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A GEOMETRIC CHARACTERIZATION OF THE WEAK-RADON NIKODYM PROPERTY IN DUAL BANACH SPACES

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ABSTRACT. We give a geometric characterization of convex, weak*-compact subsets of a dual Banach space with the weak-Radon Nikodym property as those sets in which every closed, convex subset is the weak*-closed convex hull of its x^{**}-weak*-strongly exposed points for each element x^{**} of X^{**}.

1. Introduction. After the characterization by Musial [**9**] and Janicka [**8**] of dual Banach spaces with the weak-Radon Nikodym property (that is, the Radon-Nikodym property for the Pettis integral) as the spaces with predual not containing l_1, many characteristic properties for the weak*-compact subsets of such spaces were proved (see [**7, 12**]). Many of these properties localized to provide equivalent properties for weak*-compact subsets of dual spaces [**6, 10, 11, 13**].

A convex, weak*-compact subset K of a dual Banach space X^* has the weak-Radon Nikodym property (w-RNP) if and only if it is a Pettis set [**5, 13**] or equivalently if it is weakly fragmented [**5**] (K is weakly fragmented if for every nonempty, w^*-compact subset F of K, $\varepsilon > 0$ and $x^{**} \in X^{**}$ there exists a nonempty, relatively open subset U of (F, w^*) such that $O(x^{**}, U) < \varepsilon$). Also, characteristic properties of a convex, weakly fragmented set K are that the norm-closed convex hull of F is equal to the weak*-closed convex hull of F for every weak*-compact subset F of K and that every convex, weak*-compact subset L of K is equal to the norm-closed convex hull of its extreme points [**5, 7**].

In this paper (see Theorem 8) we give a geometric characterization of convex, weak*-compact, with the w-RNP subsets of a dual Banach space as those sets in which every weak*-compact, convex subset is the weak*-closed convex hull of its *x^{**}-weak*-strongly exposed points* for each element x^{**} of X^{**}. An extreme point x^* of K is an x^{**}-weak*-strongly exposed point of K for some x^{**} in X^{**} if there exists

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an x in X such that, for every sequence (x_n^*) in K, the sequence $(x^{**}(x_n^*))$ converges to $x^{**}(x^*)$ whenever the $(x_n^*(x))$ converges to $x^*(x) = \sup\{y^*(x) : y^* \in K\}$. An example of an extreme point which is not x^{**}-weak*-strongly exposed is given (Example 2). By the same example we have that in the characterization the weak*-closure may not be replaced by the norm-closure. The proof of this theorem is based on techniques similar to those used in the proof of the analogous characterization for sets with the RNP [**4**].

2. Notations. Let Y be a topological Hausdorff space and f a real valued function on Y. For $A \subseteq Y$, the oscillation of f on A is the $O(f, A) = \sup\{|f(y) - f(x)| : x, y \in A\}$ and the oscillation of f at a point $x \in Y$ is $O(f, x) = \inf\{O(f, U) : U \subseteq Y$ is open and $x \in U\}$. Obviously, f is continuous at x if and only if $O(f, x)$ is equal to zero.

Let X be a Banach space. We denote by X^* and X^{**} the dual and second dual of X, respectively. If A is a subset of X, then we denote by norm-cl A the norm-closure of A, by w^*-cl A the weak*-closure of A and by conv A the convex hull of A. The set of the extreme points of a convex set C is denoted by ext C. If K is a bounded subset of X^*, then a w^*-slice (or w^*-open slice) of K is a set of the form $S(K, x, \varepsilon) = \{f \in K : f(x) \geq M(x, K) - \varepsilon\}$ where $x \in X$, $\varepsilon > 0$ and $M(K, x) = \sup\{f(x) : f \in K\}$.

