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BUCKLING OF A NONLINEARLY ELASTIC COLUMN: VARIATIONAL PRINCIPLES, BIFURCATION AND ASYMPTOTICS

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ABSTRACT. We study the equilibrium states of an elastic column subject to an end thrust. The column is modeled as an inextensible rod with a nonlinear moment-curvature equation. We bring various mathematical theories to bear on this problem; the calculus of variations, bifurcation theory, phase plane analysis, and singular perturbation theory for ordinary differential equations. We study the connections between the various approaches.

Many of the results presented here are not new. Our object is to show how this problem can serve to illustrate various abstract theories and to show how the various approaches compliment each other.

1. Introduction. In this paper we study the equilibrium configurations of an elastic column subject to end thrust. This problem can be posed as a nonlinear eigenvalue problem for a second order ordinary differential equation. Of course this is a very old problem. The case in which the column is modeled as an inextensible rod with the bending moment depending linearly on the curvature was solved by Euler [**9**] in 1744 in terms of elliptic integrals. Here we will consider the problem in which there is a nonlinear moment-curvature equation. In this case a fairly complete picture of the structure of the set of equilibria can be obtained by a phase plane analysis as given by Maddocks [**12**]. Here we wish to show how this problem fits into the framework of two theories of modern nonlinear analysis; the direct method of the calculus of variations and global bifurcation theory. The variational approach starts from the observation that equilibria appear as stationary points of an energy functional. The direct method of the calculus of variations consists of using methods of functional analysis to prove the existence of a (not necessarily unique) global minimum of the energy. Now we will show by other means that for some values of λ, the parameter representing the magnitude of the end loading there are multiple solutions

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of the eigenvalue problem. A natural question to ask is which of these solutions represent a global minimum of the energy? More generally, a solution is called *stable* if it represents a local minimum of the energy. It is important to identify which solutions are stable in this sense as it is generally true that such solutions are also stable in the dynamic sense. This means that if we consider the dynamic problem of the motion of the column with initial conditions which are a small perturbation of the equilibrium configuration, the configuration of the column will remain near the equilibrium for all positive time.

Bifurcation theory seeks to describe the set of solutions to nonlinear eigenvalue problems by constructing *bifurcation diagrams*. These are graphical representations of the set of solutions in which the horizontal axis represents the parameter λ and the vertical axis some functional of the solution (see Figure 1). For all nonnegative values of λ our problem has a *trivial solution* in which the column remains straight. However, at certain values of λ, which are precisely the eigenvalues of the linearized problem, branches of nontrivial solutions *bifurcate* from the trivial solution. These are curves which, in fact, exist globally. If the eigenvalues of the linearized problem are ordered as $0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$, the branch bifurcating from $(\lambda_n, 0)$ is characterized by the property that solutions lying on this branch have exactly n interior zeros. Of particular interest is the curve of solutions bifurcating from $(\lambda_0, 0)$. Nontrivial solutions correspond to bowed or buckled states of the column. The natural questions are: which of the solutions are stable and, in particular, for a given λ which solution has the minimum energy? For $\lambda > \lambda_0$ the trivial solution is easily seen to be unstable. This shows that proving existence of a solution by minimizing the energy is not a vacuous exercise. It also turns out that branches of solutions which bifurcate from $(\lambda_n, 0)$ for $n > 0$ cannot represent stable equilibria. Thus our attention focuses on the branch bifurcating from the trivial solution at $(\lambda_0, 0)$. The stability of solutions on this branch depends on the shape of the curve. It has long been known [**7**] that if the branch is as in Figure 1a, at least locally, near the bifurcation point, the solutions are unstable. This is the case of *subcritical bifurcation*. On the other hand, if the branch is as in Figure 1b, (*supercritical bifurcation*) the solutions on this branch are stable. Recent work of Maddocks [**12**] has shown that, if the ordinate of the bifurcation diagram (labeled $F(u)$ in Figure 1) is chosen appropriately, one can make global statements about stability. Figure 1c shows a bifurcating curve of solutions. (Antman and Adler [**4**] have shown that such curves are realizable.) Maddocks' results imply that solutions lying on segments AB and CD are stable while those lying on segments OA and BC are unstable.

The bifurcation diagram can be obtained by applying the results of Crandall and Rabinowitz [**8, 15**]. This theory gives us a fairly complete description of the bifurcation diagram. However, it cannot predict the precise shape of the curves, i.e., the number of "wiggles" as in Figure 1c. (It can, however, give the initial direction of the curve as in Figures 1a and 1b.) The phase plane analysis of Maddocks confirms the results of Crandall and Rabinowitz and thus serves to illustrate it.

Another interesting aspect of the problem is the behavior of solutions as $\lambda \to \infty$. Our analysis will show that $\lambda \to \infty$ along each branch, i.e., for every nonnegative integer n, for $\lambda > \lambda_n$, there exists a solution (actually two, by symmetry) having exactly n interior zeros. The asymptotic behavior of solutions as $\lambda \to \infty$ is a chapter in the theory of singular perturbations for ordinary differential equations.

The outline of the remainder of the paper is as follows. Section 2 is devoted to a statement of the problem and its variational formulation. Section 3 describes the existence theorem using the direct method of the calculus of variations. Section 4 is concerned with global bifurcation theory. Section 5 discusses the phase plane analysis while Section 6 deals with the stability results. Section 7 is concerned with the asymptotic behavior of solutions as $\lambda \to \infty$. Some concluding remarks constitute Section 8.

