ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 3, Summer 1995

STRICTLY POSITIVE DEFINITE KERNELS ON THE CIRCLE

VALDIR A. MENEGATTO

ABSTRACT. A sufficient condition is given for the strict positive definiteness and for the strict conditional negative definiteness of a real, continuous radial kernel on the circle. In addition, some necessary conditions are also given, nearly characterizing these kernels.

1. Introduction. On the unit circle S^1, let d_1 be the geodesic distance. The purpose of this paper is to address the problem of finding a continuous function $f : [0, \pi] \to \mathbf{R}$ for which $f \circ d_1$ is either a strictly positive definite or strictly conditionally negative definite kernel. Following [**1**], we say that a function $f : S^1 \times S^1 \to \mathbf{R}$ is a *positive definite kernel* if and only if

$$\sum_{i,j=1}^{n} c_i c_j f(x_i, x_j) \geq 0$$

for all $n \in \mathbf{N}$, $\{x_1, x_2, \dots, x_n\} \subset S^1$, and $\{c_1, c_2, \dots, c_n\} \subset \mathbf{R}$. We say that the function f is a *conditionally negative definite kernel* if $f(x, y) = f(y, x)$ for all $x, y \in S^1$ and

$$\sum_{i,j=1}^{n} c_i c_j f(x_i, x_j) \leq 0$$

for all $n \geq 2$, $\{x_1, x_2, \dots, x_n\} \subset S^1$ and $\{c_1, c_2, \dots, c_n\} \subset \mathbf{R}$ with $\sum_{j=1}^{n} c_j = 0$. If the above inequalities are strict whenever $x_1, x_2, \dots, x_n$ are different and at least one of the $c_1, c_2, \dots, c_n$ does not vanish, we say that the kernel f is *strictly positive* (respectively, *strictly conditionally negative*) *definite.*

Received by the editors on April 7, 1993, and in revised form on September 20, 1993.
Partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnologico–CNPq (Brazil).

Schoenberg [**5**] showed that a continuous kernel $f \circ d_1$ is positive definite if and only if f has the form

$$f(t) = \sum_{k=0}^{\infty} a_k \cos kt$$

in which $a_k \geq 0$ for all k and $\sum_{k=0}^{\infty} a_k < \infty$. Using this result, it is not hard to see that a continuous kernel $g \circ d_1$ is conditionally negative definite if and only if g has the form

$$g(t) = g(0) + \sum_{k=1}^{\infty} a_k(1 - \cos kt)$$

in which $a_k \geq 0$ for all k and $\sum_{k=1}^{\infty} a_k < \infty$. In view of the definitions above, if $f \circ d_1$ is strictly positive (respectively, strictly conditionally negative) definite, then f will have a nontrivial series representation as above. In addition, the strict positive (respectively strict conditional negative) definiteness will depend on the set of integers k for which $a_k > 0$ and not on the magnitude of the a_k.

Given distinct points $x_1, x_2, \ldots, x_n$ on S^1 and a strictly positive (respectively, strictly conditionally negative) definite kernel $f \circ d_1$, we can interpolate arbitrary data at the x_i by a function of the form

$$s(x) = \sum_{j=1}^{n} c_j f(d_1(x, x_j)).$$

This type of interpolation had been previously investigated in [**2, 3, 4, 6** and **7**]. In [**7**], it was proved that if f has a series representation $f(t) = \sum_{k \in \mathbf{N}} a_k \cos kt$, in which $a_k > 0$ for all k and $\sum_{k \in \mathbf{N}} a_k < \infty$, then $f \circ d_1$ is strictly positive definite. In Section 2 we improve this result giving nearly optimal sufficient conditions on a function f of the form $f(t) = \sum_{k \in K} a_k \cos kt$, in which $a_k > 0$ for all $k \in K$ and $\sum_{k \in K} a_k < \infty$, in order that $f \circ d_1$ be strictly positive definite. In Section 3, three completely independent necessary conditions are given but unfortunately these conditions are not equivalent to the previous one. As in the usual radial basis approach to interpolation in $\mathbf{R}^n$, completely monotonic functions can be used in interpolation on S^1. In [**3**], it was proved that if f is a nonconstant completely monotonic function, then $f \circ d_1$ is strictly positive definite. Following this sequence of ideas, in Section 4 we fix a nonconstant completely monotonic F and find necessary and sufficient conditions on a conditionally negative definite function g in order that the kernel $F \circ g \circ d_1$ be strictly positive definite. Similar results are obtained for strict conditional negative definiteness. We remark that there is a natural connection among these questions, distance geometry, and metric embedding theory but it is not our intention to explore it.

