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WEAK COMPACTNESS IN SPACES OF DIFFERENTIABLE MAPPINGS

MANUEL GONZÁLEZ AND JOAQUÍN M. GUTIÉRREZ

ABSTRACT. We characterize the weakly compact subsets (and thereby the weak convergence) in several spaces of k-times continuously differentiable mappings between real Banach spaces. As an application, we give characterizations of the Dunford-Pettis (DP) property of a Banach space F in terms of the weak sequential continuity of the composition map $(f,g) \mapsto g \circ f$, where $f : E \to F$ is a differentiable mapping and $g : F \to G$ is a linear operator. We also prove that F has the DP property if and only if whenever $(x_n) \subset F$ is weakly null and (P_n) is a weakly null sequence of polynomials from F to another space G, then $(P_n(x_n))$ converges to 0 in the weak topology of G. Finally, we derive a new proof of the fact that any weakly compact homomorphism between algebras of differentiable functions is induced by a constant mapping.

1. Introduction. Kalton [**11**] characterized the weakly compact subsets of the space $K(E,F)$ of compact operators between Banach spaces E and F. These results were extended in [**4, 5**] to the case of compact operators between locally convex spaces.

In this paper we characterize the weakly compact subsets (and thereby the weak convergence) in several spaces of k-times continuously Fréchet differentiable mappings between real Banach spaces E and F: $C^k_{wu}(E,F)$ (definition below), $C^k_c(E,F)$ and $C^k(E,F)$ (definitions in Section 5). As an application, we prove that F has the Dunford-Pettis property if and only if, for every pair of real Banach spaces E, G and integer k, the composition map:

$$T : C^k_{wu}(E,F) \times L(F,G) \to C^k_{wu}(E,G)$$

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given by $T(f,g) = g \circ f$, where $L(F,G)$ is the space of (linear bounded) operators from F to G, is weakly sequentially continuous, in the sense that it takes weakly convergent sequences into weakly convergent sequences. Let us note that taking $E = \{0\}$ and $G = \mathbf{R}$ (the real field), T becomes a map from $F \times F^*$ into $\mathbf{R}$ and we obtain the usual sequential definition of the Dunford-Pettis property: if $(x_n) \subset F$, $(\phi_n) \subset F^*$ are weakly null sequences, then $(\langle x_n, \phi_n \rangle)$ tends to zero. This result is also valid if we replace C^k_{wu} by C^k_c or C^k and extends to the differentiable case analogous characterizations obtained in [**16, 17**] for spaces of compact operators and spaces of vector valued continuous functions. Similar results are obtained for spaces of polynomials. Finally, we derive a new proof of the fact (see [**8**]) that any weakly compact homomorphism between algebras of differentiable functions is induced by a constant mapping.

$\mathbf{R}$ will be the real field, $\mathbf{N}$ the natural numbers including 0, and $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$. E, F and G will denote real Banach spaces, B_E the closed unit ball of E, and E^* its topological dual. The weak-star topology is denoted by w^* and bw^* is the finest topology on E^* that agrees with w^* on bounded subsets; bw^* is a locally convex topology [**6**, II, Section 5, Lemma 2]. $E^*_{bw^*}$ will denote the space E^* endowed with the bw^* topology. For a Hausdorff topological space X, we denote by $C(X,F)$ the space of continuous mappings from X into F. Throughout, when the range space is omitted, it is understood to be $\mathbf{R}$; for example, $C(X) = C(X,\mathbf{R})$. Unless otherwise stated, $C(X)$ is given the compact open topology whose associated weak topology will be denoted by w. The topology on $C(X)$ of pointwise convergence is referred to as τ_p. We shall need the following result:

Theorem 1 [**7**, 4.3 Corollary 2]. *Suppose $A \subset C(X)$ is uniformly bounded on all compact subsets of X. Then*:

(a) *A is* (*relatively*) *τ_p-compact if and only if it is* (*relatively*) *w-compact.*

(b) *Each sequence $(f_n) \subset A$ is τ_p-convergent if and only if it is weakly convergent.*

$C_{wu}(E,F)$ stands for the space of all maps $f : E \to F$ which are weakly uniformly continuous on all bounded subsets; in other words, for each bounded subset $B \subset E$ and $\varepsilon > 0$ there are a finite set $\{\phi_1, \dots, \phi_k\} \subset E^*$ and $\delta > 0$ such that, if $x, y \in B$ and $|\phi_i(x-y)| < \delta$, $i = 1, \dots, k$, then $||f(x) - f(y)|| < \varepsilon$.

For $n \in \mathbf{N}$ and Hausdorff locally convex spaces X, Y, we denote by $\mathcal{P}\,({}^nX, Y)$ the space of all n-homogeneous continuous polynomials from X to Y; i.e., all maps of the form $P(x) = A(x, \overset{(n)}{\dots}, x)$ for any n-linear continuous map A from $X^n = X \times \overset{(n)}{\cdots} \times X$ into Y, endowed with the topology of uniform convergence on bounded subsets. $\mathcal{P}\,({}^0X, Y)$ may be identified with Y. $\mathcal{P}\,({}^nE, F)$ is a Banach space with the norm $||P|| = \sup\{||P(x)|| : ||x|| \leq 1\}$. Given $P \in \mathcal{P}\,({}^nE, F)$, there is a unique continuous symmetric n-linear mapping $\widehat{P} : E^n \to F$ such that $\widehat{P}(x, \dots, x) = P(x)$ for all $x \in E$. We write $\mathcal{P}_{wu}({}^nE, F) = \mathcal{P}\,({}^nE, F) \cap C_{wu}(E, F)$.