2. Statement of the problem. We consider planar deformations of a column clamped at one end and subject to a compressive force at the other. We model the column as an inextensible rod. Let $\{\mathbf{i}, \mathbf{j}\}$ be a fixed orthonormal basis in the plane. We define a position vector function $\mathbf{r}$ of the real variable $s \in [0,1]$. Here s identifies material cross sections of the rod so that $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$ is the position of the material point at the centroid of the section s in the deformed configuration. The condition that the rod be inextensible is that $|\mathbf{r}'(s)| = 1$ so that

$$\mathbf{r}'(s) = \cos\theta(s)\mathbf{i} + \sin\theta(s)j, \tag{2.1}$$

where prime denotes differentiation with respect to s. We assume that

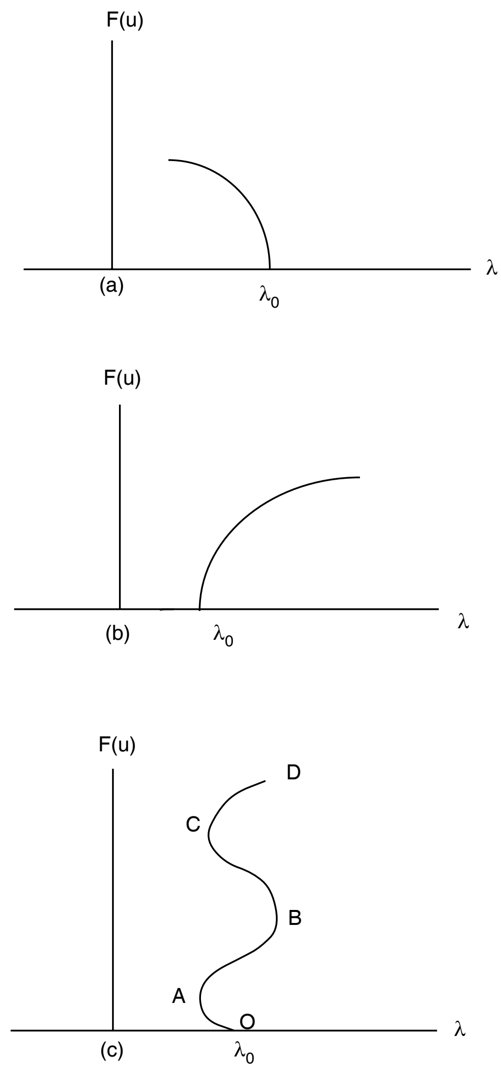

FIGURE 1. Bifurcation diagrams. Figure 1a shows an example of subcritical bifurcation; the branch of nontrivial solutions is unstable. Figure 1b shows an example of supercritical bifurcation; the branch of nontrivial solutions is stable.

in the natural (undeformed) state $\mathbf{r}(s) = s\mathbf{i}$ and the rod is clamped at $\mathbf{0}$ so that

$$\mathbf{r}(0) = \mathbf{0}, \qquad \theta(0) = 0. \tag{2.2}$$

The free end of the rod, $s = 1$, is subjected to a compressive force acting parallel to $\mathbf{i}$

$$\mathbf{f}(1) = -\lambda\mathbf{i}, \qquad \lambda > 0. \tag{2.3}$$

Let $\mathbf{n}(s)$ denote the contact force and $M(s)\mathbf{k}$, where $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, the bending moment exerted by the material of $(s, 1]$ on the material of $[0, s]$. Then the equations of equilibrium for the rod are

$$\mathbf{n}' = \mathbf{0}, \tag{2.4}$$

$$M' + (\mathbf{r}' \times \mathbf{n}) \cdot \mathbf{k} = \mathbf{0}. \tag{2.5}$$

From (2.3) and (2.4) we find

$$\mathbf{n} = -\lambda\mathbf{i}. \tag{2.6}$$

If we insert (2.6) in (2.5) and use (2.1), we find

$$M' + \lambda \sin\theta = 0. \tag{2.7}$$

The elastic properties of the rod are embodied in the moment-curvature equation relating the bending moment M to the curvature θ'. Thus, we assume that there is a differentiable function

$$R \ni \mu \to \widehat{M}(\mu) \in R$$

such that

$$M(s) = \widehat{M}(\theta'(s)). \tag{2.8}$$

We assume

$$\begin{gathered} \widehat{M}(-\mu) = -\widehat{M}(\mu), \qquad \widehat{M}(0) = 0, \qquad \widehat{M}'(0) = 1, \\ \widehat{M}_\mu > 0, \qquad \widehat{M}(\mu) \to \infty \quad \text{as } \mu \to \infty. \end{gathered} \tag{2.9}$$

(From now on we will drop the carat over the M.) The condition that the end of the rod at $s = 1$ be free is expressed by the condition

$$M(1) = 0, \tag{2.10}$$

or, in view of (2.9), this is equivalent to

$$\theta'(1) = 0. \tag{2.11}$$

Thus our boundary value problem consists of finding solutions to the boundary value problem

$$M(\theta')' + \lambda \sin\theta = 0, \qquad 0 < s < 1, \tag{2.12}$$

subject to the boundary conditions

$$\theta(0) = \theta'(1) = 0. \tag{2.13}$$

If we can solve (2.12) and (2.13), we can determine the deformed configuration of the rod from (2.1) and (2.2a), i.e.,

$$\mathbf{r}(s) = \int_0^s \cos(\theta(\sigma))\, d\sigma \mathbf{i} + \int_0^s \sin(\theta(\sigma))\, d\sigma \mathbf{j}. \tag{2.14}$$

For all values of λ, $\theta \equiv 0$ is a solution (the trivial solution) corresponding to a straight rod. Of course, the object of this paper is to study the existence of nontrivial solutions.

We define the *strain energy function* W by

$$W(p) = \int_0^p M(\tau)\, d\tau. \tag{2.15}$$

Then from (2.9),

$$\begin{gathered} W(-p) = W(p), \qquad W(0) = W'(0) = 0, \qquad W''(0) = 1, \\ W''(p) > 0, \qquad W(p)/|p| \to \infty \quad \text{as } |p| \to \infty. \end{gathered} \tag{2.16}$$

The boundary value problem (2.12), (2.13) is equivalent to finding stationary points of the energy functional

$$E(\theta) = \int_0^1 (W(\theta') + \lambda \cos\theta)\, ds \tag{2.17}$$

over the class of functions

$$V = \{\theta \mid \theta \in C^2[0,1], \theta(0) = 0\}.$$

Equation (2.12) is the Euler-Lagrange equation for this functional. The condition $\theta'(1) = 0$ is a natural boundary condition. In the next section we will prove existence of nontrivial solutions of (2.12), (2.13) by showing that for each λ, E has an absolute minimum in the class V and for sufficiently large $\lambda > 0$ this minimum is not attained at the trivial solution.