2. A sufficient condition.

For natural numbers m and n and a set A of integers, we write $m + nA$ to denote the set of all integers of the form $m + na$, with $a \in A$. An increasing sequence $\{c_k\}$ of nonnegative integers is said to be *prime* if, for every finite set P of prime numbers there exists a c_k not divisible by any element in P. Equivalently, $\{c_k\}$ is an increasing sequence not contained in any set of the form $p_1\mathbf{N} \cup p_2\mathbf{N} \cup \cdots \cup p_n\mathbf{N}$, where $p_1, p_2, \ldots, p_n$ are prime numbers. Any infinite increasing sequence of prime numbers is a trivial example of a prime sequence. Theorem 2.1 below shows that if K contains certain arithmetic sequences and $f(t) = \sum_{k\in K} a_k \cos kt$, with $a_k > 0$ for all $k \in K$ and $\sum_{k\in K} a_k < \infty$, then $f \circ d_1$ is strictly positive definite. We write $\mathbf{N}_l$ to denote the set of all nonnegative integers not exceeding l.

Theorem 2.1. *Let $f(t) = \sum_{k\in K} a_k \cos kt$, in which $K \subset \mathbf{N}$, $a_k > 0$ for all k, and $\sum_{k\in K} a_k < \infty$. In order that $f \circ d_1$ be a strictly positive definite kernel it is sufficient that K have a subset of the form $\cup_{k=0}^{\infty}(b_k + c_k\mathbf{N}_k)$, in which $\{b_k\} \cup \{c_k\} \subset \mathbf{N}$ and $\{c_k\}$ is a prime sequence.*

Proof. Suppose that K has a subset as described in the theorem, and let $x_1, x_2, \ldots, x_n$ be distinct points on S^1. Write $x_j = (\cos\phi_j, \sin\phi_j)$ where the ϕ_j are distinct modulo 2π, and let A be the matrix with entries $A_{ij} = f(d_1(x_i, x_j))$. For distinct μ and ν, $(\phi_\mu - \phi_\nu)/2\pi \in \mathbf{R}\backslash\mathbf{Z}$. Let $p_{\mu\nu}$ be a prime number chosen in the following way: if $(\phi_\mu - \phi_\nu)/2\pi = m_1/m_2$ with m_1 and m_2 relatively prime, we let $p_{\mu\nu}$ be a prime dividing m_2; otherwise, we let $p_{\mu\nu}$ be any prime. It is now seen that $\exp(il\phi_\mu) \neq \exp(il\phi_\nu)$ whenever $l \in \mathbf{N}\backslash p_{\mu\nu}\mathbf{N}$. Define $P = \{p_{\mu\nu} : 1 \le \mu < \nu \le n\}$. Since $\{c_k\}$ is a prime sequence, there is an index k_0 such that c_{k_0} is not divisible by any element of P. By enlarging the set P, if necessary, we can assume that $k_0 \geq n$. Now $\exp(ic_{k_0}\phi_\mu) \neq \exp(ic_{k_0}\phi_\nu)$ for $1 \leq \mu < \nu \leq n$, and $b_{k_0} + c_{k_0}\mathbf{N}_{k_0} \subset K$. As a consequence of this, the $n \times n$ matrix B with entries

$$B_{\mu\nu} = \exp(i(b_{k_0} + \mu c_{k_0})\phi_\nu) = \exp(i(b_{k_0} + c_{k_0})\phi_\nu)[\exp(ic_{k_0}\phi_\nu)]^{\mu-1}$$

is nonsingular because it is a Vandermonde-like matrix corresponding to n distinct points

$$\exp(ic_{k_0}\phi_1), \exp(ic_{k_0}\phi_2), \dots, \exp(ic_{k_0}\phi_n).$$

From the equation

$$\begin{aligned} c^t Ac &= \sum_{\mu,\nu=1}^{n} c_\mu c_\nu \sum_{k\in K} a_k \cos kd_1(x_\mu, x_\nu) \\ &= \sum_{k\in K} a_k \sum_{\mu,\nu=1}^{n} c_\mu c_\nu \cos k(\phi_\mu - \phi_\nu) \\ &= \sum_{k\in K} a_k \left| \sum_{\mu=1}^{n} c_\mu \exp(ik\phi_\mu) \right|^2 \end{aligned}$$

we now see that $c^t Ac > 0$ unless the function $h(t) = \sum_{\mu=1}^{n} c_\mu \exp(i\phi_\mu t)$ vanishes on K. Because of the nonsingularity of B, it follows that h vanishes on K if and only if $c_1 = c_2 = \cdots = c_n = 0$. Therefore, $c^t Ac = 0$ only if $c = 0$. $\square$

Remark. The appearance of Vandermonde matrices in the proof above suggests the optimality of the condition presented in the theorem.

