \, d\theta,$$

and hence

$$\int_0^{2\pi} [\rho^2(\theta) - \rho^{*2}(\theta)] \, d\theta = 0.$$

As the integrand is $2\pi$-periodic, this implies the existence of at least two values of $\theta \in [0, 2\pi[$ for which $\rho(\theta) = \rho^*(\theta)$, i.e., the existence of at least two fixed points for $H$. Notice that this elementary version of the Poincaré-Birkhoff theorem was already used by Morris [**28**] in 1965 to prove the existence of infinitely many $2\pi$-periodic solutions of the equation

$$x'' + 2x^3 = e(t),$$

when $e$ is continuous, $2\pi$-periodic and has mean value zero.

The conditions of the Poincaré-Birkhoff theorem are rather difficult to check in applications, and mathematicians have looked for more easily applicable extensions. One of them, due to Jacobowitz [**21**], develops a remark already made by Poincaré [**30**] and has been used by Jacobowitz [**21**], Butler [**6, 4**] and Hartman [**20**] to study the $T$-periodic solutions of 'unforced' superlinear second order equations. We state it in terms of the lifted map $\widetilde{H}$.

**Lemma 8.** *Assume that*

$$\widetilde{H} : S = \mathbf{R} \times ]0, R[ \longrightarrow \widetilde{H}(S),$$
$$(\theta, \rho) \longmapsto (\theta + g(\theta, \rho), h(\theta, \rho))$$

*is the lift of an area preserving homeomorphism such that*

$$\liminf_{\rho \to 0+} g(\theta, \rho) = 0, \quad \text{uniformly in } \theta,$$
$$g(\theta, R) < 0, \qquad \theta \in \mathbf{R},$$

*and*

$$\lim_{\rho \to 0+} h(\theta, \rho) = 0, \quad \text{uniformly in } \theta.$$

*Then $H$ has at least two fixed points.*

This Jacobowitz theorem was then used by W. Ding [**12**] to obtain a version of the Poincaré-Birkhoff theorem which is well-suited for the application to the periodic solutions of forced superlinear equations, and may be close to the one that Butler was thinking about when considering the extension of its results to the forced case (see [**6**]).

**Lemma 9.** *Let $H : B[R] \subset \mathbf{R}^2 \to H(B[R])$ be an area-preserving homeomorphism such that $H(A) \subset \mathbf{R}^2 \backslash \{0\}$ where $A = B[R] \backslash B(r)$.*

*Assume that* $0 \in T(B(r))$ *and that a lifting* $\widetilde{H}$ *of* $H$ *satisfies the twist condition*

$$g(\theta, r).g(\theta, R) < 0,$$

*for all* $\theta \in \mathbf{R}$*. Then* $H$ *has at least two fixed points in* $A$ *which correspond to two fixed points* $(\theta_i, \rho_i)$*,* $i = 1, 2$*, of* $\widetilde{H}$ *such that* $g(\theta_i, \rho_i) = 0$*,* $i = 1, 2$*.*

**8. Hamiltonian second order superlinear equations: harmonic solutions.** In this section we shall describe some results of T. Ding and Zanolin [**10**] about the use of the W. Ding version of the Poincaré-Birkhoff theorem given above to the existence of infinitely harmonic and subharmonic solutions of equation (9). For simplicity, we shall assume that $g$ is locally Lipschitzian, so that the local uniqueness of the Cauchy problem holds. This condition can be dropped at the expense of an approximation procedure (see [**10**]).

