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BOURGAIN ALGEBRAS ON THE MAXIMAL IDEAL SPACE OF H^∞

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ABSTRACT. Let B be a Douglas algebra. For another Douglas algebra D, by considering the integral representation, there exists the corresponding closed subspace $\hat{D}$ of $C(M(B))$ the space of continuous functions on maximal ideal space of B. Let $[\hat{D}]_{M(B)}$ be the closed subalgebra of $C(M(B))$ generated by $\hat{D}$. In this paper we describe the algebra $[\hat{D}]_{M(B)}$ and determine the Bourgain algebra of $[\hat{D}]_{M(B)}$ relative to $C(M(B))$.

1. Introduction. The concept of Bourgain algebras was introduced by Cima and Timoney (see [**2**] and [**6**]). Let $\mathcal{A}$ be a Banach algebra with identity, and let $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$. The *Bourgain algebra* $\mathcal{B}_b$ *relative to* $\mathcal{A}$ is the space of f in $\mathcal{A}$ such that $||ff_n+\mathcal{B}|| \to 0$ $(n \to \infty)$ for every sequence $\{f_n\}_n$ in $\mathcal{B}$ converging weakly to zero. Cima and Timoney proved that $\mathcal{B}_b$ is a closed subalgebra of $\mathcal{A}$ containing $\mathcal{B}$. We shall write $\mathcal{B}_{bb}$ for $(\mathcal{B}_b)_b$. For other recent papers on Bourgain algebras, the reader is referred to [**4, 5, 9, 10, 13, 17, 20, 22, 23**].

Let H^∞ be the space of boundary functions of bounded analytic functions on the open unit disk Δ. With the essential supremum norm, H^∞ is a subalgebra of L^∞ on the unit circle T. A closed subalgebra B of L^∞ containing H^∞ is called a Douglas algebra. Let C denote the space of continuous functions on T. As Sarason showed, the algebra $H^\infty + C$ is a Douglas algebra (see [**21**] for a discussion of this algebra). In [**4**], Cima, Janson and Yale proved that $(H^\infty)_b = (H^\infty)_{bb} = H^\infty + C$ relative to L^∞. In [**10**], the authors and Mortini studied Bourgain algebras of Douglas algebras B and showed that $B_b = B_{bb}$ relative to L^∞. In [**17**] the second author determined the Bourgain algebra of the disk algebra A and proved that $A_b = A_{bb}$ relative to L^∞. These are all studies of Bourgain algebras relative to L^∞ on T.

In what follows we denote the set of nonzero multiplicative linear functionals of a Douglas algebra B by $M(B)$. With the weak*-topology,

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$M(B)$ is a compact Hausdorff space, and we think of $M(L^\infty)$ as a subset of $M(B)$ (and we may think of $M(B)$ as a subset of $M(H^\infty)$). Furthermore, it can be shown that $M(H^\infty + C) = M(H^\infty)\backslash\Delta$. Writing X for $M(L^\infty)$, we know that X is the Shilov boundary for every Douglas algebra. Using the Gelfand transform, we may think of H^∞ as a subset of the space of continuous functions on $M(H^\infty)$, denoted $C(M(H^\infty))$, and we identify L^∞ with the space of continuous functions on X, denoted $C(X)$. For a point x in $M(H^\infty)$, there is a representing measure μ_x on X such that $\int_X f\, d\mu_x = f(x)$ for every $f \in H^\infty$. For an L^∞ function g, we let

$$\hat{g}(x) = \int_X g\, d\mu_x \quad \text{for every } x \in M(H^\infty).$$

Then $\hat{g}$ is a continuous function of $M(H^\infty)$ [**16**, p. 93], and for $g \in B$ we see that $\hat{g}$ coincides with the Gelfand transform of g on $M(B)$, and $(gh)^\wedge = \hat{g}\hat{h}$ on $M(B)$ for $h \in B$. In [**9**], Ghatage, Sun and Zheng studied Bourgain algebras on $M(H^\infty)$. Letting $C(\bar{\Delta})$ denote the algebra of continuous functions on the closed unit disk, their result may be stated as $(H^\infty)_b = (H^\infty)_{bb} = H^\infty + C(\bar{\Delta})$ relative to $C(M(H^\infty))$.