The following corollary contains similar results for strict conditional negative definiteness. Since there are no new ideas involved, we omit its proof.

Corollary 2.2. *Let* $g(t) = g(0) + \sum_{k\in K} a_k(1 - \cos kt)$, *in which* $K \subset \mathbf{N}\backslash\{0\}$, $a_k > 0$ *for all* k, *and* $\sum_{k\in K} a_k < \infty$. *In order that* $g \circ d_1$ *be a strictly conditionally negative definite kernel, it is sufficient that*

K have a subset of the form $\cup_{k=0}^{\infty}(b_k + c_k\mathbf{N}_k)$, where $\{b_k\} \cup \{c_k\} \subset \mathbf{N}$ and $\{c_k\}$ is a prime sequence.

It should be noted that the hypothesis on the sequence $\{c_k\}$ in the previous results can be changed into any of the following:

(H1) $\{c_k\}$ is increasing and there is no prime number dividing infinitely many c_k.

(H2) $\{c_k\}$ is increasing and any two c_k are relatively prime.

Indeed, this follows from:

Lemma 2.3. *Let $\{c_k\}$ be an increasing sequence of nonnegative integers. The following assertions are equivalent*:

(A1) *$\{c_k\}$ is a prime sequence.*

(A2) *$\{c_k\}$ has a subsequence $\{c_{k_n}\}$ such that no prime number divides infinitely many c_{k_n}.*

(A3) *$\{c_k\}$ has a subsequence $\{c_{k_m}\}$ in which any two c_{k_m} are relatively prime.*

Proof. Obviously, (A3) implies (A2). Assume that $\{c_k\}$ has a subsequence $\{c_{kn}\}$ such that no prime number divides infinitely many c_{k_n}. If $\{c_k\}$ is not a prime sequence, then there is a finite set P of prime numbers such that each c_k is divisible by at least one element of P. Since P is finite and $\{c_{k_n}\}$ is increasing, at least one element in P divides infinitely many c_{k_n}, a contradiction. Thus, (A2) implies (A1). Finally, if $\{c_k\}$ is a prime sequence, a subsequence $\{c_{k_n}\}$ such that any two c_{k_m} are relatively prime can be constructed inductively in the following way: Let c_{k_1} be any positive c_k. Let P_1 be the set of all prime numbers dividing every element c_k not larger than c_{k_1}. From our assumption on $\{c_k\}$, there is a c_{k_2} not divisible by any element of P_1. Let P_2 be the set of all prime numbers dividing every element c_k not larger than c_{k_2}. Again, there is a c_{k_3} not divisible by any element in P_2. Because $\{c_k\}$ is increasing, we can proceed in this way to obtain a subsequence $\{c_{k_m}\}$ of $\{c_k\}$ such that no prime dividing a c_{k_m} divides any preceding element in the sequence. Obviously, such subsequence has the property state in (A_3). Thus, (A_1) implies (A_3). $\square$

3. Necessary conditions. In this section we obtain necessary conditions for strict positive (respectively, strict conditional negative) definiteness, trying somehow to match the sufficient condition just obtained.

Lemma 3.1. *Let $g(t) = g(0) = \sum_{k\in K} a_k(1-\cos kt)$, in which $K \subset \mathbf{N}\backslash\{0\}$ is a finite set of cardinality N and $a_k > 0$ for all $k \in K$. Then for any $n > 2 + 2(-N + \sum_{k\in K} 2^{k+1})$ and any set of points $x_1, x_2, \dots, x_n$ on S^1, the $n\times n$ matrix A with entries $A_{ij} = g(d_1(x_i, x_j))$ has rank not exceeding $n/2 - 1$.*

Proof. Assume the hypotheses and let $x_1, x_2, \dots, x_n$ be points on S^1. For each k in K put $g_k(t) := 1 - \cos kt$. By elementary trigonometry, there are real numbers $b_{k0}, b_{k1}, b_{k2}, \dots, b_{kk}$ such that $g_k(t) = \sum_{l=0}^{k} b_{kl}\cos^l t$. Hence,

$$g_k(d_1(x_i, x_j)) = \sum_{l=0}^{k} b_{kl}\langle x_i, x_j\rangle^l, \qquad 1 \le i, j \le n.$$

The Gram matrix with entries $\langle x_i, x_j\rangle$ is nonnegative definite and has rank not exceeding 2. Hence, each matrix $(\langle x_i, x_j\rangle^l)$, $0 \le l \le k$, has rank not exceeding 2^l. Thus, each matrix $(g_k(d_1(x_i, x_j)))$ has rank not exceeding $\sum_{l=0}^{k} 2^l = (2^{k+1} - 1)$. Since A has entries given by $g(0) + \sum_{k\in K} a_k g_k(d_1(x_i, x_j))$, its rank cannot exceed $1 + \sum_{k\in K}(2^{k+1} - 1) = 1 + (-N + \sum_{k\in K} 2^{k+1})$. $\square$

Lemma 3.1 reveals that if $g \circ d_1$ is strictly conditionally negative definite then the series representing g cannot be a finite sum. A refinement of this fact is now obtained. The ideas for this line of proof can be found in [**4**].