For $k \in \mathbf{N}$ and X as above, $C^k(X, F)$ stands for the space of all k-times continuously Fréchet differentiable mappings from X to F (see [**12**]), endowed with the topology τ_u^k of uniform convergence of the mappings and their derivatives, up to the order k, on compact subsets. We denote by $C_{wu}^k(E, F)$ the subspace of those $f \in C^k(E, F)$ such that for $0 \leq j \leq k$, we have $d^jf \in C_{wu}(E, \mathcal{P}_{wu}({}^jE, F))$, where d^jf is the jth derivative map of f, and $d^0f = f$. It will be endowed with the topology τ_b^k generated by the seminorms

$$p_n(f) = \sup\{||d^jf(x)|| : ||x|| \leq n,\ 0 \leq j \leq k\}.$$

The space $(C_{wu}^k(E, F), \tau_b^k)$ is Fréchet [**3**, Proposition 3].

In Section 5 we introduce the spaces $(C_c^k(E, F), \tau_u^k)$ and $(C^k(E, F), \tau_c^k)$ of differentiable mappings, for which our results are also valid, and we outline the corresponding proofs. We have chosen to give the proofs in detail for the case of $C_{wu}^k(E, F)$ which is a bit more complicated. All these spaces were introduced in infinite dimensional approximation theory. It is worth noting that, for $\dim(E) < \infty$, we have $C_{wu}^k(E, F) = C^k(E, F)$ for all F and that an operator from E to F belongs to $C_{wu}^k(E, F)$ if and only if it is compact [**3**, Proposition 2.5].

The space $C_{wu}^k(E, F)$ was introduced in [**3**] and has been extensively studied by several authors [**1, 2, 8, 9, 10**]. Another reason for our interest in this space is that it has complex analogs of some relevance to the Michael problem on automatic continuity of complex valued homomorphisms on a complex Fréchet algebra. Namely, let E be a complex Banach space, $F = \mathbf{C}$ the complex field, and $k = \infty$, and call the corresponding space $H_{wu}(E)$. It is known [**14**, p. 244] that if every scalar valued homomorphism on $H_{wu}(E)$ is continuous then so is every scalar valued homomorphism on any complex Fréchet algebra.

The following result will also be needed:

Theorem 2 [**1**, Theorem 2.4]. *Every mapping $f \in C^k_{wu}(E,F)$ can be extended in a unique way to a mapping $\tilde{f} : E^{**} \to F$, so that $\tilde{f} \in C^k(E^{**}_{bw^*}, F)$. Moreover, $f \mapsto \tilde{f}$ defines a linear isometry of Fréchet spaces.*

2. Weakly compact subsets of $C^k_{wu}(E,F)$. In this section we characterize the weakly compact subsets of $C^k_{wu}(E,F)$, first for $k = 0$ and then for arbitrary k. Let $U = E^{**}_{bw^*}$ and $V = (B_{F^*}, w^*)$. We define the map $\chi : C_{wu}(E,F) \to C(U \times V)$ taking $f \in C_{wu}(E,F)$ into the function χ_f given by

$$\chi_f(u,v) = \langle \tilde{f}(u), v\rangle, \quad \text{for each } u \in U \text{ and } v \in V,$$

where $\tilde{f}$ is the extension of f to $C(E^{**}_{bw^*}, F)$ (Theorem 2).

Proposition 3. *The mapping χ defines a linear isometry from the Fréchet space $C_{wu}(E,F)$ onto a closed linear subspace of $C(U \times V)$.*

Proof. We first show that χ is well defined. Indeed, given $f \in C_{wu}(E,F)$, let $(u_\alpha, v_\alpha) \subset U \times V$ be a net converging to (u,v). For $\varepsilon > 0$, there is an α_0 such that we have $||\tilde{f}(u_\alpha) - \tilde{f}(u)|| < \varepsilon/2$ and $|\langle \tilde{f}(u), v_\alpha - v\rangle| < \varepsilon/2$, whenever $\alpha \geq \alpha_0$. Then:

$$\begin{aligned}|\chi_f(u_\alpha, v_\alpha) - \chi_f(u,v)| &= |\langle \tilde{f}(u_\alpha), v_\alpha\rangle - \langle \tilde{f}(u), v\rangle| \\ &\leq |\langle \tilde{f}(u_\alpha), v_\alpha\rangle - \langle \tilde{f}(u), v_\alpha\rangle| \\ &\quad + |\langle \tilde{f}(u), v_\alpha\rangle - \langle \tilde{f}(u), v\rangle| \\ &\leq ||v_\alpha|| \cdot ||\tilde{f}(u_\alpha) - \tilde{f}(u)|| + |\langle \tilde{f}(u), v_\alpha - v\rangle| \\ &< \varepsilon, \quad \alpha \geq \alpha_0,\end{aligned}$$

so $\chi_f \in C(U \times V)$. Easily, χ is linear and injective. Moreover, denoting by (q_n) the seminorms that generate the topology of $C(U \times V)$, we have for each $f \in C_{wu}(E,F)$:

$$\begin{aligned} q_n(\chi_f) &= \sup\{|\langle \tilde{f}(u), v\rangle| : u \in nB_{E^{**}}; v \in B_{F^*}\} \\ &= \sup\{||\tilde{f}(u)|| : u \in nB_{E^{**}}\} \\ &= \sup\{||f(x)|| : x \in nB_E\} \\ &= p_n(f). \qquad \square \end{aligned}$$

We say that a net $(f_\alpha) \subset C_{wu}(E,F)$ converges to f in the w_0 topology if $\langle \tilde{f}_\alpha(u), v\rangle$ converges to $\langle \tilde{f}(u), v\rangle$ for all $u \in E^{**}$ and $v \in F^*$.