3. Existence via the calculus of variations. In this section we will prove existence of a solution of (2.12), (2.13) by proving the existence of a minimizer of the energy E in a Sobolev space and by showing that, in fact, this solution is in $C^2[0,1]$ and satisfies (2.12), (2.13). We will show the existence of a minimizer of the energy by applying a theorem of functional analysis.

Theorem 3.1 [**17**]. *Let X be a reflexive Banach space and E a coercive, sequentially weakly lower semicontinuous functional on X. Then E attains its minimum on X.*

Recall that E is coercive if for each $\alpha \in R$ the set $\{z \in X \mid E(z) \leq \alpha\}$ is bounded.

To apply Theorem 3.1, we define

$$X = \{\theta \mid \theta \in H^1[0,1];\ \theta(0) = 0\} \tag{3.2}$$

where $H^1[0,1]$ is the Sobolev space of functions in $L_2[0,1]$ whose distributional derivatives are in $L_2[0,1]$, or equivalently, the space of absolutely continuous functions with square integrable first derivatives.

We argue as in [**16**]. We first consider the mapping

$$\theta \to \int_0^1 W(\theta'(s))\, ds. \tag{3.3}$$

We wish to show that this map is sequentially weakly lower semicontinuous, i.e., that for each $c \in R$ the set

$$F(c) = \left\{\theta \mid \int_0^1 W(\theta'(s))\, ds \leq c\right\}$$

is closed in the weak topology of X. Since, by (2.16), W is strictly convex each set $F(c)$ is convex. Since a strongly closed convex set is weakly closed it is sufficient to show that $F(c)$ is strongly closed. Thus, let $\{\theta_k\} \in F(c)$ and suppose $\theta_k \to \theta$ in X. Then $\theta'_k \to \theta'$ in $L_2[0,1]$. Then $\{\theta'_k\}$ has a subsequence, again denoted by $\{\theta'_k\}$ which converges almost everywhere to θ'. Since W is continuous, Fatou's lemma implies

$$\begin{aligned}\int_0^1 W(\theta'(s))\,ds &= \int_0^1 \lim W(\theta'_k(s))\,ds \\ &\le \liminf \int_0^1 W(\theta'_k(s))\,ds \le c,\end{aligned}$$

showing that the mapping (3.3) is sequentially weakly lower semicontinuous. Next we observe that the mapping

$$\theta \to \int_0^1 \cos(\theta(s))\,ds$$

is continuous from $H^1[0,1]$ to R (in the strong topology). This is because if $\theta_k \to \theta$ in $H^1[0,1]$ then, in fact, $\theta_k \to \theta$ uniformly on $[0,1]$. For $\theta \in X$ we have the representation

$$\theta(s) = \int_0^s \theta'(\sigma)\,d\sigma. \tag{3.4}$$

Thus the functional $E(\theta)$ is sequentially weakly lower semicontinuous on X. The coercivity of E follows easily from the growth condition on W, (2.16). Thus, we may apply Theorem 3.1 to conclude:

Theorem 3.5. *The functional $E(\theta)$ defined by* (2.17) *attains its minimum over the space X defined by* (3.2).

We now show that a minimizer of E is in fact in $C^2[0,1]$ and satisfies (2.12), (2.13). For this we let $u = \theta'$ so that

$$\theta(s) = \int_0^s u(\sigma)\,d\sigma.$$

The function u is in $L_2[0,1]$. The minimization problem can then be restated as

$$\begin{aligned}\widehat{E}(u) &= \int_0^1 \left\{ W(u(s)) + \lambda \cos\left(\int_0^s u(\sigma)\, d\sigma \right) \right\} ds \\ &= \min\{\widehat{E}(v) \mid v \in L_2[0,1]\}.\end{aligned}$$

The first step in the regularity theory is to show that $u \in L_\infty[0,1]$. The most convenient way to do this is to use the convexity of W. The following is a variant of an argument given in [**5**]. Let $v \in L_2[0,1]$ be such that

$$\int_0^1 W(v(s))\, ds < \infty. \tag{3.6}$$

For $0 < t < 1$, by the convexity of W,

$$W((1-t)u + tv) < (1-t)W(u) + tW(v) \tag{3.7}$$

so $(1-t)u + tv$ has finite energy and

$$\widehat{E}(u) \le \widehat{E}((1-t)u + tv). \tag{3.8}$$

If we use (3.7) in (3.8) we find for $0 < t < 1$

$$\begin{aligned}\int_0^1 W(u(s))\, ds \le \int_0^1 & W(v(s))\, ds \\ &+ \frac{\lambda}{t} \int_0^1 \left[\cos\left(\int_0^s \{(1-t)u(\sigma) + tv(\sigma)\}\, d\sigma \right) \right. \\ &\left. \qquad\qquad - \cos \int_0^s u(\sigma)\, d\sigma \right] ds.\end{aligned} \tag{3.9}$$

We let $t \to 0$. An easy argument based on the Lebesgue dominated convergence theorem shows that the second term on the right converges to

$$\lambda \int_0^1 \sin\left\{ \int_0^s u(\sigma)\, d\sigma \right\} \int_0^s (v(\tau) - u(\tau))\, d\tau\, d\sigma.$$

By reversing the order of integration this can be rewritten as

$$\lambda \int_0^1 v(\tau) - u(\tau))\, d\tau \int_\tau^1 \sin\left(\int_0^s u(\sigma)\, d\sigma \right) ds. \tag{3.10}$$

Hence the result of letting $t \to 0$ in (3.9) is

$$\int_0^1 \Big\{ W(u(s)) - W(v(s)) + \lambda(v(s) - u(s)) \cdot \int_s^1 \sin\Big(\int_0^\sigma u(\tau)\,d\tau\Big)\,d\sigma \Big\}\,ds \le 0. \tag{3.11}$$

Equation (3.11) holds for every v satisfying (3.6). Now we choose v. Let Ω be a measurable subset of $[0,1]$, and define

$$v = \begin{cases} 0 & \text{in } \Omega, \\ u & \text{in } [0,1] - \Omega. \end{cases} \tag{3.12}$$

This v satisfies (3.6). We use this v in (3.11). We find

$$\int_\Omega \Big\{ W(u(s)) - \lambda u(s) \int_s^1 \sin\Big(\int_0^\sigma u(\tau)\,d\tau\Big)\,d\sigma \Big\}\,ds \le 0. \tag{3.13}$$

Since Ω is arbitrary, (3.13) implies that

$$W(u(s)) \le \lambda u(s) \int_s^1 \sin\Big(\int_0^\sigma u(\tau)\,d\tau\Big)\,d\sigma \quad \text{a.e.} \tag{3.14}$$

Hence, in particular, $W(u(s)) \le \lambda |u(s)|$ almost everywhere in $[0,1]$. By (2.16) this implies that $u \in L_\infty[0,1]$.