Theorem 3.2. *Let $g(t) = g(0) + \sum_{k=1}^{\infty} a_k(1-\cos kt)$, in which $a_k \ge 0$ for all k and $\sum_{k=1}^{\infty} a_k < \infty$. A necessary condition in order that $g \circ d_1$ be a strictly conditionally negative definite kernel is that $a_k > 0$ for infinitely many odd and infinitely many even integers k.*

Proof. We first assume that $a_{2k} > 0$ for only finitely many k and prove that $g \circ d_1$ is not strictly conditionally negative definite. Set

$M := \max\{k : a_{2k} > 0\}$, and let N denote the cardinality of the set $K := \{k : 0 \le k \le 2M, a_k > 0\}$. Let n be any positive integer such that $n > 1 - N + \sum_{k\in K} 2^{k+1}$. Choose $2n$ distinct and pairwise antipodal points $x_1, x_2, \ldots, x_{2n}$ on S^1, and define matrices A, B and C by

$$\begin{aligned} A_{ij} &= -\sum_{k>2M} a_k \cos kd_1(x_i, x_j) \\ B_{ij} &= g(0) + \sum_{k>2M} a_k + \sum_{k\in K} a_k(1 - \cos kd_1(x_i, x_j)) \\ C_{ij} &= A_{ij} + B_{ij}. \end{aligned}$$

If $1 \le i < j \le 2n$, the vector $v^{ij} \in \mathbf{R}^{2n}$ having 1 as its ith component, 1 as its jth component, and 0 components elsewhere, is in the null space of A. Hence, the rank of A is not larger than n. By Lemma 3.1, the rank of B does not exceed $n-1$. Thus, C has rank not exceeding $2n-1$ or, equivalently, C is singular. This implies that $g \circ d_1$ is not strictly conditionally negative definite. The other half of the proof when $a_{2k+1} > 0$ for only finitely many k is similar, and we omit the details. $\square$

The condition in the previous theorem is not sufficient to guarantee strict conditional negative definiteness. Indeed, consider $g(t) = \sum_{k\in K} a_k(1-\cos kt)$, in which $K = 3\mathbf{N}$, $a_k > 0$ for all k, and $\sum_{k\in K} a_k < \infty$. If x_1 and x_2 are two points on S^1 such that $d_1(x_1, x_2) = 2\pi/3$, then the 2×2 matrix with entries $g(d_1(x_i, x_j))$ is the zero matrix.

Lemma 3.3. *If n is a positive integer and ϕ is not an odd multiple of π, then*

$$\sum_{j=1}^{2n} (-1)^j \exp(ij\phi) = i \sin n\phi \sec\frac{\phi}{2} \exp\frac{i(2n+1)\phi}{2}.$$

Proof. By elementary trigonometry, we have

$$\sum_{j=1}^{2n} \cos j\phi = \sum_{j=1}^{2n} \sin\frac{\phi}{2} \cos j\phi \csc\frac{\phi}{2}$$

$$= \sum_{j=1}^{2n} \frac{1}{2}\left[\sin\frac{(2j+1)\phi}{2} - \sin\frac{(2j-1)\phi}{2}\right]\csc\frac{\phi}{2}$$
$$= \frac{1}{2}\left[\sin\frac{(4n+1)\phi}{2} - \sin\frac{\phi}{2}\right]\csc\frac{\phi}{2}$$
$$= \sin(n\phi)\cos\frac{(2n+1)\phi}{2}\csc\frac{\phi}{2}.$$

Similarly,

$$\sum_{j=1}^{2n} \sin j\phi = \sin(n\phi)\sin\frac{(2n+1)\phi}{2}\csc\frac{\phi}{2}.$$

Hence,

$$\sum_{j=1}^{2n} \exp ij\phi = \sin(n\phi)\csc\frac{\phi}{2}\exp\frac{i(2n+1)\phi}{2}.$$

The result now follows by changing ϕ into $\phi+\pi$ in the last equation. $\square$