Let us first consider the existence of harmonic solutions. The proof is based upon the following lemmas, whose proof is quite similar to that of the corresponding results of the approach using degree theory given above. They only require the following assumption upon $g$:

$$\lim_{|x| \to \infty} g(x) \operatorname{sign} x = +\infty. \tag{19}$$

**Lemma 10.** *If condition* (19) *holds, then all solutions of the Cauchy problem for* (9) *exist over* $\mathbf{R}$*.*

**Lemma 11.** *Let* $T_0 \geq R$ *be fixed. If condition* (19) *holds, then for each* $R_1 > 0$ *there exists* $R_2 > R_1$ *such that for each solution* $u$ *of*

$$u_1' = u_2, \qquad u_2' = -g(u_1) + e(t), \tag{20}$$

*such that* $|u(0)| \leq R_1$ *(respectively,* $|u(0)| \geq R_2$*) one has* $\max_{t \in [0, T_0]} |u(t)| \leq R_2$ *(respectively,* $\min_{t \in [0, T_0]} |u(t)| \geq R_2$*).*

One can therefore use polar coordinates $(\rho(t; v), \Theta(t; v))$ for the solution $u(t)$ of (20) with $u(0) = v$ whenever $|v| \geq r_0$ and $r_0 > 0$ large enough so that $|u(t)| > 0$ for all $t \in [0, T_0]$.

**Lemma 12.** *There exists* $d \geq r_0$ *such that, for every solution* $u$ *of* (20) *with* $|u(0)| = d$*, one has* $(d/dt)\Theta(t; u(0)) < 0$ *for all* $t \in [0, T_0]$*. Moreover, there exists a nondecreasing and continuous function* $\beta : [d, +\infty[ \to ]0, +\infty[$ *such that for any solution of* (20) *with* $|u(0)| = r$*, one has* $(d/dt)\Theta(t; u(0)) \geq -\beta(r)$*, whenever* $t \in [0, T_0]$.

If $r > 0$ is large enough, one can therefore define the following integers

$$\mathbf{n}_*(r) = \max\left\{n \in \mathbf{N} : n \leq \inf_{|u(0)|=r} \frac{|\Theta(T; u(0)) - \Theta(0; u(0))|}{2\pi}\right\},$$

$$n^*(r) = \min\left\{n \in \mathbf{N} : n \geq \sup_{|u(0)|=r} \frac{|\Theta(T; u(0)) - \Theta(0; u(0))|}{2\pi}\right\}.$$

Thus, each solution of (20) with initial values on $\partial B(r)$ makes at least $n_*(r)$ and at most $n^*(r)$ clockwise rotations around the origin during a period of time $T$.

One then has the following existence theorem.

**Theorem 4.** *Assume that condition* (19) *holds and that there exists* $r_1 \neq r_2 \geq d$ *such that*

$$\mathbf{n}_*(r_2) - \mathbf{n}^*(r_1) \geq 2.$$

*Then system* (20) *has at least one* $T$*-periodic solution* $u$ *with*

$$\min\{r_1, r_2\} \leq |u(0)| \leq \max\{r_1, r_2\}.$$

*Proof.* We consider, say, the case where $r_1 < r_2$. Using the lemmas above, it is not difficult to see that the Poincaré operator $P : B[r_2] \to P(B[r_2])$ is an area preserving homeomorphism such that, for $A = B[r_2] \backslash B(r_1)$, $P(A) \subset \mathbf{R}^2 \backslash \{0\}$ and $0 \in P(B(r_1))$. Also,

$$|\Theta(T; v) - \Theta(0; v)| = \Theta(0; v) - \Theta(T; v).$$

From the hypothesis we can find an integer $\mathbf{n}$ such that

$$\mathbf{n}_*(r) < \mathbf{n} < \mathbf{n}^*(r),$$

and $\varepsilon_0 \in ]0, 1/2[$ such that

$$|\Theta(T; v) - \Theta(0; v)| \leq 2\pi(\mathbf{n} - \varepsilon_0) \quad \text{for all } |v| = r_1,$$
$$|\Theta(T; v) - \Theta(0; v)| \geq 2\pi(\mathbf{n} + \varepsilon_0) \quad \text{for all } |v| = r_2.$$

Moreover, we can choose a lifting $\widetilde{P}$ of $P$ such that if $\widetilde{P}(\theta, \rho) = P(v)$, then

$$g(\theta, \rho) = \Theta(T; v) - \Theta(0; v) + 2\pi\mathbf{n},$$

so that the 'twist' condition will hold, and the conclusion follows from W. Ding's version of the Poincaré-Birkhoff theorem. □

An easy consequence of Theorem 4 is the following corollary.