Theorem 4. *A family $A \subset C_{wu}(E,F)$ is weakly compact if and only if it is uniformly bounded on all bounded subsets of E and w_0-compact.*

Proof. Suppose $A \subset C_{wu}(E,F)$ is weakly compact; then A is bounded, which means uniformly bounded on all bounded subsets of E. Since w_0 is coarser than the weak topology of $C_{wu}(E,F)$, A is w_0-compact.

Conversely, suppose $A \subset C_{wu}(E,F)$ is uniformly bounded on all bounded subsets of E and w_0-compact. If $(f_\alpha) \subset C_{wu}(E,F)$ is a w_0-null net, then $\langle \tilde{f}_\alpha(u), v\rangle = \chi_{f_\alpha}(u,v)$ tends to zero for each $u \in E^{**}$ and $v \in B_{F^*}$, so χ is a continuous mapping from $(C_{wu}(E,F), w_0)$ into $(C(U \times V), \tau_p)$, and $\chi(A)$ is τ_p-compact. From Theorem 1, $\chi(A)$ is w-compact. Hence, by Proposition 3, A is weakly compact. $\square$

Corollary 5. *A sequence $(f_n) \subset C_{wu}(E,F)$ is weakly convergent to $f \in C_{wu}(E,F)$ if and only if it is uniformly bounded on all bounded subsets of E and $\langle \tilde{f}_n(u), v\rangle$ converges to $\langle \tilde{f}(u), v\rangle$ for each $u \in E^{**}$ and $v \in B_{F^*}$.*

We shall now deal with the space $C^k_{wu}(E,F)$ for $k \in \mathbf{N}^*$. The following property will be used without explicit mention: given $f \in C^k_{wu}(E,F)$ with extension $\tilde{f} \in C^k(E^{**}_{bw^*}, F)$, whose derivatives are $d^j f \in C_{wu}(E, \mathcal{P}_{wu}(^j E, F))$ and $d^j \tilde{f} \in C(E^{**}_{bw^*}, \mathcal{P}(^j E^{**}_{bw^*}, F))$, $1 \le j \le k$, respectively, then $d^j \tilde{f}$ is the extension of $d^j f$ to $E^{**}_{bw^*}$ (see [**1**, Theorem 2.4]).

We say that a net $(f_\alpha) \subset C^k_{wu}(E,F)$ converges to f in the w_k topology whenever $\langle d^j \tilde{f}_\alpha(u), \psi\rangle$ converges to $\langle d^j \tilde{f}(u), \psi\rangle$ for all $u \in E^{**}$,

$0 \leq j \leq k$, and $\psi \in \mathcal{P}_{wu}({}^j E, F)^*$. For $k = \infty$, we let $j \in \mathbf{N}$. Since $f \mapsto \langle d^j \tilde{f}(u), \psi \rangle$ defines a linear continuous form on $C^k_{wu}(E, F)$, the w_k topology is coarser than the weak topology.

For each $f \in C_{wu}(E, \mathcal{P}_{wu}({}^j E, F))$ and $u \in E^{**}$ we have $\tilde{f}(u) \in \mathcal{P}_{wu}({}^j E, F)$ where $\tilde{f}$ is the extension of f to E^{**}; $\tilde{f}(u)$ has a unique extension to a polynomial $\hat{f}(u) \in \mathcal{P}({}^j E^{**}_{bw^*}, F)$. We say that a net $(f_\alpha) \subset C_{wu}(E, \mathcal{P}_{wu}({}^j E, F))$ converges to f in the $\hat{w}_j$ topology whenever $\langle \hat{f}_\alpha(u)(h), v \rangle$ converges to $\langle \hat{f}(u)(h), v \rangle$ for each $u \in E^{**}$, $h \in B_{E^{**}}$, and $v \in B_{F^*}$.

Write $W = (B_{E^{**}}, w^*)$ and, for U and V as above, define the map

$$T : C_{wu}(E, \mathcal{P}_{wu}({}^j E, F)) \to C(U \times W \times V)$$

taking f into a function Tf given by $Tf(u, h, v) = \langle \hat{f}(u)(h), v \rangle$ for $u \in U$, $h \in W$ and $v \in V$.