Thus if η is any test function, $t \in R$ and $v = \theta' + t\eta'$ then (3.6) is satisfied. It is easy to see that the map $t \to E(\theta + t\eta)$ is differentiable and its derivative at $t = 0$ must be zero, i.e.,

$$\int_0^1 \{\eta'(s) M(\theta'(s)) - \lambda \sin\theta(s)\eta(s)\}\,ds = 0. \tag{3.15}$$

So we see that θ is a weak solution of (2.12). But it is standard that weak solutions of (2.12) are, in fact, classical solutions. Thus, if we take η in (3.15) to be a smooth function not vanishing at $s = 1$ and integrate by parts we see that $\theta'(1) = 0$.

Thus, we have proven existence of a solution of (2.12), (2.13) for each λ which, however, may be trivial. However, for λ sufficiently large, the solution we found by this method is nontrivial.

Proposition 3.16. *Let*

$$\lambda > \pi^2/4. \tag{3.17}$$

Then the trivial solution $\theta \equiv 0$ is not stable, i.e., it does not represent a local minimum of the energy.

Proof. Let

$$\eta = \sin \frac{\pi}{2} s, \tag{3.18}$$

$$F(t) = E(t\eta) = \int_0^1 \{W(t\eta') + \lambda \cos(t\eta)\}\, ds.$$

Then

$$F'(t) = \int_0^1 \{M(t\eta')\eta' - \lambda \sin(t\eta)\eta\}\, ds.$$

Of course, $F'(0) = 0$. We compute

$$\begin{aligned} F''(t) &= \int_0^1 \{M'(t\eta')(\eta')^2 - \lambda \cos(t\eta)\eta^2\}\, ds, \\ F''(0) &= \int_0^1 \{(\eta')^2 - \lambda\eta^2\}\, ds. \end{aligned} \tag{3.19}$$

(Recall that by (2.16) $M'(0) = 1$.) If we insert η given by (3.18) into (3.19) and use (3.17), we find

$$F''(0) = \frac{1}{2}\left(\frac{\pi^2}{4} - \lambda\right) < 0.$$

Thus for λ satisfying (3.17) the trivial solution is not even a local minimum of the energy. Thus, there must be nontrivial solutions of (2.12) and (2.13) for all λ satisfying (3.17). In fact, we will show in the next section that if

$$\lambda > (2n+1)^2\pi^2/4 \tag{3.20}$$

there are at least $2n+2$ nontrivial solutions of (2.12), (2.13). A question naturally arises as to which of these solutions minimize the energy. One of the main goals of this paper is to provide an answer to this question. □

4. Global bifurcation. In this section we consider (2.12) and (2.13) from the point of view of global bifurcation theory. This theory, developed mainly by Crandall and Rabinowitz [**8, 15**] in the 1970's finds its main applications in the area of boundary value problems such as (2.12), (2.13). This theory deals with equations of the form

$$u = \lambda Lu + H(\lambda, u) \tag{4.1}$$

where L is a compact linear operator on a Banach space E and H is a nonlinear compact map from $R \times E$ into E with $H = o(||u||)$ near $u = 0$ uniformly on bounded λ intervals. The set of solutions of (4.1) consists of pairs (λ, u) in $R \times E$.

For every $\lambda \in R$ $(\lambda, 0)$ is a solution of (4.1). These are the trivial solutions. A point $(\mu, 0)$ is a *bifurcation point* if every neighborhood of $(\mu, 0)$ in $R \times E$ contains nontrivial solutions. A necessary condition for $(\mu, 0)$ to be a bifurcation point is that μ^{-1} belong to the spectrum of L. We say that μ is an *eigenvalue* of L if there exists a nonzero $v \in E$ such that $v = \lambda Lv$. The *multiplicity* of μ is the dimension of $\cup_{n=1}^{\infty} \ker(I - \mu L)^n$ where $\ker A$ denotes the kernel of A. Since L is compact, the eigenvalues of L are isolated and have finite multiplicity. The basic result is

Theorem 4.2 [**15**]. *If μ is an eigenvalue of L of odd multiplicity, then $(\mu, 0)$ is a bifurcation point for* (4.1).

This is a purely local result. Actually much more is true, namely the following global result. Let S denote the closure of the set of nontrivial solutions of (4.1) in $R \times E$.

Theorem 4.3 [**15**]. *If μ is an eigenvalue of L of odd multiplicity, then S contains a maximal connected subset $\Sigma(\mu)$,* (*called a branch through* $(\mu, 0)$) *which contains $(\mu, 0)$ and is either*

(i) *unbounded or*

(ii) *contains $(\nu, 0)$ where $\nu \neq \mu$ is an eigenvalue of L.*

In the application of the theory to (2.12) and (2.13), we will show that, in fact, alternative (ii) cannot occur. Thus, branches of solutions bifurcating from $(\mu, 0)$ and $(\nu, 0)$ where μ and ν are distinct eigenvalues of L are disjoint. The natural question to ask is whether these branches are, in fact, curves. In the next section we will show that they are. In general, we have the following local result.

Theorem 4.4 [**15**]. *If μ is a* simple *eigenvalue of L (i.e., has multiplicity* 1) *and $H(\lambda, u) = \lambda N(u)$ where N is continuously Frechet differentiable near $u = 0$, then near $(\mu, 0), \Sigma(\mu)$ consists of a continuous curve of solutions.*

We now show that we can transform (2.12) and (2.13) into the form (4.1) and that we may apply Theorems 4.1, 4.2 and 4.3.