**Corollary 2.** *Assume that condition* (19) *holds and that*

$$\lim_{r \to +\infty} \mathbf{n}_*(r) = +\infty. \tag{21}$$

*Then system (20) has a sequence* $(u_j)$ *of* $T$*-periodic solutions such that*

$$\lim_{j \to \infty} \min_{t \in \mathbf{R}} |u_j(t)| = +\infty.$$

It remains, of course, to find conditions upon $g$ which insure that (21) holds. A delicate analysis given in [**10**] shows that it is the case under a condition which is weaker than the superlinearity of $g$ and stronger than the superquadratic character of $G$. This condition makes use of the time-map $\tau(c)$ (introduced above) associated to the solution $\xi(.; c)$ of the autonomous equation

$$x'' + g(x) = 0$$

such that $x(0) = c > 0$ and $x'(0) = 0$. The use of the properties of $\tau(c)$ in the study of periodic solutions of forced second order differential equations was introduced by Opial [**29**]. The condition to add to (19) to be sure that (21) holds is that

$$\lim_{c \to +\infty} \tau(c) = 0, \tag{22}$$

and it is checked in [**10**] that (19) and (22) imply that $G(x)/x^2 \to +\infty$ as $|x| \to \infty$, and that the superlinearity of $g$ implies (22).

Thus, one has the following corollary.

**Corollary 3.** *If*

$$g(x)/x \longrightarrow +\infty \quad \text{as } |x| \longrightarrow \infty,$$

*then system* (20) *has a sequence* $(u_j)$ *of* $T$*-periodic solutions such that*

$$\lim_{j\to\infty} \min_{t\in\mathbf{R}} |u_j(t)| = +\infty.$$

**9. Hamiltonian second order superlinear equations: subharmonic solutions.** As the forcing term $e$ is also $mT$-periodic for any integer $m \geq 2$, it is not difficult in general to mimic the reasonings made to prove the existence of $T$-periodic solutions to obtain the existence of $mT$-periodic solutions. But those solutions could as well be the previous $T$-periodic one repeated $m$ times, and hence one must find ways to exclude this situation. The following result, announced above, is very useful in this respect.

**Lemma 13.** *Let $m \geq 2$ be an integer and $\zeta > m$ a prime number. Let $u$ be an $mT$-periodic solution of* (20) *which turns $\zeta$ times* (*in the same sense*) *around the origin during the interval* $[0, mT]$. *Then $u$ is a subharmonic solution of order $m$ of* (20).

*Proof.* We have to show that, for each integer $1 \leq k \leq m-1$, $u$ is not $kT$-periodic. If $u$ is $kT$-periodic for some integer $1 \leq k \leq m-1$, then, on the interval $[0, kT]$, $u$ turns in the same direction $l < \zeta$ times around the origin. Now $u$ is also $kmT$-periodic and, on the interval $[0, kmT]$, $u$ will turn in the same direction $k\zeta = ml$ times around the origin. Thus, $\zeta$ divides $ml$ and hence must divide $m$ or $l$, which is impossible as $\zeta > m$ and $\zeta > l$. □

We can repeat all the reasoning of the previous section and obtain the following results where, in the definition of $\mathbf{n}_*(r)$, $\mathbf{n}^*(r)$ and of the Poincaré operator, one has replaced $T$ by $mT$.

**Theorem 5.** *Assume that condition* (19) *holds and that there exists* $r_1 \neq r_2 \geq d$ *and a prime number* $\zeta > m$ *such that*

$$\mathbf{n}_*(r_2) > \zeta > \mathbf{n}^*(r_1).$$

*Then system* (20) *has at least one* $mT$*-periodic solution* $u$ *with*

$$\min\{r_1, r_2\} \leq |u(0)| \leq \max\{r_1, r_2\},$$

*which is not* $kT$*-periodic for any* $1 \leq k \leq m-1$.