Lemma 6. (a) *T defines a linear isometry of the Fréchet space $C_{wu}(E, \mathcal{P}_{wu}({}^j E, F))$ onto a closed linear subspace of $C(U \times W \times V)$.*

(b) *T is also an isomorphism when these spaces are given the topologies $\hat{w}_j$ and τ_p, respectively.*

Proof. (a) Let us see that T is well defined. If $(u_\alpha, h_\alpha, v_\alpha) \subset U \times W \times V$ is a net converging to (u, h, v) and $\varepsilon > 0$, then there is an α_0 such that, for $\alpha \geq \alpha_0$, $||\hat{f}(u_\alpha) - \hat{f}(u)|| < \varepsilon/3$, $||\hat{f}(u)(h_\alpha) - \hat{f}(u)(h)|| < \varepsilon/3$ and $|\langle \hat{f}(u)(h), v_\alpha - v \rangle| < \varepsilon/3$. Hence,

$$\begin{aligned} |\langle \hat{f}(u_\alpha)(h_\alpha), v_\alpha \rangle - \langle \hat{f}(u)(h), v \rangle| &\leq |\langle \hat{f}(u_\alpha(h_\alpha), v_\alpha \rangle - \langle \hat{f}(u)(h_\alpha), v_\alpha \rangle| \\ &\quad + |\langle \hat{f}(u)(h_\alpha), v_\alpha \rangle - \langle \hat{f}(u)(h), v_\alpha \rangle| \\ &\quad + |\langle \hat{f}(u)(h), v_\alpha \rangle - \langle \hat{f}(u)(h), v \rangle| \\ &\leq ||v_\alpha|| \cdot ||\hat{f}(u_\alpha) - \hat{f}(u)|| \cdot ||h_\alpha||^j \\ &\quad + ||v_\alpha|| \cdot ||\hat{f}(u)(h_\alpha) - \hat{f}(u)(h)|| \\ &\quad + |\langle \hat{f}(u)(h), v_\alpha - v \rangle| \\ &< \varepsilon. \end{aligned}$$

It is easy to check that T is linear and injective. If (p_n) and (q_n) are the seminorms generating the topology of $C_{wu}(E, \mathcal{P}_{wu}({}^j E, F))$ and

$C(U\times W\times V)$, respectively, then we have for each $f\in C_{wu}(E,\mathcal{P}_{wu}(^jE,F))$:

$$\begin{aligned} q_n(Tf) &= \sup\{|\langle \hat{f}(u)(h),v\rangle| : u\in nB_{E^{**}}; h\in B_{E^{**}}; v\in B_{F^*}\} \\ &= \sup\{||\hat{f}(u)|| : u\in nB_{E^{**}}\} \\ &= \sup\{||\tilde{f}(u)|| : u\in nB_{E^{**}}\} \\ &= p_n(f). \end{aligned}$$

(b) is straightforward from the definition of $\hat{w}_j$. □

Given a family $A\subset C^k_{wu}(E,F)$, we denote

$$A_j := \{d^j f : f\in A\}\subset C_{wu}(E,\mathcal{P}_{wu}(^jE,F)), \qquad 0\le j\le k.$$

Lemma 7. *Given $k\in \mathbf{N}^*\backslash\{0\}$, let $A\subset C^k_{wu}(E,F)$ be a bounded family. Each A_j, $0\le j\le k$, is $\hat{w}_j$-compact in $C_{wu}(E,\mathcal{P}_{wu}(^jE,F))$ if and only if it is w_0-compact.*

Proof. Since A is bounded, A_j is uniformly bounded on all bounded subsets of E. If A_j is $\hat{w}_j$-compact then, by Lemma 6(b), $T(A_j)$ is τ_p-compact. Now, from Theorem 1, $T(A_j)$ is weakly compact. By Lemma 6(a), A_j is weakly compact and, hence, w_0-compact. Since w_0 is finer than $\hat{w}_j$, the converse is obvious. □

Theorem 8. *For $k\in\mathbf{N}$ and $A\subset C^k_{wu}(E,F)$, the following assertions are equivalent*:

(a) *A is weakly compact*;

(b) *A is w_k-compact and A_j is uniformly bounded on all bounded subsets of E for each $0\le j\le k$*;

(c) *A_j is uniformly bounded on all bounded subsets of E and w_0-compact in $C_{wu}(E,\mathcal{P}_{wu}(^jE,F))$ for each $0\le j\le k$*;

(d) *A_j is uniformly bounded on all bounded subsets of E and $\hat{w}_j$-compact in $C_{wu}(E,\mathcal{P}_{wu}(^jE,F))$ for each $0\le j\le k$.*

Proof. (c) $\Leftrightarrow$ (d) by the previous lemma.

(a) $\Rightarrow$ (b) is clear since w_k is coarser than the weak topology.

We shall prove the other parts by complete induction. For the C^0-maps, the result holds (Theorem 4). Suppose it holds for the C^p-maps, $0 \le p \le r-1 < k$. Define the linear map:

$$T : C^r_{wu}(E,F) \to C(E^{**}_{bw^*}, \mathcal{P}\,({}^r E^{**}_{bw^*}, F)) \times C^{r-1}_{wu}(E,F)$$

taking f into $Tf = (d^r \tilde{f}, f)$.