Theorem 3.4. *Let $g(t) = g(0) + \sum_{k\in K} a_k(1-\cos kt)$, in which K is a subset of $\mathbf{N}\backslash\{0\}$, $a_k > 0$ for all k, and $\sum_{k\in K} a_k < \infty$. In order that $g \circ d_1$ be a strictly conditionally negative definite kernel, it is necessary that for each positive integer n, the set $n(1+2\mathbf{N}) \cap K$ be infinite.*

Proof. Assume that $g \circ d_1$ is strictly conditionally negative definite. We first prove that for each positive integer n the set $n(1+2\mathbf{N}) \cap K$ is nonempty. Suppose, on the contrary, that there is a positive integer n such that $n(1+2\mathbf{N}) \cap K = \varnothing$. In view of Theorem 3.2, we can assume that $n \geq 2$. Choose $2n$ distinct points $x_1, x_2, \dots, x_{2n}$ on S^1 such that $d_1(x_j, x_{j+1}) = \pi/n$ for $1 \leq j \leq 2n-1$. If B is the matrix with entries $B_{ij} = g(d_1(x_i,x_j))$ and c is the vector in $\mathbf{R}^{2n}$ having $(-1)^j$ as its jth component for $1 \leq j \leq 2n$, we have that

$$\begin{aligned} c^t Bc = &\sum_{\mu,\nu=1}^{2n} (-1)^\mu(-1)^\nu g(0) \\ &+ \sum_{\mu,\nu=1}^{2n} (-1)^\mu(-1)^\nu \sum_{k\in K} a_k(1-\cos kd_1(x_\mu,x_\nu)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in K} a_k \sum_{\mu,\nu=1}^{2n} (-1)^\mu(-1)^\nu\left(1-\cos\left(k(\mu-\nu)\frac{\pi}{n}\right)\right)\\
&= -\sum_{k\in K} a_k \sum_{\mu,\nu=1}^{2n} (-1)^\mu(-1)^\nu \cos\frac{k(\mu-\nu)\pi}{n}\\
&= -\sum_{k\in K} a_k\left[\left(\sum_{\mu=1}^{2n}(-1)^\mu\cos\frac{\mu k\pi}{n}\right)^2+\left(\sum_{\mu=1}^{2n}(-1)^\mu\sin\frac{\mu k\pi}{n}\right)^2\right]\\
&= -\sum_{k\in K} a_k\left|\sum_{\mu=1}^{2n}(-1)^\mu \exp\frac{i\mu k\pi}{n}\right|^2.
\end{aligned}
$$

From our assumption on n, it follows that $k\pi/n$ is not an odd multiple of π for all $k \in K$. Hence, by Lemma 3.3, $\sum_{\mu=1}^{2n}(-1)^\mu \exp(i\mu k\pi/n) = 0$ for all k in K, whence $c^t B c = 0$. Since $c \neq 0$, this contradicts our assumption on g. Next, in order to prove that each set $n(1+2\mathbf{N})\cap K$ is infinite, we assume that $n(1+2\mathbf{N})\cap K$ is finite for some positive integer n, and we reach a contradiction. Let $n(1+2l)$ denote the largest element in $n(1+2\mathbf{N})\cap K$. It follows from the first part of the proof that the set $n(3+2l)(1+2\mathbf{N})\cap K$ is nonempty. From the inclusion $n(3+2l)(1+2\mathbf{N})\cap K \subset n(1+2\mathbf{N})\cap K$, there is a nonnegative integer m such that $n(3+2l)(1+2m)\in n(1+2\mathbf{N})\cap K$ and $n(3+2l)(1+2m) > n(1+2l)$, contradicting our choice of l. $\square$

It is immediately seen that any set K satisfying the condition stated in the previous theorem contains infinitely many odd and infinitely many even multiples of any positive integer. On the other hand, the condition is not sufficient to guarantee strict conditional negative definiteness as the example after Theorem 3.2 shows.

Lemma 3.5. *If $p_1, p_2, \ldots, p_n$ are distinct prime numbers, then the numbers of the form $\sum_{j=1}^n \varepsilon_j/p_j$, where $\varepsilon_j = 0$ or 1, are all distinct modulo $\mathbf{Z}$.*

Proof. It suffices to prove that any number of the form $\sum_{j=1}^n \delta_j/p_j$, where $\delta_j = -1$ or 1, is not an integer. Suppose that $\sum_{j=1}^n \delta_j/p_j$ is an integer for some δ_j as above. We can write $\delta_1 p_2 \cdots p_n + p_1 N =$

$p_1 p_2 \cdots p_n M$ for some integers M and N, and this shows that p_1 is a divisor of the product $p_2 p_3 \cdots p_n$, a contradiction. $\square$

A subset K of $\mathbf{N}$ is said to be *generated* by a set P of prime numbers if every element in $K\backslash\{0,1\}$ is a multiple of an element in P, and P is the smallest set of prime numbers with this property. If P is an infinite set and K is generated by P, then K is said to be *infinitely generated* by P.