**Corollary 4.** *Assume that condition* (19) *holds and that*

$$\lim_{r \to +\infty} \mathbf{n}_*(r) = +\infty.$$

*Then, for each integer* $m \geq 2$, *system* (20) *has a sequence* $(u_j)$ *of subharmonic solutions of order* $m$ *such that*

$$\lim_{j \to \infty} \min_{t \in \mathbf{R}} |u_j(t)| = +\infty.$$

**Corollary 5.** *If*

$$g(x)/x \longrightarrow +\infty \quad \textit{as } |x| \longrightarrow \infty,$$

*then, for each integer* $m \geq 2$, *system* (20) *has a sequence* $(u_j)$ *of subharmonic solutions of order* $m$ *such that*

$$\lim_{j \to \infty} \min_{t \in \mathbf{R}} |u_j(t)| = +\infty.$$

A classical reasoning shows, moreover, that if $e$ is not constant, then each solution $u_j$ has $mT$ as a minimal period.

Using again W. Ding's version of the Poincaré-Birkhoff theorem, T. Ding and Zanolin [**11**] have also proved the existence of subharmonic solutions for differential equations of the form (9) when the time-map $\tau(c)$ associated to the corresponding autonomous equation is such that

$$\tau(c) \longrightarrow +\infty \quad \text{as } c \longrightarrow +\infty.$$

This happens in particular for sublinear nonlinearities, i.e., when

$$g(x)/x \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty.$$

Similar results can also be obtained in such situations using critical point theory, and we shall describe some recent ones in the next sections. The idea of using critical point theory to prove the existence of subharmonic solutions to differential equations is due to Rabinowitz [**32**].

**10. Hamiltonian second order sublinear equations: harmonic solutions.** Lazer [**22**] proved in 1968, by a clever use of the Schauder fixed point theorem, a result implying that if $g$ is continuous and satisfies the following conditions:

(B) $g$ is *bounded*,

(LL) $g^- := \limsup_{x\to-\infty} g(x) < \bar{e} := (1/T)\int_0^T e(t)\,dt < g_+ := \liminf_{x\to+\infty} g(x)$,

then equation (9) has at least one $T$-periodic solution.

Condition (LL) is usually called a *Landesman-Lazer* condition, and it implies each of the following three conditions:

(ALP) $G(x) - \bar{e}x \to +\infty$ as $|x| \to \infty$,

(S) There exists $r > 0$ such that $(g(x) - \bar{e})x > 0$ for $|x| \geq r$,

(SQ) $2G(x)/x^2 \to 0$ as $|x| \to \infty$.

The (ALP) condition was first introduced in the setting of Dirichlet boundary value problems by Ahmad, Lazer and Paul [**1**]. (S) is a *sign condition* for $g - \bar{e}$ and (SQ) is a *subquadratic growth condition* for the potential $G$.

In the autonomous case,

$$x'' + g(x) = \bar{e}, \tag{23}$$

where the real number $\bar{e}$ is such that conditions (ALP) and (S) hold, one can check that the orbits of solutions with initial conditions $(c, 0)$ with $c > 0$ large enough are closed simple curves, and the corresponding time map $\tau(c)$ is such that

$$\tau(c) \to +\infty \quad \text{as } c \to +\infty.$$

Hence, equation (23) will admit subharmonics of order $m$ for all sufficiently large $m$. On the other hand, condition (ALP) implies that $G(.)-\bar{e}(.)$ has a critical point $x_0$, which is of course a $T$-periodic solution of (23). We shall show, using critical point theory that those results still hold if we add to the right-hand member of (23) any $T$-periodic continuous function having mean value zero.