As E we take $C^1[0,1]$, the space of functions which, along with their first derivatives, are continuous on $[0,1]$. For $f \in C^1[0,1]$ we define

$$||f|| = \max_{0 \le s \le 1} |f(s)| + \max_{0 \le s \le 1} |f'(s)|. \tag{4.5}$$

In order to convert the problem (2.12), (2.13) to the form (4.1) with L and H compact, we rewrite (2.12) as

$$\theta'' + \frac{\lambda \sin\theta}{M'(\theta)} = 0. \tag{4.6}$$

Let $G(s,t)$ be the Green's function for $\mathcal{L}\, v = -v''$ with the boundary conditions (2.13);

$$G(s,t) = \begin{cases} s, & 0 \le s \le t \le 1, \\ t, & 0 \le t \le s \le 1. \end{cases} \tag{4.7}$$

Then (4.6) with the boundary condition (2.13) is equivalent to

$$\theta(s) = \lambda \int_0^1 G(s,t) \frac{\sin\theta(t)}{M'(\theta'(t))}\, dt. \tag{4.8}$$

It is easy to check that the expression on the left of (4.8) considered as an operator on $C^1[0,1]$ has the form (4.1). The linearization of (4.8) about the trivial solution is equivalent to the linearization of (2.12), viz.

$$\theta'' + \lambda\theta = 0, \tag{4.9}$$

along with the boundary conditions (2.13). This problem has eigenvalues

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 \pi^2, \qquad n = 0, 1, 2, 3, \dots \tag{4.10}$$

with the corresponding eigenfunctions

$$\phi_n(s) = \sin\left(n + \frac{1}{2}\right)\pi s. \tag{4.11}$$

Each of the eigenvalues is simple. Therefore, we may apply Theorems 4.2, 4.3 and 4.4 to conclude that through each point $(\lambda_n, 0)$ there passes a connected family C_n of solutions of (2.12), (2.13) which is, at least locally, a curve. Now in fact, C_n and C_m are disjoint if $n \neq m$. This follows from the fact that nontrivial solutions belonging to C_n inherit their nodal structure from (4.11) and hence may be characterized by having exactly n interior zeros (i.e., zeros on $(0, 1)$). In the next section we will show that C_n is globally a curve. Since alternative (ii) of Theorem 4.3 is ruled out, C_n must be unbounded. We now show that on each C_n, λ is unbounded from above, i.e., for every $\lambda > \lambda_n$ there exists a nontrivial solution of (2.12), (2.13) having exactly n interior zeros. We do this by ruling out all other possibilities.

Lemma 4.12. *If $\lambda \leq 0$ (2.12), (2.13) has only the trivial solution.*

Proof. If $\lambda = 0$, this follows directly from the equation which can be integrated to obtain $M(\theta') =$ constant. Since $\theta'(1) = 0$ the constant is zero and $\theta' \equiv 0$. Since $\theta(0) = 0$, $\theta \equiv 0$. Otherwise, we note that (2.12), (2.13) has a first integral

$$F(\theta'(s)) - \lambda \cos\theta(s) = -\lambda\cos\gamma \tag{4.13}$$

where

$$F(p) = pM(p) - W(p) \tag{4.14}$$

and

$$\gamma = \theta(1). \tag{4.15}$$

We note that F has the following properties:

$$\begin{gathered} F(0) = 0, \qquad F(p) > 0 \quad \text{for } p \neq 0, \\ F \text{ is even and convex,} \qquad F(p) \to \infty \quad \text{as } |p| \to \infty. \end{gathered} \tag{4.16}$$

Now, for a nontrivial solution $F(\theta'(0)) > 0$ but

$$F(\theta'(0)) = \lambda(1 - \cos\gamma) \leq 0$$

if $\lambda < 0$, a contradiction, proving the lemma. If we assume

$$W''(p) \geq W(p)/p \quad \text{for } p \neq 0, \tag{4.17}$$

then, as is shown in [**12**] a sharp lower bound on the set of λ for which nontrivial solutions exist is λ_0. This is the case in which C_0 is as in Figure 1b (supercritical bifurcation). However, if (4.17) is violated, it is easy to construct examples in which C_0 is as in Figure 1a (subcritical bifurcation) and there exist nontrivial solutions for $\lambda < \lambda_0$. However, as we are in the process of showing, if we follow the curve in Figure 1a it must eventually cross every line $\lambda = c$ for $c > \lambda_0$. $\square$

Lemma 4.18. *For a fixed $\lambda > 0$, there exists a constant $C(\lambda)$ such that if θ is a solution of* (2.12), (2.13), *then*

$$||\theta||_{C^1} \leq C(\lambda). \tag{4.19}$$

Proof. From (4.13) it follows that $F(\theta') \leq 2\lambda$. Since F is coercive this implies that there is a $C' = C'(\lambda)$ such that

$$|\theta'(s)| \leq C'(\lambda). \tag{4.20}$$

The bound (4.19) then follows from (3.4). $\square$

Remark 4.21. Our phase plane analysis of the next section shows that we have the a priori bound

$$|\theta(s)| < \pi, \qquad 0 \leq s \leq 1. \tag{4.22}$$

It follows from Lemmas 4.12 and 4.18 that, along C_n, λ must be unbounded from above as all other possibilities have been ruled out. Since, if $\theta(s)$ is a solution of (2.12), (2.13), then so is $-\theta(s)$ nontrivial solutions occur in pairs, and we have

Theorem 4.22. *Let λ satisfy* (3.20)*. Then there are at least $2n+2$ nontrivial solutions of* (2.12)*,* (2.13)*.*

5. Phase plane analysis. In this section we show how the problem (2.12), (2.13) can be attacked directly using phase plane analysis. This analysis confirms the results obtained by bifurcation theory and shows that the sets C_n defined in Section 4 are indeed curves. The results of this section are taken from Maddock's paper [**12**].