Definition 1. Let X be a Banach space, K a w^*-compact, convex subset of X^* and $x^{**} \in X^{**}$. An *extreme point* x^* of K is an x^{**}-weak*-strongly exposed point of K (written $x^* \in x^{**}$-w^*-strexp K) if and only if there exists an $x \in X$ which x^{**}-w^*-strongly exposes x^*. This means that $M(K, x) = x^*(x)$ and for every $\varepsilon > 0$ there exists a slice $S(K, x, \delta)$ of K with $O(x^{**}, S(K, x, \delta)) < \varepsilon$. Equivalently, x^* is x^{**}-weak*-strongly exposed by x if and only if for every sequence (x_n^*) in K such that $x_n^*(x) \to x^*(x) = M(K, x)$ we have $x^{**}(x_n^*) \to x^{**}(x^*)$. We denote by x^{**}-w^*-$SE(K)$ the set of elements of X which x^{**}-w^*-strongly expose an element of K. It is easy to see that $x \in x^{**}$-w^*-$SE(K)$ if and only if for every $\varepsilon > 0$ there exists a slice $S(K, x, \delta)$ of K with $O(x^{**}, S(K, x, \delta)) < \varepsilon$.

The following example shows that there exist extreme points which are not x^{**}-weak*-strongly exposed for some $x^{**} \in X^{**}$.

Example 2. Let X denote the Banach space c_0. Then $X^* = \mathbf{l}_1$ and $X^{**} = \mathbf{l}^\infty$. Let e_n, $n \in \mathbf{N}$ be the unit vectors in $\mathbf{l}^1$ and K the weak*-closure of the convex hull of $\{e_n : n \in \mathbf{N}\}$. Since the w^*-limit of (e_n) is 0, we have that $0 \in K$. Moreover, 0 is an extreme point of K. But 0 is not an x^{**}-weak*-strongly exposed point of K for $x^{**} = (-1, -1, \dots) \in \mathbf{l}^\infty$, because $\lim_n x^{**}(e_n) = -1 \neq 0$.

The following lemma is influenced by the analogous lemma of Bishop [**2**].

Lemma 3. *Let K be a w^*-compact subset of a dual space X^* and $x^{**} \in X^{**}$. If for every $\delta > 0$ and $x \in X$ there exists a $y \in X$ such that $||x - y|| < \delta$ and y determines a slice $S(K, y, a)$ of K with $O(x^{**}, S(K, y, a)) < \delta$, then $K = w^*\text{- clconv}\,(x^{**} - \operatorname{strexp} K)$. Moreover, x^{**}-w^*-$SE(K)$ is a dense G_δ subset of X.*

Proof. For every $\varepsilon > 0$, let O_ε be the set of all $x \in X$ which determine a slice S of K with $O(x^{**}, S) < \varepsilon$. Then O_ε is open, since for every $x \in O_\varepsilon$ and every slice $S(K, x, a)$ of K there is a $\delta > 0$ such that $S(K, y, a/2) \subseteq S(K, x, a)$ whenever $y \in X$ and $||y - x|| < \delta$. Also O_ε is dense in X by hypothesis. Hence by the Baire category theorem the set $\cap_{n=1}^\infty O_{1/n}$ is dense and G_δ in X. It is immediate that x^{**}-w^*-$SE(K) = \cap_{n=1}^\infty O_{1/n}$.

If $K_1 = w^*\text{-clconv}\,(x^{**} - w^* - \operatorname{strexp} K)$ is a proper subset of K, then from the separation theorem we can find a w^*-slice $S(K, x, a)$ of K which is disjoint from K_1. Since x^{**}-w^*-$SE(K)$ is dense in X there exists a y in $x^{**} - w^* - SE(K)$ such that $S(K, y, a\backslash 2) \subseteq S(K, x, a)$. If $x^* \in K$ is $x^{**}-w^*$-strongly exposed by y, then $x^* \in K_1 \cap S(K, y, a/2) \subseteq K_1 \cap S(K, x, a)$, a contradiction. □

The following lemma is a version of the superlemma [**1, 3**] and the proof is analogous.