T is an injective isomorphism. Indeed, a sequence $(f_n) \subset C^r_{wu}(E,F)$ is τ^r_b-null if and only if $(d^r \tilde{f}_n)$ is null in the compact open topology and (f_n) is null in $(C^{r-1}_{wu}(E,F), \tau^{r-1}_b)$. T is also an isomorphism when $C^r_{wu}(E,F)$ is given the w_r topology and the range space is endowed with the product $w_0 \times w_{r-1}$. Indeed, let $(f_\alpha) \subset C^r_{wu}(E,F)$ be a net; then $f_\alpha \to 0$ in w_r if and only if $d^r \tilde{f}_\alpha \to 0$ in w_0 and $f_\alpha \to 0$ in w_{r-1}.

(b) $\Rightarrow$ (c). If A is w_r-compact, then $T(A)$ is $w_0 \times w_{r-1}$-compact and its projections $A_r \subset C(E^{**}_{bw^*}, \mathcal{P}\,({}^r E^{**}_{bw^*}, F))$ and $A \subset C^{r-1}_{wu}(E,F)$ are w_0 and w_{r-1} compact, respectively. By the induction hypothesis, the sets $A_0, \dots, A_{r-1}$ are w_0-compact.

(c) $\Rightarrow$ (a). If A_r is uniformly bounded on all bounded subsets of E and w_0-compact, then it is weakly compact in $C(E^{**}_{bw^*}, \mathcal{P}\,({}^r E^{**}_{bw^*}, F))$ (Theorem 4). If the sets $A_0, \dots, A_{r-1}$ are w_0-compact and uniformly bounded, then, by the induction hypothesis, A is weakly compact in $C^{r-1}_{wu}(E,F)$. Therefore, $T(A) \subset A_r \times A$ is relatively weakly compact and so is A in $C^r_{wu}(E,F)$. Now we need only show that A is weakly closed, which is true since the inclusion map $C^r_{wu}(E,F) \hookrightarrow C^{r-1}_{wu}(E,F)$ is continuous. □

Theorem 9. *For a family $A \subset C^\infty_{wu}(E,F)$, the following assertions are equivalent*:

(a) *A is weakly compact*;

(b) *A is w_∞-compact and A_j is uniformly bounded on all bounded subsets of E for each $j \in \mathbf{N}$*;

(c) *A_j is uniformly bounded on all bounded subsets of E and w_0-compact in $C_{wu}(E, \mathcal{P}_{wu}({}^j E, F))$ for each $j \in \mathbf{N}$*;

(d) *A_j is uniformly bounded on all bounded subsets of E and $\hat{w}_j$-compact in $C_{wu}(E, \mathcal{P}_{wu}({}^j E, F))$ for each $j \in \mathbf{N}$.*

Proof. (a) $\Rightarrow$ (b) since w_∞ is coarser than the weak topology.

(b) $\Rightarrow$ (c). Let $j \in \mathbf{N}$. By the continuity of the inclusion map $(C^\infty_{wu}(E,F), w_\infty) \hookrightarrow (C^j_{wu}(E,F), w_j)$, A is w_j-compact in $C^j_{wu}(E,F)$. Applying part (b) $\Rightarrow$ (c) of Theorem 8, we get the w_0-compactness of A_j in $C_{wu}(E, \mathcal{P}_{wu}({}^jE,F))$.

(c) $\Leftrightarrow$ (d) by Lemma 7.

(c) $\Rightarrow$ (a). Consider the operator $T : C^\infty_{wu}(E,F) \to \prod_{j=0}^\infty C^j_{wu}(E,F)$ given by $Tf = (f, f, \dots)$. Clearly, T is an injective topological isomorphism. By Theorem 8, A is weakly compact in $C^k_{wu}(E,F)$ for each $k \in \mathbf{N}$, so $T(A) \subset A \times A \times \dots$ is relatively weakly compact in $\prod_{j=0}^\infty C^j_{wu}(E,F)$ and so is A in $C^\infty_{wu}(E,F)$. On the other hand, by the continuity of the inclusion map $C^\infty_{wu}(E,F) \hookrightarrow C^k_{wu}(E,F)$, A is weakly closed in $C^\infty_{wu}(E,F)$, and this finishes the proof. □

The arguments used above also prove the following:

Theorem 10. *For $k \in \mathbf{N}\backslash\{0\}$, and $f, (f_n)$ in $C^k_{wu}(E,F)$, the following assertions are equivalent*:

(a) *(f_n) is weakly convergent to f;*

(b) *for each $0 \le j \le k$, $(d^j f_n)$ is uniformly bounded on all bounded subsets of E and $\langle d^j \tilde{f}_n(u), \psi\rangle$ converges to $\langle d^j \tilde{f}(u), \psi\rangle$ for all $u \in E^{**}$ and $\psi \in \mathcal{P}_{wu}({}^jE,F)^*$;*

(c) *for each $0 \le j \le k$, $(d^j f_n)$ is uniformly bounded on all bounded subsets of E and $\langle d^j \tilde{f}_n(u)(h), v\rangle$ converges to $\langle d^j \tilde{f}(u)(h), v\rangle$ for all $u \in E^{**}$, $h \in B_{E^{**}}$ and $v \in B_{F^*}$.*

This theorem is likewise valid for $k = \infty$, with obvious modifications.

3. Characterizations of the Dunford-Pettis property. In this section we apply the preceding results to obtain characterizations of the Dunford-Pettis property. We note that $K(F,G)$ is a subspace of $C^k_{wu}(F,G)$ for any $k \in \mathbf{N}^*$ [**3**, Proposition 2.5].