We let $u = \theta$, $v = \theta'$ and write (2.12) as the system

$$u' = v, \qquad v' = \frac{\lambda \sin u}{M'(v)}. \tag{5.1}$$

Equation (5.1) has rest points at $(u, v) = (n\pi, 0)$, $n = 0, \pm 1, \pm 2, \dots$. It also has the first integral

$$F(v) - \lambda \cos u = -\lambda \cos \gamma \tag{5.2}$$

where F is given by (4.14) and γ by (4.15). The phase plane picture is given in Figure 2. The curve corresponding to $\gamma = \pi$, i.e.,

$$F(v) - \lambda(\cos u + 1) = 0 \tag{5.3}$$

is a separatrix connecting the rest points at $(-\pi, 0)$ and $(\pi, 0)$. Orbits of (5.1) lie on the curves (5.2). Since F is an even function of v, the phase plane picture is symmetric in both axes. For a fixed λ a nontrivial solution exists and lies in C_n precisely when an orbit which starts on the v axis at $s = 0$ arrives at the u axis at $s = 1/(2n+1)$, i.e., the "time" it takes to make a quarter turn around the origin is $1/(2n+1)$. In order to show that such orbits actually exist we must show that the parameter γ can be chosen in $(0, \pi)$ in such a way as to bring this about.

We will consider solutions with $\theta'(0) > 0$. By symmetry there will be a corresponding solution with $\theta'(0) < 0$ differing only in sign.

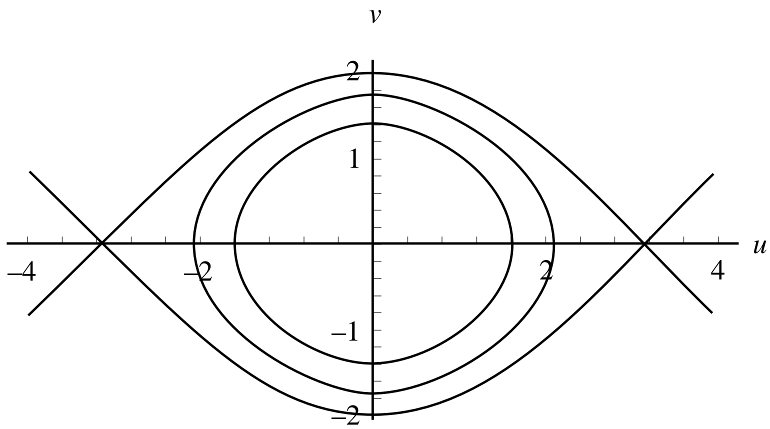

FIGURE 2. Phase plane for equation (5.1).

Since we are considering a piece of the orbit for which $v = \theta' \geq 0$ we may rewrite (5.2) as

$$d\theta/ds = G(\lambda(\cos\theta - \cos\gamma)) \tag{5.4}$$

where G is the inverse of the restriction of F to $[0, \infty)$. Thus, the "time" it takes for the orbit to go from $(0, \theta'(0))$ to $(\gamma, 0)$ is given by

$$T(\gamma, \lambda) = \int_0^\gamma \frac{d\theta}{G(\lambda(\cos\theta - \cos\gamma))}. \tag{5.5}$$

Recall that we have a nontrivial solution of (2.12), (2.13) lying on C_n precisely when

$$T(\gamma, \lambda) = \frac{1}{2n+1}. \tag{5.6}$$

Our results follow from the theorem proved by Maddocks. (The notation used here is different from that used in [**12**].)

Theorem 5.7 [**12**]. *The function T defined by* (5.5) *has the following properties*:

T is a differentiable function of (γ, λ) on $(0, \pi) \times (0, \infty)$ with

$$\lim_{\gamma \to \pi} T(\gamma, \lambda) = \infty, \qquad \lim_{\gamma \to 0} T(\gamma, \lambda) = \frac{\pi}{2\sqrt{\lambda}}, \tag{5.8}$$

$$\frac{\partial T(\gamma, \lambda)}{\partial \lambda} < 0. \tag{5.9}$$

If (4.16) *holds, then*

$$\frac{\partial T(\gamma, \lambda)}{\partial \gamma} > 0. \tag{5.10}$$

From this theorem we can immediately read off the following results:

1. If

$$\lambda > \frac{\pi^2}{4}(2n+1)^2 = \lambda_n \tag{5.11}$$

there exists a nontrivial solution of (2.12), (2.13) in C_n.

2. If in addition (4.17) and thus (5.10) holds, the solution in C_n is unique (up to sign). In this case the bifurcation is supercritical. By the implicit function theorem, (5.6) can be solved for γ as a function of λ on (λ_n, ∞). This also shows that C_n is a curve.)

3. In any case (5.9) shows that (5.6) can be solved for λ as a function of γ on $(0, \pi)$, again showing that C_n is a curve.

If (4.17) is violated (it is easy to construct such W's) then the curve can assume shapes as in Figures 1a and 1c.

Thus we have confirmed the theoretical analysis of Section 4. We next turn to the issue of stability.

6. Stability. In this section we consider the stability of solutions on the branches of nontrivial solutions which bifurcate from the trivial solution. Recall that a solution is stable if it is a local minimum of the energy E defined by (2.17). A solution θ is stable if the second variation of E at θ is a positive definite quadratic form [**12, 13**], i.e.,

$$\int_0^1 \{M'(\theta'(s))\eta'(s)^2 + \lambda \cos(\theta(s))\eta(s)^2\}\, ds > 0 \tag{6.1}$$

for all $\eta \in V$ which do not vanish identically. Condition (6.1) is equivalent to the requirement that the linear, self adjoint eigenvalue problem

$$-(M'(\theta')\eta')' + \lambda \cos\theta\eta = \mu\eta, \tag{6.2}$$

$$\eta(0) = 0, \qquad \eta'(1) = 0 \tag{6.3}$$

have only positive eigenvalues.

The following result is classical (see, e.g., [**13**, Section 6]).