Theorem 3.6. *Let $g(t) = g(0) + \sum_{k\in K} a_k(1-\cos kt)$, in which $K \subset \mathbf{N}\backslash\{0\}$, $a_k > 0$ for all $k \in K$, and $\sum_{k\in K} a_k < \infty$. In order that $g \circ d_1$ be a strictly conditionally negative definite kernel, it is necessary that K be infinitely generated.*

Proof. We assume that K is generated by a finite set $P = \{p_1, p_2, \dots, p_n\}$ of prime numbers and show that $g \circ d_1$ is not strictly conditionally negative definite. It is easily seen that the nonzero function

$$h(t) = (\exp(i2\pi t/p_1) - 1)(\exp(i2\pi t/p_2) - 1)\cdots(\exp(i2\pi t/p_n) - 1)$$

vanishes on K. Direct computation reveals that we can write h in the form

$$h(t) = \sum_{j=1}^{2^n} c_j \exp(i\phi_j t), \qquad 0 = \phi_1 < \phi_2 < \cdots <= \phi_{2^n},$$
$$\sum_{j=1}^{2n} c_j = 0.$$

By Lemma 3.5 the ϕ_j are all distinct modulo 2π. Hence, the set $\{\phi_j : 1 \le j \le 2^n\}$ defines 2^n distinct points on S^1, namely,

$$x_j := (\cos\phi_j, \sin\phi_j), \qquad 1 \le j \le 2^n.$$

Writing $B_{ij} = g(d_1(x_i, x_j))$, and taking account of all the above, we obtain

$$\begin{aligned}\sum_{\mu,\nu=1}^{2^n} c_\mu c_\nu B_{\mu\nu} &= \sum_{\mu,\nu=1}^{2^n} c_\mu c_\nu \Big[g(0) + \sum_{k\in K} a_k(1-\cos k(\phi_\mu - \phi_\nu))\Big]\\ &= -\sum_{k\in K} a_k \sum_{\mu,\nu=1}^{2^n} c_\mu c_\nu \cos k(\phi_\mu-\phi_\nu)\\ &= -\sum_{k\in K} a_k \Bigg[\bigg(\sum_{\mu=1}^{2^n} c_\mu \cos k\phi_\mu\bigg)^2 + \bigg(\sum_{\mu=1}^{2^n} c_\mu \sin k\phi_\mu\bigg)^2\Bigg]\\ &= -\sum_{k\in K} a_k |h(k)|^2 = 0.\end{aligned}$$

Since the vector $c = (c_j)$ is nonzero, $g \circ d_1$ is not strictly conditionally negative definite. □

The conditions stated in Theorems 3.4 and 3.6 are together still not sufficient to guarantee strict conditional negative definiteness. In fact, let $g(t) = \sum_{k\in K} a_k(1-\cos kt)$, in which $K = (1+2\mathbf{N}) \cup 4\mathbf{N}$, $a_k > 0$ for all $k \in K$, and $\sum_{k\in K} a_k < \infty$. Set $\alpha = \sum_{k=1}^\infty a_{2k+1}$. Any four equally spaced points x_1, x_2, x_3 and x_4 on S^1 produces the following 4×4 matrix

$$(g(d_1(x_i,x_j))) = \begin{pmatrix} 0 & \alpha & 2\alpha & \alpha\\ \alpha & 0 & \alpha & 2\alpha\\ 2\alpha & \alpha & 0 & \alpha\\ \alpha & 2\alpha & \alpha & 0\end{pmatrix}$$

which is obviously singular. Observe that the entries in the interpolation matrix above depend only on the odd part $\sum_{k=1}^\infty a_{2k+1}(1-\cos(2k+1)t)$ of g. This suggests that K does not contain enough even numbers. This remark is the key for our last necessary condition.