Cima, Janson and Yale's theorem, $(H^\infty)_b = H^\infty + C$ relative to L^∞, had two different generalizations. One is the study of Bourgain algebras relative to $L^\infty = C(M(L^\infty))$, and the other is the study of Bourgain algebras relative to $C(M(H^\infty))$. The purpose of this paper is to unify these studies. For a Douglas algebra D, let $\hat{D} = \{\hat{f}; f \in D\}$. Then $\hat{D}$ is a closed subspace of $C(M(H^\infty))$, but $\hat{D}$ is not necessarily an algebra. In this paper we study Bourgain algebras of one Douglas algebra relative to another. To do this, we look at the closed subalgebra of $C(M(B))$ generated by $\hat{D}$, denoted here by $[\hat{D}]_{M(B)}$. This allows us to study the Bourgain algebra $([\hat{D}]_{M(B)})_b$ relative to $C(M(B))$. The Bourgain algebra of D relative to L^∞ is denoted by D_b. When $B = L^\infty$, our study is the same as [**10**]. When $B = H^\infty$, our study is on the same situation as [**9**]. Hence, our investigation of Bourgain algebras on $M(B)$ includes both of these cases.

Although the notation is rather cumbersome, we obtain results about familiar algebras. In Section 2 we refine a theorem of P. Jones [**18**] to show that the interpolating Blaschke products which are invertible in D separate the points of $M(H^\infty)\backslash M(D)$. As an application we can describe the algebra $[\hat{D}]_{M(B)}$. In Section 3, we give some basic results on Bourgain algebras which can be applied in a variety of situations. Using these results, in Section 4 we determine $([\hat{D}]_{M(B)})_b$ and we show that in this case, as in many other cases, $([\hat{D}]_{M(B)})_b = ([\hat{D}]_{M(B)})_{bb}$. If we consider the special case in which $B = D = H^\infty$, we obtain Ghatage, Sun and Zheng's theorem. In Section 5 we study the disk algebra A and show that $(\hat{A}|_{M(B)})_b = (\hat{A}|_{M(B)})_{bb}$.

2. A separation theorem. First we give some notation and definitions. For a compact Hausdorff space, Y, we denote by $C(Y)$ the space of continuous functions on Y. For a subset E of Y and a function f in $C(Y)$, we put

$$||f||_E = \sup\{|f(x)|; x \in E\}.$$

Let $\mathcal{A}$ be a closed subalgebra of $C(Y)$. For a sequence $\{f_n\}_n$ in $\mathcal{A}$, $f_n \to 0$ weakly in $\mathcal{A}$ if and only if $\{f_n\}_n$ is sup-norm bounded and $f_n \to 0$ pointwise on Y. When this condition is satisfied, we say that $f_n \to 0$ weakly on Y. If we denote by ∂ the Shilov boundary for $\mathcal{A}$, then for a sequence $\{f_n\}_n$ in $\mathcal{A}$ we have that $f_n \to 0$ weakly on Y if and only if $f_n \to 0$ weakly on ∂. A closed subset E of Y is called an antisymmetric set for $\mathcal{A}$ if the restriction $f|_E$, $f \in \mathcal{A}$, is a real function then $f|_E$ is constant [**7**, p. 60].

Let $\{z_n\}_n$ be a sequence in Δ with $\Sigma(1 - |z_n|) < \infty$. The function

$$\psi(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \qquad z \in \Delta$$

is called a Blaschke product. Moreover, if

$$\inf_k \prod_{n: n \neq k} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| > 0,$$

then ψ is said to be interpolating. A function q in H^∞ is called inner if $|q| = 1$ on $X = M(L^\infty)$. Blaschke products are typical inner functions. The Chang-Marshall theorem [**3, 19**] says that every Douglas algebra B is generated in L^∞ by H^∞ and complex conjugates of interpolating Blaschke products ψ with $\bar{\psi} \in B$. We also have

$$\begin{aligned} M(B) &= \{x \in M(H^\infty); |\hat{\psi}(x)| = 1 \text{ for interpolating Blaschke} \\ &\qquad\qquad \text{products } \psi \text{ with } \bar{\psi} \in B\} \\ &= \{x \in M(H^\infty); |\hat{q}| = 1 \text{ for inner functions } q \text{ with } \bar{q} \in B\}. \end{aligned}$$

Hence, if B_1 and B_2 are Douglas algebras with $B_1 \subset B_2$, then $M(B_2) \subset M(B_1)$. We use this theorem several times. Since X is the Shilov boundary for every Douglas algebra B, if $f \in B$ we have $||\hat{f}||_{M(B)} = ||f||_X$. Hence, for a closed subset E such that $X \subset E \subset M(B)$, $\hat{B}|_E$ is a closed subalgebra of $C(E)$. [**8**] is a nice reference for this material.