Lemma 4. *Let X be a Banach space, K, K_0 and K_1 be w^*-compact, convex subsets of X^*, $\varepsilon > 0$ and $x_1^{**}, \dots, x_n^{**} \in X^{**}$ with $||x_i^{**}|| = 1$ for $i = 1, \dots, n$. Suppose that*:

1. *K_0 is a subset of K and $O(x_i^{**}, K_0) < \varepsilon$ for every $i = 1, \dots, n$.*
2. *K is not a subset of K_1.*
3. *K is a subset of* $\mathrm{conv}\,(K_0 \cup K_1)$.

Then there exists a w^*-slice S of K which contains a point of K_0 and $O(x_i^{**}, S) < \varepsilon$ for every $i = 1, \dots, n$.

Proposition 5. *Let C and K be w^*-compact and convex subsets of a dual space X^*, $x^{**} \in X^{**}$ and $\varepsilon > 0$. If K has the w-RNP and $K \backslash C \neq \varnothing$, then there exists a w^*-slice S of* $\mathrm{conv}\,(K \cup C)$ *such that $S \cap K \neq \varnothing$ and $O(x^{**}, S) < \varepsilon$.*

Proof. Let $J = \mathrm{conv}\,(K \cup C)$. Obviously, J is a w^*-compact and convex subset of X^*. Also, let $D = \{x^* \in J$: there is an $x \in X$ such that $x^*(x) = M(J, x) > M(C, x)\}$. Then $\varnothing \neq D \subseteq K$ and w^*-clconv $(D \cup C) = J$ (for more details see [**4**, (3.5.2)]). Since K is weakly fragmented there exists [**10**] a w^*-slice S^1 of $D^1 = w^*$-clconv D such that $S^1 \cap D \neq \varnothing$ and $O(x^{**}, S^1) < \varepsilon/3$. Let $K_0 = w^*$-clconv $(S^1 \cap D)$ and $K_1 = w^*$-clconv $[(D \backslash S_1) \cup C]$. Then the sets J, K_0, K_1 satisfy the hypotheses of Lemma 4. Hence, we can find a w^*-slice S of J such that $S \cap K \neq \varnothing$ and $O(x^{**}, S) < \varepsilon$. □

Lemma 6. *Let X be a Banach space and $x \in x$ with $||x|| = 1$. For $t > 0$ denote by V_t the set $\{x^* \in X^* : x^*(x) = 0$ and $||x^*|| \leq t\}$. Assume that $x_0^*, y^* \in X^*$, $x_0^*(x) > y^*(x)$ and $||x_0^* - y^*|| \leq t/2$. If $y \in X$, $||y|| = 1$ and $x_0^*(y) > M(y^* + V_t, t)$, then $||x - y|| \leq 2/t ||x_0^* - y^*||$.*

For the proof, see [**4**, Lemma 3.3.3].

Theorem 7. *Let K be a w^*-compact, convex subset of a Banach space X^* and $x^{**} \in X^{**}$. If K has the w-RNP, then $K = w^*$-*clconv $(x^{**} - w^* - \mathrm{strexp}\, K)$. *Moreover, x^{**}-w^*-$SE(K)$ is dense and G_δ in X.*

Proof. It is sufficient to check the hypotheses of Lemma 3. Let $0 < \delta < 1$ and $x \in X$ with $||x|| = 1$. Since K is bounded, there exists a $y^* \in X^*$ such that $y^*(x) < x^*(x) - 1$ for every $x^* \in K$. Let $V = \{x^* \in X^* : x^*(x) = 0 \text{ and } ||x^*|| \leq 2M/\delta\}$ where $M = \sup\{||x^* - y^*|| : x^* \in K\}$ and let $C = y^* + V$. Then $K \cap C = \varnothing$, hence $K \backslash C \neq \varnothing$ and from Proposition 5, there exists a w^*-slice $S = S(J, y, a)$ of $J = \operatorname{conv}(K \cup C)$ such that $x_0^* \in S \cap K \neq \varnothing$ and $O(x^{**}, S) < \delta$. It is easy to check that $S \cap C = \varnothing$ and $M(K, y) = M(J, y)$. Since $K \subseteq J$ we have that $S(K, y, a) \subseteq S(J, y, a)$ and hence $O(x^{**}, S(K, y, a) < \delta$. Finally, from Lemma 6 $||x - y|| \leq \delta/M||y^* - x_0^*|| \leq \delta$. □