In the nonautonomous case, we first consider the existence of at least one $T$-periodic solution of the equation (9) when $e$ is $T$-periodic, written as $e(t) = \bar{e} + \tilde{e}(t)$, with $\bar{e} = (1/T)\int_0^T e(t)\,dt$ and $\int_0^T \tilde{e}(t)\,dt = 0$. It is standard (see, e.g., [**27**]) that the $T$-periodic solutions of (9) are the critical points of the action functional $\varphi_T$ defined over

$$H_T^1 = \{x : \mathbf{R} \to \mathbf{R} : x \text{ is } T\text{-periodic, absolutely continuous and } x' \in L^2(0,T)\}$$

(with the usual Sobolev norm $||x||$) by

$$\varphi_T(x) = \int_0^T \left[\frac{(x'(t))^2}{2} - G(x(t)) + e(t)x(t)\right] dt.$$

This functional is of class $C^1$ and

$$\varphi_T'(x)y = \int_0^T [x'(t)y'(t) - g(x(t))y(t) + e(t)y(t)]\,dt,$$

for every $x$ and $y \in H_T^1$. The geometry of this functional is described by the following lemma. For each $x \in H_T^1$, we write $x = \bar{x} + \tilde{x}$, with $\bar{x} = (1/T)\int_0^T x(t)\,dt$, so that $H_T^1 = \overline{H} \oplus \widetilde{H}$.

**Lemma 14.** *If conditions* (B) *and* (ALP) *hold, then* $\varphi_T(\bar{x}) \to -\infty$ *as* $|\bar{x}| \to \infty$ *in* $\overline{H}$ *and*

$$\varphi_T(\tilde{x}) \geq ||\tilde{x}||_{L^2}^2/2 - C||\tilde{x}||_{L^2},$$

*for all* $\tilde{x} \in \widetilde{H}$ *and some* $C > 0$, *so that* $\varphi_T(\tilde{x}) \to +\infty$ *as* $||\tilde{x}'||_{L^2} \to \infty$.

*Proof.* We have

$$\varphi_T(\bar{x}) = -[G(\bar{x}) - \overline{ex}]T,$$

and the first conclusion follows from condition (ALP), and

$$\begin{aligned}\varphi_T(\tilde{x}) &= \int_0^T \left[\frac{(\tilde{x}'(t))^2}{2} - G(\tilde{x}(t)) + \tilde{e}(t)\tilde{x}(t)\right] dt \\ &\geq \frac{||\tilde{x}'||_{L^2}^2}{2} - C||\tilde{x}||_{L^2},\end{aligned}$$

using condition (B), and the last assertion follows from the Wirtinger inequality (see, e.g., [**27**]). ◻

We are therefore in a situation where the following saddle point theorem of Rabinowitz [**31**] could be applied.

**Lemma 15.** *Let $E = V \oplus W$ be a Banach space, with $V$ nontrivial and finite dimensional, and $\varphi \in C^1(E, \mathbf{R})$ a functional satisfying the Palais-Smale condition (i.e., every sequence $(x_n)$ in $E$ with $(\varphi(u_n))$ bounded and $(\varphi'(u_n))$ converging to zero has a convergent subsequence). If there exists $R > 0$ such that*

$$\max_{\partial B(R) \cap V} \varphi < \inf_W \varphi, \tag{24}$$

*then $\varphi$ has a critical point $x \in E$ with critical value*

$$\varphi(x) = \inf_{\gamma \in \Gamma} \max_{\xi \in B[R] \cap V} \varphi(\gamma(\xi)),$$

*where $\Gamma = \{\gamma \in C(B[R] \cap V, E) : \gamma|_{\partial B(R) \cap V} = \text{identity}\}$.*

We have, therefore, the following extension of the Lazer existence theorem (see, e.g., [**27**, Theorem 4.8]).

**Theorem 6.** *Assume that conditions* (B) *and* (ALP) *hold. Then equation* (9) *has at least one $T$-periodic solution $x_T$ such that*

$$\varphi_T(x_T) = \inf_{\gamma \in \Gamma} \max_{\xi \in B[R] \cap \overline{H}} \varphi_T(\gamma(\xi)),$$

*where $\Gamma_T = \{\gamma \in C(B[R] \cap \overline{H}, H_T^1) : \gamma|_{\partial B(R) \cap \overline{H}} = \text{identity}\}$.*