Theorem 6.4. *The number of negative eigenvalues of* (6.2), (6.3) *is equal to the number of interior zeros of the solution y of the initial value problem*

$$-(M'(\theta')y')' + \lambda \cos\theta y = 0, \tag{6.5}$$

$$y(1) = 1, \qquad y'(1) = 0. \tag{6.6}$$

Theorem 6.7. *Solutions lying on C_n for $n > 0$ are unstable.*

Proof. If $\theta(s)$ is a nontrivial solution of (2.12), (2.13) then $\theta'(s)$ satisfies (6.5). (Just differentiate (2.12).) If θ has $n \geq 1$ interior zeros, then θ' has at least one interior zero. Of course, $\theta'(1) = 0$. Since zeros of solutions of (6.5) interlace each other, the solution y of (6.5), (6.6) must have at least one interior zero. The result follows from an application of Theorem 6.4. $\square$

Theorem 6.7 tells us that for $\lambda > \lambda_0$ the minimum energy solution must lie on C_0 and therefore we again see that this branch must intersect each line $\lambda = c$ for $c > \lambda_0$.

We now turn to the branch C_0. In the context of their bifurcation theory, Crandall and Rabinowitz [**7**] have shown that if the bifurcation is subcritical (Fig. 1a), then, at least near the bifurcation point, solutions on C_0 are unstable. This is a purely local result. The definitive work on the subject is that of Maddocks [**12**]. He is able to analyze quite completely the stability of solutions along a branch

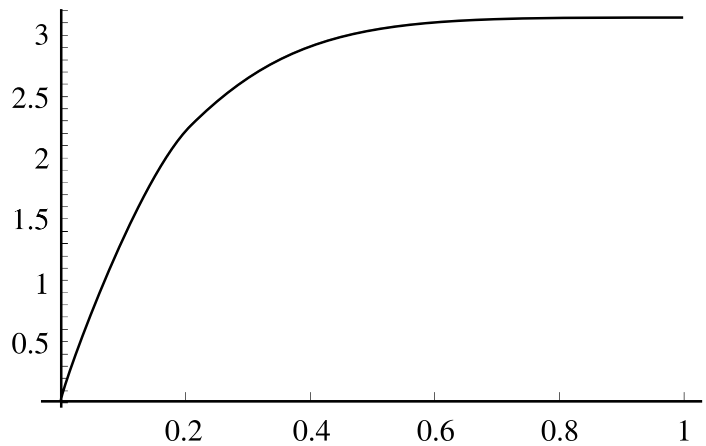

FIGURE 3. Solution on C_0 for $W(p) = p^2/2$, $\lambda = 50$.

such as C_0. The crucial idea in this paper is that when the ordinate in the bifurcation diagram is chosen appropriately stability depends on the shape of the curve. His results are global. The relevant result as applied to our problem is

Theorem 6.8 [**12**]. *On C_0 the forward going segments, i.e., those segments for which $d\gamma/d\lambda > 0$ are stable while the backward going segments are unstable.*

Hence in Figure 1c, if $F(u) = \gamma = \theta(1)$, then the solutions on segments AB and CD are stable while solutions on segments OA and BC are unstable.

So we see that the answer to the question of the location of the minimum energy solutions is that they must lie on one of the forward going segments of C_0.

7. Asymptotic behavior. We now consider the behavior of solutions as $\lambda \to \infty$ along the branch C_n. It is interesting to trace the evolution of a solution as it moves along C_n. The branch C_n begins at $(\lambda_n, 0)$, along the branch $\lambda \to \infty$ although perhaps not

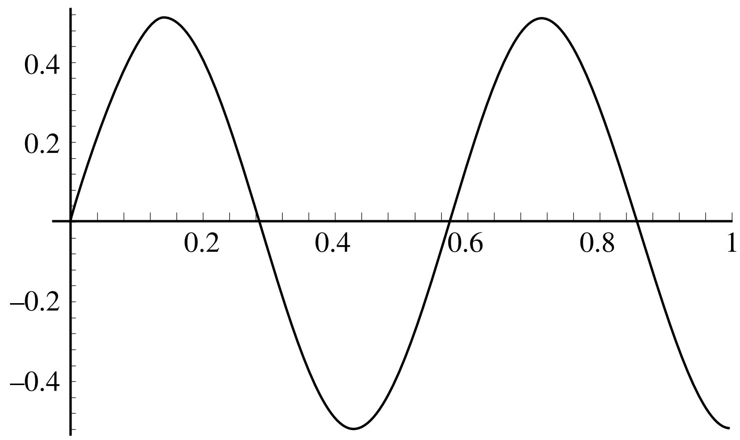

FIGURE 4a. Solution on C_3 for $W(p) = p^2/2$, $\lambda = 125$. Note: $\lambda_3 = (7\pi/2)^2 = 120.9$.

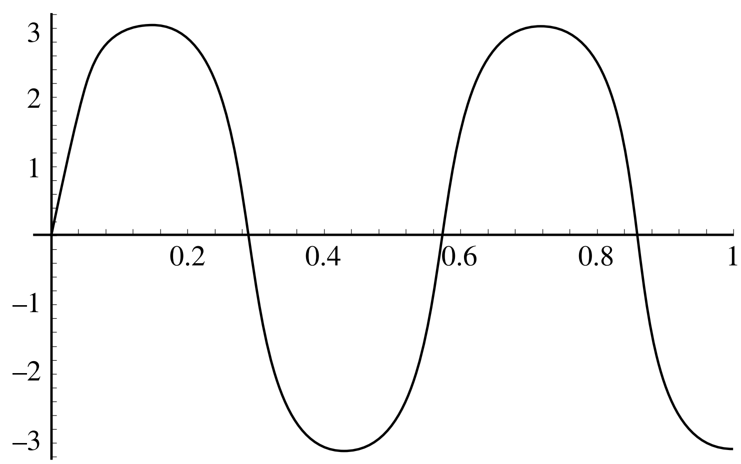

FIGURE 4b. Solution on C_3 for $W(p) = p^2/2$, $\lambda = 1000$.