Theorem 3.7. *Let $g(t) = g(0) + \sum_{k\in K} a_k(1-\cos kt)$, in which $K \subset \mathbf{N}\backslash\{0\}$, $a_k > 0$, and $\sum_{k\in K} a_k < \infty$. In order that $g \circ d_1$ be a strictly conditionally negative definite kernel, it is necessary that $L =: \{k : 2k \in K\}$ be infinitely generated.*

Proof. We proceed as in the proof of Theorem 3.6. We assume that L is generated by a finite set $P = \{p_1, p_2, \dots, p_{n-1}\}$ of prime numbers and prove that $g \circ d_1$ is not strictly conditionally negative definite. The nonzero function

$$h(t) = (\exp(i\pi t) + 1)\left(\exp\frac{i2\pi t}{p_1} - 1\right)\left(\exp\frac{i2\pi t}{p_2} - 1\right) \cdots \left(\exp\frac{i2\pi t}{p_{n-1}} - 1\right)$$

vanishes on K. Using Lemma 3.5, we can write $h(t) = \sum_{j=1}^{2^n} c_j \exp(i\phi_j t)$ with $\sum_{j=1}^{2^n} c_j = 0$ and all the ϕ_j distinct modulo 2π. Setting $x_j = (\cos\phi_j, \sin\phi_j)$ for $1 \leq \mu \leq 2^n$, we have

$$\sum_{i,j=1}^{2^n} c_i c_j g(d_1(x_i, x_j)) = \sum_{k \in K} a_k |h(k)|^2 = 0.$$

This shows that $g \circ d_1$ is not strictly conditionally negative definite because $c \neq 0$. $\square$

At this point it is important to emphasize that the necessary conditions presented in Theorems 3.4, 3.6 and 3.7 are independent of each other, as the sets $2\mathbf{N} \cup (3 + 6\mathbf{N}), 4\mathbf{N} \cup (1 + 2\mathbf{N})$, and $(4 + 2\mathbf{N}) \cup \{\text{prime numbers}\}$ show. We do not know whether those three conditions, together, are sufficient to guarantee strict conditional negative definiteness. On the other hand, we were unable to construct a counterexample. For instance, we were unable to determine whether or not the function $g(t) = \sum_{k \in K} a_k(1 - \cos kt)$, in which $K = (4 + 2\mathbf{N}) \cup (3 + 6\mathbf{N}) \cup \{\text{prime numbers}\}$, $a_k > 0$ for all $k \in K$, and $\sum_{k \in K} a_k < \infty$ defines a strictly conditionally negative definite kernel $g \circ d_1$.

Corollary 3.8. *Let $f(t) = \sum_{k \in K} a_k \cos kt$, in which $K \subset \mathbf{N}$, $a_k > 0$ and $\sum_{k \in K} a_k < \infty$. In order that $f \circ d_1$ be a strictly positive definite kernel, it is necessary that*

(a) *For each positive integer n, the set $n(1 + 2\mathbf{N}) \cap K$ be infinite,*

(b) *K be infinitely generated,*

(c) *$L = \{k : 2k \in K\}$ be infinitely generated.*

4. Examples. We denote by $\mathcal{CM}$ the set of all nonconstant continuous functions $F : [0,\infty) \to [0,\infty)$ which are completely monotonic on $(0,\infty)$, and by $\mathcal{DM}$ the set of continuous functions $F : [0,\infty) \to [0,\infty)$ which are differentiable on $(0,\infty)$ and such that F' is a nonconstant completely monotonic function on $(0,\infty)$. Recall that a function $F : [0,\infty) \to [0,\infty)$ is said to be *completely monotonic* on $(0,\infty)$ if and only if $(-1)^j F^{(j)}(t) \geq 0$ for $j \in \mathbf{N}$, and $t > 0$. Lemma 4.1 below can be found in a more general formulation in [**4**].

Lemma 4.1. *Let $F : [0,\infty) \to [0,\infty)$ be a continuous function and $g \circ d_1$ be a* (*continuous*) *conditionally negative definite kernel on S^1. The following assertions hold*:

(a) *If $F \in \mathcal{CM}$, then $F \circ g \circ d_1$ is strictly positive definite if and only if $g(t) > g(0)$ for $t \in (0,\pi]$.*

(b) *If $F \in \mathcal{DM}$, then $F \circ g \circ d_1$ is strictly conditionally negative definite if and only if $g(t) > g(0)$ for $t \in (0,\pi]$.*

Using this result, it is not hard to see (this is Theorem 4.9 in [**4**]) that any function F as in (a), respectively, (b), above is such that $F \circ d_1$ is strictly positive, respectively, strictly conditionally negative, definite. Another class of examples can be obtained with the help of the following lemma.