*Proof.* We take, of course, $E = H^1_T$, $V = \overline{H}$, $W = \widetilde{H}$ in Rabinowitz theorem. The existence of $R > 0$ such that condition (24) holds easily follows from Lemma 14. It remains, therefore, to prove the Palais-Smale condition. To this end, if $(x_n)$ is a sequence verifying the assumptions of the Palais-Smale condition, then we have

$$\begin{aligned} M||\tilde{x}_n|| &\geq \varphi_T'(x_n)\tilde{x}_n \\ &= \int_0^T [(x_n'(t))^2 - g(x_n(t))\tilde{x}_n(t) + e(t)\tilde{x}_n(t)]\, dt \\ &\geq ||x_n'||^2_{L^2} - C||\tilde{x}_n||_{L^2}, \end{aligned}$$

and hence, using the Wirtinger inequality and the fact that, on $\widetilde{H}$, the Sobolev norm $||x||$ and the norm $||x'||_{L^2}$ are equivalent, we see that $(\tilde{x}_n)$ is bounded. On the other hand, we have

$$\begin{aligned} \varphi_T(x_n) = \frac{||x_n||^2}{2} &+ \int_0^T \tilde{e}(t)\tilde{x}_n(t)\, dt \\ &- \int_0^T [G(\overline{x_n} + \tilde{x}_n(t)) - \overline{ex_n}]\, dt, \end{aligned}$$

and, as $(\varphi_T(x_n))$ is bounded, the same must be true for $(\int_0^T [G(\overline{x_n} + \tilde{x}_n(t)) - \overline{ex_n}]\, dt)$. If $(\overline{x_n})$ is not bounded, we therefore get a contradiction using condition (ALP). Thus $(x_n)$ is bounded, and a standard argument, given, for example, in [**27**, Proposition 4.1], implies the existence of a convergent subsequence. □

**11. Hamiltonian second order sublinear equations: subharmonic solutions.** We shall now describe a recent argument of Fonda and Lazer [**16**] showing how to use the Rabinowitz saddle point theorem to obtain the existence of subharmonic solutions for (9) in the sublinear case. One needs the following standard lemma, a special case of a result of [**23**].

**Lemma 16.** *If conditions* (B) *and* (S) *hold, then the set of possible $T$-periodic solutions of equation* (9) *is a priori bounded.*

*Proof.* Let $x$ be any possible $T$-periodic solution of (9). Then

$$x''\tilde{x} + g(x)\tilde{x} = e(t)\tilde{x},$$

and hence,

$$-\int_0^T (x'(t))^2\,dt + \int_0^T g(x(t))\tilde{x}(t)\,dt = \int_0^T \tilde{e}(t)\tilde{x}(t)\,dt,$$

so that, using condition (B), we get $||x'||_{L^2} \le C$, and hence $\max_{t\in\mathbf{R}} |\tilde{x}(t)| \le C'$ by Sobolev inequality. We also have

$$\frac{1}{T}\int_0^T [g(x(t)) - e(t)]\,dt = 0,$$

and hence no $T$-periodic solution $x$ can be such that $|\bar{x}| \ge r + C'$. The *a priori* bound for $\bar{x}$ and for $||x'||_{L^2}$ easily implies an *a priori* bound for $||x||$. □

**Theorem 7.** *Assume that conditions* (B), (S) *and* (ALP) *hold. Then equation* (9) *has at least one $T$-periodic solution, and there exists $m_0 > 1$ such that, for each prime integer $m \ge m_0$, equation* (9) *has a subharmonic solution of order $m$.*

*Proof.* For each integer $k \ge 2$, one can repeat the argument of Theorem 6 for the functional $\varphi_{kT}$ defined over

$$H^1_{kT} = \{x : \mathbf{R} \to \mathbf{R} : x \text{ is } kT\text{-periodic, absolutely continuous, } x' \in L^2(0,kT)\}$$

(with the usual Sobolev norm $||x||_{kT}$) by

$$\varphi_{kT}(x) = \int_0^{kT} \left[\frac{(x'(t))^2}{2} - G(x(t)) + e(t)x(t)\right] dt.$$