Theorem 11. *The following assertions are equivalent*:

(a) *F has the Dunford-Pettis* (DP) *property*;

(b) *for every* E, G *and* $k \in \mathbf{N}^*$, *the map*:

$$T : C^k_{wu}(E,F) \times L(F,G) \to C^k_{wu}(E,G)$$

given by $T(f,g) = g \circ f$, *is weakly sequentially continuous*;

(c) *the statement of* (b) *holds for some* $k \in \mathbf{N}^*$;

(d) *for every* E, G *and* $k \in \mathbf{N}^*$, *if one of the sequences* $(f_n) \subset C^k_{wu}(E,F)$ *and* $(g_n) \subset L(F,G)$ *is weak Cauchy and the other one is weakly null, then* $(g_n \circ f_n)$ *is weakly null.*

Proof. (c) $\Rightarrow$ (a). Taking $E = \{0\}$ and $G = \mathbf{R}$, T becomes a map $F \times F^* \to \mathbf{R}$ given by $T(y,\phi) = \langle y, \phi\rangle$ for $y \in F$ and $\phi \in F^*$, and we obtain the definition of the DP property.

(a) $\Rightarrow$ (b). Given $k \in \mathbf{N}$, it is enough to consider weakly null sequences $(f_n) \subset C^k_{wu}(E,F)$ and $(g_n) \subset L(F,G)$. For $0 \le j \le k$ and $x \in E$, we have:

$$||d^j(g_n \circ f_n)(x)|| = ||g_n \circ d^j f_n(x)|| \le ||g_n|| \cdot ||d^j f_n(x)||.$$

Hence $\{d^j(g_n \circ f_n) : n \in \mathbf{N}\}$ is uniformly bounded on all bounded subsets of E. Now, for each $0 \le j \le k$, $u \in E^{**}$, $h \in B_{E^{**}}$ and $\psi \in F^*$, we have $\langle d^j \tilde{f}_n(u)(h), \psi\rangle \xrightarrow{n} 0$, so the sequence $(d^j \tilde{f}_n(u)(h))_n \subset F$ is weakly null. On the other hand, the map $L(F,G) \to L(F^{**}, G^{**})$ taking A into its second adjoint A^{**} is continuous, and so are the functionals on $L(F^{**}, G^{**})$ given by $S \mapsto \langle w, Sz\rangle$, for each $z \in F^{**}$ and $w \in G^*$. Hence, $(w \circ g_n^{**})$ is a weakly null sequence in F^*. By the DP property of F,

$$\lim_n \langle g_n \circ d^j \tilde{f}_n(u)(h), w\rangle = \lim_n \langle d^j \tilde{f}_n(u)(h), w \circ g_n^{**}\rangle = 0;$$

and so, by Theorem 10, $(g_n \circ f_n)$ is weakly null in $C^k_{wu}(E,G)$. Slight modifications are needed for $k = \infty$.

(b) $\Rightarrow$ (c) is obvious.

(a) $\Leftrightarrow$ (d) follows easily from (a) $\Leftrightarrow$ (b) and the remark that in a linear topological space a sequence (x_n) is Cauchy if and only if for all subsequences (m_k), $(n_k) \subset \mathbf{N}$, $(x_{m_k} - x_{n_k})$ is null. □

Theorem 12. *The following assertions are equivalent*:

(a) *F has the* DP *property*;

(b) *for every G and $k \in \mathbf{N}$, the map $T : F \times \mathcal{P}(^kF, G) \to G$ given by $T(x, P) = P(x)$, is weakly sequentially continuous*;

(c) *the statement of* (b) *holds for some $k \in \mathbf{N}$ and $G = \mathbf{R}$.*

Proof. (a) $\Rightarrow$ (b). By induction on k. For $k = 1$, the result is contained in Theorem 11. Suppose it holds for $k-1$. Take weakly null sequences $(x_n) \subset F$ and $(P_n) \subset \mathcal{P}(^kF, G)$. The operator

$$\mathcal{P}(^kF, G) \to \mathcal{P}(^{k-1}F, L(F, G))$$

taking P to $\overline{P}$, given by $\overline{P}(x)(y) = \widehat{P}(y, x, \dots, x)$, is an isomorphism. Hence, $(\overline{P}_n)$ is weakly null in $\mathcal{P}(^{k-1}F, L(F, G))$. By the induction hypothesis, $(\overline{P}_n(x_n))$ is weakly null in $L(F, G)$, and by Theorem 11, $(P_n(x_n)) = (\overline{P}_n(x_n)(x_n))$ is weakly null in G.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). Suppose F does not have the DP property. Then we can find weakly null sequences $(x_n) \subset F$ and $(\phi_n) \subset F^*$ such that $|\langle x_n, \phi_n \rangle| \geq 1$ for all $n \in \mathbf{N}$. Define $P_n(x) := \langle x, \phi_n \rangle^k$. Then (P_n) is a weakly null sequence in $\mathcal{P}_{wu}(^kF) \subseteq \mathcal{P}(^kF)$. Moreover, $|P_n(x_n)| \geq 1$. □

The following Corollary is now easy.