Combining the above we have the following characterization:

Theorem 8. *Let K be a w^*-compact, convex subset of a dual Banach space X^*. Then the following are equivalent*:

1. *K has the weak-Radon Nikodym property.*
2. *Each w^*-compact, convex subset C of K satisfies*:

$$C = w^* - \operatorname{clconv}(x^{**} - w^* - \operatorname{strexp} C)$$

for every x^{**} in X^{**}.

3. *For each w^*-compact, convex subset C of K and each $x^{**} \in X^{**}$ the set $x^{**} - w^* - SE(C)$ is dense and G_δ in X.*

Proof. 1 ⇒ 2 and 3. If K has the w-RNP, then each w^*-compact, convex subset C of K has the same property. Hence, from Theorem 7 $C = w^* - \operatorname{clconv}(x^{**} - w^* - \operatorname{strexp} C)$ and $x^{**} - w^* - SE(C)$ is dense and G_δ in X for every $x^{**} \in X^{**}$.

3 ⇒ 1. We will prove that K is weakly fragmented. Let F be a w^*-compact subset of K, $x^{**} \in X^{**}$ and $\varepsilon > 0$. If $C = w^*$-$\operatorname{clconv} F$ then x^{**}-w^*-$SE(C) \neq \varnothing$ from 3. Hence there exists a w^*-slice S of C with $O(x^{**}, S) < \varepsilon$. Of course, $S \cap F$ is a nonempty relatively open subset of (F, w^*) and $O(x^{**}, S \cap F) < \varepsilon$. Hence, K is weakly fragmented.

2 ⇒ 1. Let F be a w^*-compact subset of K and $x^{**} \in X^{**}$. If $C = w^*$-$\operatorname{clconv} F$, then from 2 we have that x^{**}-w^*-$\operatorname{strexp} C \neq \varnothing$. Hence x^{**}-w^*-$SE(C) \neq \varnothing$. The proof is continued as in 3 ⇒ 1. □

Corollary 9. *A dual Banach space X^* has the w-RNP if and only if every convex, w^*-compact subset of X is the w^*-closed convex hull of its x^{**}-weak*-strongly exposed points for every x^{**} in X^{**}.*

Remark 10. It is not true that every w^*-compact, convex, with the w-RNP subset K of X^* is equal to the norm-closed convex hull of its x^{**}-w^*-strongly exposed points for every x^{**} in X^{**}. For example, let $X^* = \mathbf{l}^1$ and $K = w^*\text{-clconv}\,\{e_n : n \in \mathbf{N}\}$. As we prove in Example 2, $0 \in K$ and 0 is not an x^{**}-weak*-strongly exposed point of K for $x^{**} = (-1, -1, \dots)$. The Milman theorem gives that $\operatorname{ext} K \subseteq \{e_n : n \in \mathbf{N}\} \cup \{0\}$, hence $x^{**}\text{-}w^*\text{-strexp}\, K \subseteq \{e_n : n \in \mathbf{N}\}$. Since K has the w-RNP we have $K = w^*\text{-clconv}\,(x^{**}\text{-}w^*\text{-strexp}\, K)$, but $K \neq \|\ \ \| - \text{clconv}\,(x^{**} - w^* - \text{strexp}\, K)$ since $0 \notin \|\ \ \|\text{-clconv}\,\{e_n : n \in \mathbf{N}\}$.

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