One obtains in this way a critical point $x_{kT}$ of $\varphi_{kT}$ with critical value

$$\varphi_{kT}(x_{kT}) = \inf_{\gamma\in\Gamma_{kT}} \max_{\xi\in B[R_k]\cap\overline{H_k}} \varphi_{kT}(\gamma(\xi)),$$

where $\Gamma_{kT} = \{\gamma \in C(B[R_k] \cap \overline{H_k}, H^1_{kT}) : \gamma|_{\partial B(R_k)\cap\overline{H_k}} = \text{identity}\}$, and $\overline{H_k} \oplus \widetilde{H}_k$ is the usual splitting of $H^1_{kT}$ in constant functions and functions of mean value zero. For any possible $T$-periodic solution $x$ of (9), we have

$$\varphi_{kT}(x) = k\varphi_T(x),$$

and therefore, by Lemma 16,

$$|\varphi_{kT}(x)/k| \le C.$$

We shall therefore show that the critical points $x_{kT}$ are not $T$-periodic by showing that

$$\varphi_{kT}(x_{kT})/k \to -\infty$$

as $k \to \infty$. This will be the case, by the characterization of their critical values, if we can find a sequence of mappings $\gamma_k \in \Gamma_k$ such that

$$\max_{\xi \in B[R_k] \cap \overline{H_k}} \frac{\varphi_{kT}(\gamma_k(\xi))}{k} \to -\infty \quad \text{as } k \to \infty.$$

Such a sequence is given by the mappings from $B[k] \cap \overline{H_k} \simeq [-k, k]$ into $H^1_{kT}$ defined by

$$\gamma_k(\xi)(.) = \xi + \left(1 - \frac{|\xi|}{k}\right) 2k \sin \frac{\omega}{k}(.).$$

Careful estimates of $\varphi_{kT}$ which can be found explicitly in [**16**] imply that this sequence has the required property. When $m$ is prime and sufficiently large, the corresponding critical point $x_{mT}$ is $mT$-periodic and not $kT$-periodic for any $1 \le k \le m-1$. □

**Corollary 6.** *Assume that the conditions* (B) *and* (LL) *hold. Then equation* (9) *has at least one $T$-periodic solution and there exists $m_0 > 1$ such that for each prime integer $m \ge m_0$, equation* (9) *has a subharmonic solution of order $m$.*

*Remark.* By a similar proof also given in [**15**], one can show that the conclusion of Theorem 7 holds when condition (B) in Theorem 7 is replaced by condition (SQ) and

(B′) $\limsup_{|x|\to\infty} g(x)/x < \omega^2$.

*Remark.* In a very recent paper, Fonda and Ramos [**16**] have refined the arguments above to prove the existence of a sequence of $kT$-periodic solutions of (9) whose amplitudes and minimal periods tend to infinity if conditions (LL) hold, together with conditions

($\mathrm{SQ}_-$) $\lim_{x\to-\infty} 2G(x)/x^2 = 0$,

and

($\mathrm{S}_-$) $(g(x)-\bar{e})\operatorname{sgn} x \geq -C$,

or if conditions (ALP), (S) and ($\mathrm{SQ}_-$) hold. In the same paper, they have also proved that if the conditions (S$'$), (LL) and

(BB$''$) $\lim_{|x|\to\infty} g(x)/x = 0$,

hold, then there exists $k_0 \geq 2$ such that for each integer $k \geq k_0$, each $\mu > 0$ and each $\nu > 0$ such that $1/\sqrt{\mu} + 1/\sqrt{\nu} > 2k/\omega$, the equation

$$x'' + \mu x^+ - \nu x_- + g(x) = e(t),$$

has at least a $T$-periodic solution and a $kT$-periodic solution which is not $T$-periodic. The mountain pass lemma is used instead of the saddle point theorem. Applications are given to simplified models of the Tacoma bridge. Results in this direction have also recently been obtained by Fonda, Schneider and Zanolin using the Poincaré-Birkhoff theorem [**17**].

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