Corollary 13. *The following assertions are equivalent*:

(a) *F has the* DP *property*;

(b) *for every G and $k \in \mathbf{N}$, if one of the sequences $(x_n) \subset F$ and $(P_n) \subset \mathcal{P}(^kF, G)$ is weak Cauchy and the other one is weakly null, then $(P_n(x_n))$ is weakly null in G*;

(c) *the statement of* (b) *holds for some $k \in \mathbf{N}$ and $G = \mathbf{R}$.*

Theorem 14. *The following assertions are equivalent*:

(a) *F has the DP property*;

(b) *for every E, G and $k \in \mathbf{N}$, the map*:

$$T : K(E, F) \times \mathcal{P}(^kF, G) \to \mathcal{P}_{wu}(^kE, G),$$

given by $T(f, P) = P \circ f$, *is weakly sequentially continuous*;

(c) *the statement of* (b) *holds for some* $k \in \mathbf{N}$;

(d) *for every* E, G *and* $k \in \mathbf{N}$, *if one of the sequences* $(f_n) \subset K(E, F)$ *and* $(P_n) \subset \mathcal{P}(^k F, G)$ *is weak Cauchy, and the other one is weakly null, then* $(P_n \circ f_n)$ *is weakly null.*

Proof. (a) $\Rightarrow$ (b). The map $\mathcal{P}(^k F, G) \to L(G^*, \mathcal{P}(^k F))$ taking P into P^*, where $P^*(\psi) = \psi \circ P$ for all $\psi \in G^*$, is a linear isometry. Given weakly null sequences $(f_n) \subset K(E, F)$ and $(P_n) \subset \mathcal{P}(^k F, G)$, we have for $u \in E^{**}$ and $\psi \in G^*$,

$$\langle P_n \circ f_n^{**}(u), \psi \rangle = \langle f_n^{**}(u), P_n^*(\psi) \rangle \overset{n}{\to} 0,$$

by Theorem 12, since $(P_n^*(\psi)) \subset \mathcal{P}(^k F)$ and $(f_n^{**}(u)) \subset F$ are weakly null sequences. Hence, by Corollary 5, $(P_n \circ f_n) \subset \mathcal{P}_{wu}(^k E, G)$ is weakly null.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). Suppose F does not have the DP property. Choose weakly null sequences $(x_n) \subset F$ and $(\phi_n) \subset F^*$ such that $|\langle x_n, \phi_n \rangle| \geq 1$. Consider $E = G = \mathbf{R}$. To get a contradiction, it is enough to define $f_n(\lambda) := \lambda x_n$ and $P_n(y) := \langle y, \phi_n \rangle^k$.

(a) $\Leftrightarrow$ (d) is as in Theorem 11. $\square$

The last three results are also valid for complex Banach spaces.

4. Weakly compact homomorphisms. We now apply the characterizations obtained in Section 2 to give a proof of the fact that any weakly compact homomorphism between algebras of differentiable functions is induced by a constant mapping. This was first shown in [**8**].

Given $k \in \mathbf{N}$, when we say that an algebra homomorphism $A : C^k(E) \to C^k(F)$ is continuous, it is understood that both $C^k(E)$ and $C^k(F)$ are endowed with the topology τ_u^k. It is shown in [**9**] that these homomorphisms are of the form $Af := f \circ \varphi$ where $\varphi : F \to E$ satisfies $\phi \circ \varphi \in C^k(F)$ for each $\phi \in E^*$ (and so, if $\dim(E) < \infty$, then $\varphi \in C^k(F, E)$). For a wide class of Banach spaces E (including the separable spaces and their duals), all these homomorphisms are automatically continuous [**9**]. A homomorphism is (*weakly*) *compact* if it takes bounded subsets into relatively (weakly) compact subsets.

Theorem 15. *For $k \in \mathbf{N}$, let $A : C^k(\mathbf{R}) \to C^k(\mathbf{R})$ be a nonzero algebra homomorphism, and let $\varphi \in C^k(\mathbf{R})$ be its inducing function. Then the following assertions are equivalent*:

(a) *φ is constant*;

(b) *A has one-dimensional rank*;

(c) *A is compact*;

(d) *A is weakly compact.*

Proof. (a) $\Rightarrow$ (b). If $\varphi(y) = x_0$ for every $y \in \mathbf{R}$, then $Af = f(x_0) \cdot \mathbf{1}$ for each $f \in C^k(\mathbf{R})$, where $\mathbf{1}$ is the constant function with value 1.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (a). Suppose φ is not constant. For $k \geq 1$, we may assume there exist $a < b$ such that φ is increasing on $[a, b]$ and (by the inverse function theorem), φ^{-1} is of class C^k in $[\varphi(a), \varphi(b)]$. To simplify notation, let $a = -1$, $b = 2$. Choose a strictly increasing function $\chi \in C^\infty(\mathbf{R})$ such that

$$\begin{cases} \chi(x) = x & \text{for } x \in [\varphi(0), \varphi(1)] \\ \lim_{x\to+\infty} \chi(x) = \varphi(2) \\ \lim_{x\to-\infty} \chi(x) = \varphi(-1) \end{cases}$$

Define $\psi := \varphi^{-1} \circ \chi \in C^k(\mathbf{R})$ and

$$g_n(y) := \begin{cases} \int_0^y \int_0^{t_k} \cdots \int_0^{t_2} \sin 2^n \pi t_1 \, dt_1 \dots dt_k & \text{for } y \geq 0 \\ 0 & \text{for } y < 0. \end{cases}$$

Since the sequence $(g_n) \subset C^k(\mathbf{R})$ is bounded, a standard application of the chain rule yields that the sequence $(f_n) := (g_n \circ \psi)$ is bounded too.