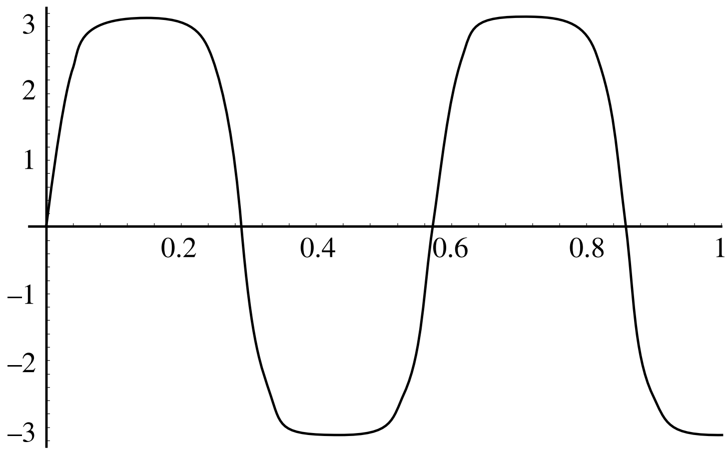

FIGURE 4c. Solution on C_3 for $W(p) = p^2/2$, $\lambda = 2000$.

monotonically. Near the initial point the solution can be represented as a small multiple of the eigenfunction (4.11) plus a smaller perturbation. This is illustrated in Figure 4a. We now wish to analyze the solution for very large λ. We may rewrite (2.12) as

$$\varepsilon M'(\theta') + \sin\theta = 0 \tag{7.1}$$

where $\varepsilon = 1/\lambda$. We recognize (7.1) as a singular perturbation problem. This problem is treated in [**14**]. Formally, for small ε, we obtain a zero order approximation by setting $\varepsilon = 0$, i.e., $\sin\theta = 0$ so that $\theta = 0$ or $\pm\pi$. What we may expect is that the solution will be constant except for *transition layers* in which the solution jumps from one constant to another in a very short s interval. More specifically, referring to the phase plane picture, Figure 2, since $(\pm\pi, 0)$ are saddle points, orbits which are close to the separatrix must spend much of the "time" in a neighborhood of these points. So for a solution lying on C_0, for large λ, in the phase plane representation the solution will move quickly to the neighborhood of $(\pi, 0)$ (Figure 3). In this case, the transition layer is at $s = 0$ and θ is near π for most of the interval. If we denote by $\theta(s,\lambda)$ the solution lying on C_0,

$$\lim_{\lambda\to\infty} \theta(s,\lambda) = \pi, \qquad 0 < s \le 1. \tag{7.3}$$

Similarly, if $\theta(s,\lambda)$ is a solution lying on C_n, $n \geq 1$, then for large λ, $\theta(s,\lambda)$ undergoes a succession of jumps from π to $-\pi$ and from $-\pi$ to π, the jumps occurring at $s = 2k/(2n+1)$ for $k = 1, \dots, n$. So, for example, for $\theta(s,\lambda)$ on C_1,

$$\lim_{\lambda\to\infty} \theta(s,\lambda) = \begin{cases} \pi, & 0 < s < 2/3, \\ -\pi, & 2/3 < s \leq 1, \end{cases} \tag{7.4}$$

and for $\theta(s,\lambda)$ on C_2

$$\lim_{\lambda\to\infty} \theta(s,\lambda) = \begin{cases} \pi, & 0 < s < 2/5, \\ -\pi, & 2/5 < s < 4/5, \\ \pi, & 4/5 < s \leq 1. \end{cases} \tag{7.5}$$

Figure 4 shows the evolution of a solution on C_3. For λ near $\lambda_3 = (7\pi/2)^2$ the solution looks like a multiple of

$$\phi_3(s) = \sin\frac{7\pi}{2}s.$$

As λ increases, we see the curve flattening out on top and transition layers developing (Figure 4b). As λ continues to increase, the flattening process becomes more pronounced (Figure 4c). Figure 5 shows the shape of the rod for large λ corresponding to the solution pictured in Figure 4c. This solution is constructed by using (2.14). As one can see, this solution is highly nonphysical. But recall that it lies on an unstable branch so we wouldn't expect to see it in nature.

8. Conclusions. In this paper we have tried to show how the techniques of modern nonlinear analysis may be brought to bear on a classical problem. Of course these techniques can be applied to more complicated models in which the rod is assumed to be nonlinearly elastic and can undergo stretch, twist and shear, as well as being subject to a variety of forces [**6, 16, 18**]. These problems can be three dimensional as well as planar. In some problems there is an obstruction to the use of bifurcation theory. Because of the existence of symmetries in the problem, the hypothesis of Theorem 4.4 fails. The problem is that the eigenvalues are not simple. This has led to a study of "bifurcation in the presence of symmetry" [**10, 11**], which is an effort to surmount this difficulty. Sometimes one can prove bifurcation results

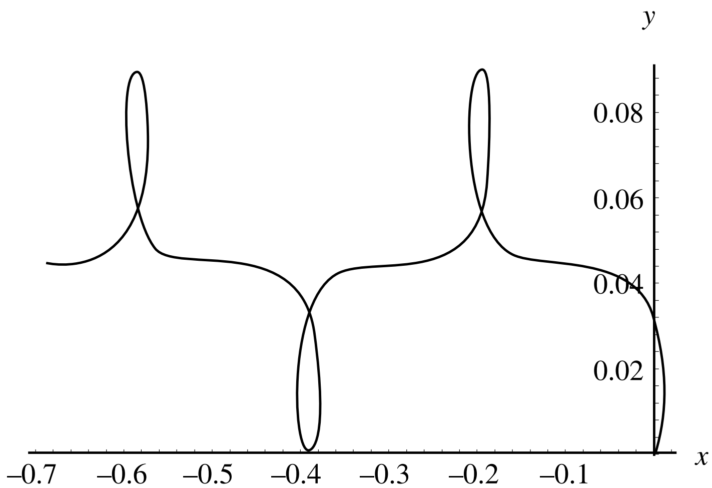

FIGURE 5. Unphysical "shape of the rod" corresponding to the solution depicted in Figure 4c.

in cases in which the Crandall-Rabinowitz theory fails by an argument which directly exploits the variational structure of the problem [**19**]. Another area of investigation is multiparameter bifurcation theory [**1, 2, 11**].

These problems can also be investigated numerically. In a typical numerical study the boundary value problem is solved by a shooting method. The bifurcating branches of solutions can be traced using a numerical continuation method [**3**]. In fact, even with all the analytical tools at our disposal, much insight can be gained by performing numerical experiments.

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