Lemma 4.2. *Let $g(t) = g(0) + \sum_{k=1}^{\infty} a_k(1-\cos kt)$, in which $a_k \geq 0$ for all k and $\sum_{k=1}^{\infty} a_k < \infty$. In order that the equation $g(t) = g(0)$ have only the zero solution in $[0,\pi]$ it is necessary and sufficient that $a_k > 0$ for a set of relatively prime indices k.*

Proof. First assume that $a_k > 0$ for relatively prime integers $k_1, k_2, \ldots, k_n$, and suppose that $g(t_0) = g(0)$ for some $t_0 > 0$. By our assumption on t_0, it follows that $\cos k_i t_0 = 1$ for $1 \leq i \leq n$. Hence, there are positive integers $\mu_1, \mu_2, \ldots, \mu_n$ such that $k_i t_0 = 2\pi\mu_i$ for $1 \leq i \leq n$. Since the k_i are relatively prime, there are integers $\nu_1, \nu_2, \ldots, \nu_n$ such that $\sum_{j=1}^{n} \nu_j k_j = 1$. Using this we obtain

$$t_0 = t_0 \sum_{j=1}^{n} \nu_j k_j = 2\pi \sum_{j=1}^{n} \mu_j \nu_j$$

and so

$$\mu_i = k_i \frac{t_0}{2\pi} = k_i \sum_{j=1}^{n} \mu_j \nu_j, \qquad 1 \le i \le n.$$

The last equality above reveals that $\sum_{j=1}^{n} \mu_j \nu_j$ is a positive integer and consequently $\mu_i \ge k_i$ for $1 \le i \le n$. It is now clear that $k_i t_0 = 2\pi\mu_i \ge 2\pi k_i$ for $1 \le i \le n$. In particular, $t_0 \ge 2\pi$.

Conversely, suppose that $t = 0$ is the only solution of $g(t) = g(0)$ in $[0, \pi]$. Order the set of all k for which $a_k > 0$, say $k_1, k_2, \dots$. Let γ_i denote the greatest common divisor of $k_1, k_2, \dots, k_i$. Clearly, $1 \le \gamma_{i+1} \le \gamma_i$ for all i. So there is a smallest index i_0 such that $\gamma_{i_0} = \gamma_{i_0+j}$ for $j = 1, 2, \dots$. If $\gamma_{i_0} > 1$, then the point $t_0 = 2\pi/\gamma_{i_0}$ is such that $0 < t_0 \le \pi$ and $g(t_0) = g(0)$, a contradiction. Thus, $\gamma_{i_0} = 1$ and $k_1, k_2, \dots, k_{i_0}$ are relatively prime. $\square$

Theorem 4.3. *let $F : [0, \infty) \to [0, \infty)$ be a continuous function and let $g(t) = g(0) + \sum_{k=1}^{\infty} a_k(1 - \cos kt)$, in which $g(0) \ge 0$, $a_k \ge 0$ for all k, and $\sum_{k=1}^{\infty} a_k < \infty$. The following assertions hold*:

(a) *If $F \in \mathcal{CM}$, then $F \circ g \circ d_1$ is strictly positive definite if and only if $a_k > 0$ for a set of relatively prime indices k.*

(b) *If $F \in \mathcal{DM}$, then $F \circ g \circ g_1$ is strictly conditionally negative definite if and only if $a_k > 0$ for a set of relatively prime indices k.*

As an example we can take $g(t) = 2\sin^2 t/2 = 1 - \cos t$. The kernel $g \circ d_1$ is the Euclidean chord distance on S^1 and it is not strictly conditionally negative definite. Using Theorem 4.3, we see that $(1/\sqrt{1+g}) \circ d_1$, $\exp(-g) \circ d_1$, and $\ln[(2+g)/(1+g)] \circ d_1$ are strictly positive definite while $\ln(1+g) \circ d_1$ and $\sqrt{1+g} \circ d_1$ are strictly conditionally negative definite.

REFERENCES

1. C. Berg, J.P.R. Christensen and P. Ressel, *Harmonic analysis on semigroups*, Springer Verlag, New York, 1984.

2. W.A. Light and E.W. Cheney, *Interpolation by periodic radial basis functions*, J. Math. Anal. Appl. **168** (1992), 111–130.

3. V.A. Menegatto, *Interpolation on spherical spaces*, Ph.D. dissertation, University of Texas at Austin, 1992.

4. ———, *Strictly positive definite kernels on the Hilbert sphere*, Appl. Anal. **55** (1994), 91–101.

5. I.J. Schoenberg, *Positive definite functions on spheres*, Duke Math. J. **9** (1942), 96–108.

6. Y. Xu and E.W. Cheney, *Interpolation by periodic radial functions*, Comput. Math. Appl. **24** (1992), 201–215.

7. ———, *Strictly positive definite functions on spheres*, Proc. Amer. Math. Soc. **116** (1992), 977–982.

Departamento de Matemática, ICMSC–Universidade de São Paulo, Caixa Postal 668, 13560-970 São Carlos SP, Brasil

email address: menegatt@icmsc.usp.br.ansp