Letting $h_n := f_n \circ \varphi$ for $n \in \mathbf{N}$, the sequence (h_n) has no subsequence weakly converging to some $h \in C^k(\mathbf{R})$. Otherwise, we should have $d^k h_n(x) \xrightarrow{n} d^k h(x)$ for all $x \in [0, 1]$ (Theorem 10). However, $d^k h_n(x) =$

$\sin 2^n \pi x$ for $x \in [0,1]$, and it is an elementary consequence of the dominated convergence theorem that this sequence has no pointwise convergent subsequence [**18**, 7.20].

For $k = 0$, we may assume that $\varphi(\mathbf{R}) \supset [0,1]$. Then it is enough to define $f_n(t) := \sin 2^n \pi t$ for $t \in \mathbf{R}$. $\square$

Now it is not difficult to prove that Theorem 15 is also valid for homomorphisms $C^k_{wu}(E) \to C^k_{wu}(F)$ (see [**8**]).

5. Weakly compact subsets in other spaces of differentiable mappings. The results given in the preceding sections are also valid in other spaces of differentiable mappings treated in the literature.

Let $C^k_c(E,F)$ be the vector space, introduced in [**3**], of those $f \in C^k(E,F)$ such that for each integer $1 \le j \le k$ and $x \in E$ we have $d^j f(x) \in \mathcal{P}_{wu}({}^j E, F)$. It will be endowed with the topology τ^k_u.

We denote by τ^k_c the compact-compact topology of order k on $C^k(E,F)$, introduced by Llavona [**13**] and Prolla [**15**], generated by the seminorms:

$$p_{K,L}(f) = \sup\{||f(x)||, ||d^j f(x)(y)|| : x \in K,\ y \in L,\ 1 \le j \le k\},$$

where K, L are compact subsets of E.

All the results of Section 2 are valid (with obvious modifications) if we replace $C^k_{wu}(E,F)$ by $(C^k_c(E,F), \tau^k_u)$ or $(C^k(E,F), \tau^k_c)$. Here we only state the main changes in the definitions.

1. *Case of* $(C^k(E,F), \tau^k_c)$. The map χ of Proposition 3 would now be $\chi : C(E,F) \to C(E \times V)$, taking f into χ_f given by $\chi_f(x,v) = \langle f(x), v\rangle$ for $x \in E$ and $v \in V$.

A net $(f_\alpha) \subset C^k(E,F)$ converges to f in the w_k topology whenever $\langle d^j f_\alpha(x), \psi\rangle$ converges to $\langle d^j f(x), \psi\rangle$ for all $x \in E$, $0 \le j \le k$ and $\psi \in (\mathcal{P}({}^j E, F), \tau_{co})^*$, where τ_{co} denote the topology of uniform convergence on compact subsets of E.

A net $(f_\alpha) \subset C(E, (\mathcal{P}({}^j E, F), \tau_{co}))$ converges to $f$in the $\hat{w}_j$ topology if $\langle f_\alpha(x)(h), v\rangle$ converges to $\langle f(x)(h), v\rangle$ for all $x, h \in E$ and $v \in V$.

The map T of Lemma 6 is now $T : C(E, (\mathcal{P}({}^jE, F), \tau_{co})) \to C(E \times E \times V)$ taking f into Tf given by $Tf(x,h,v) = \langle f(x)(h), v\rangle (x, h \in E, v \in V)$.

In the statement of Theorem 8, we should say that A_j is uniformly bounded on all compact subsets of E, in the sense that for compact subsets $K, L \subset E$, there exists $\lambda > 0$ such that $||d^j f(x)(y)|| \le \lambda$ for all $x \in K$, $y \in L$ and $f \in A$.

2. *Case of* $(C_c^k(E,F), \tau_u^k)$. Everything is as in 1, with the exception of the following two definitions:

A net $(f_\alpha) \subset C(E, \mathcal{P}_{wu}({}^jE,F))$ converges to f in the $\hat{w}_j$ topology whenever $\langle \hat{f}_\alpha(x)(h), v\rangle$ converges to $\langle \hat{f}(x)(h), v\rangle$ for all $x \in E$, $h \in B_{E^{**}}$ and $v \in V$, where $\hat{f}(x)$ is the extension of $f(x)$ to E^{**}.

The map T of Lemma 6 is now $T : C(E, \mathcal{P}_{wu}({}^jE,F)) \to C(E \times W \times V)$ taking f into Tf given by $Tf(x,h,v) = \langle \hat{f}(x)(h), v\rangle$ for all $x \in E$, $h \in W$ and $v \in V$.

Theorem 11 is also valid if we replace C_{wu}^k by C_c^k or C^k. Similarly, Theorem 15 is valid for continuous homomorphisms $C^k(E) \to C^k(F)$ and $C_c^k(E) \to C_c^k(F)$, where the algebras may be endowed with one of the topologies τ_u^k and τ_c^k.

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Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, 39071 Santander, Spain

Departamento de Matemática Aplicada, ETS de Ingenieros Industriales, Universidad Politécnica de Madrid, C. José Gutiérrez Abascal 2, 28006 Madrid, Spain