For points x and y in $M(H^\infty)$, we put

$$\rho(x,y) = \sup\{|\hat{f}(y)|; f \in H^\infty, ||f||_X = 1, \hat{f}(x) = 0\}.$$

The set $P(x) = \{\zeta \in M(H^\infty); \rho(x,\zeta) < 1\}$ is called a Gleason part. The open disk $\Delta = P(0)$ is a typical Gleason part. Of course, by the Corona Theorem, Δ is dense in $M(H^\infty)$. When $P(x) \neq \{x\}$, we call x a nontrivial point. By Hoffman's theorem [**16**] there is a continuous one to one map L_x from Δ onto $P(x)$ such that $L_x(0) = x$ and $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. The map L_x is called the Hoffman map. For $f \in H^\infty$, we denote by

$$Z(\hat{f}) = \{x \in M(H^\infty); \hat{f}(x) = 0\}.$$

For each point x in $Z(\hat{f})$, we may consider the order of the zero of f at x. If x is a trivial point, we say that the order of the zero of f at x is infinite. If x is a nontrivial point, we can define the order of the zero at x by considering the function $\hat{f} \circ L_x$ at 0 in the usual way. When f has a zero of infinite order at x, for each integer n we can write f as the product of n functions in H^∞ vanishing at x [**8**, p. 379, Lemma 3.3]. For a subset E of $M(H^\infty)$, we denote by $\mathrm{cl}\, E$ the weak*-closure of E in $M(H^\infty)$. If ψ is an interpolating Blaschke product with zeros $\{z_n\}_n$, then $Z(\hat{\psi}) = \mathrm{cl}\,\{z_n\}_n$ [**15**, p. 205], and every point in $Z(\hat{\psi})$ is nontrivial. Conversely, for every nontrivial point x there is an interpolating Blaschke product ψ such that $\hat{\psi}(x) = 0$.

In [**18**], P. Jones proved that interpolating Blaschke products separate the points in $M(H^\infty)$. The following is a similar type of separation theorem for Douglas algebras.

Theorem 2.1. *The interpolating Blaschke products which are invertible in a Douglas algebra B separate the points of $M(H^\infty)\backslash M(B)$.*

Proof. Let x and y be two different points in $M(H^\infty)\backslash M(B)$. Since H^∞ separates the points in $M(H^\infty)$, there is a function f in H^∞ with

$$\hat{f}(x) = 0, \qquad \hat{f}(y) \neq 0 \quad \text{and} \quad ||f||_X < 1. \tag{1}$$

Since $x \notin M(B)$, by the Chang-Marshall theorem there exists an inner function u invertible in B with $|\hat{u}(x)| < 1$. Let $\alpha = \hat{u}(x)$, and let

$$v = (u - \alpha)/(1 - \bar{\alpha} u) \quad \text{on } T. \tag{2}$$

Since $|\hat{u}| = 1$ on $M(B)$, we see that v is invertible in B. Furthermore, v is also inner and $\hat{v}(x) = 0$. We now construct an inner function w invertible in B such that

$$\hat{w}(x) = 0 \quad \text{and} \quad \hat{w}(y) \neq 0. \tag{3}$$

We will then use w to obtain an interpolating Blaschke product b invertible in B with $\hat{b}(x) \neq \hat{b}(y)$.

Case 1. If $\hat{u}(x) \neq \hat{u}(y)$, let $w = v$. In this case it is clear that w is an inner function invertible in B satisfying (3).

Case 2. If $\hat{u}(x) = \hat{u}(y)$, consider the function $f + v$. In this case, by (2), $\hat{v}(y) = 0$. By (1), for $\zeta \in X$, we have

$$|f(\zeta) + v(\zeta)| \geq |v(\zeta)| - |f(\zeta)| > 0.$$

So $f + v$ is invertible in L^∞. By the strong logmodularity of H^∞ [**8**, p. 201], there is an invertible function g in H^∞ such that $|g| = |(f+v)^{-1}|$ on X. Let $w = g(f + v)$. Then w is an inner function. Since v is invertible in B, $|\hat{v}| = 1$ on $M(B)$. Hence, for ξ in $M(B)$,

$$\begin{aligned} |\hat{w}(\xi)| &= |\hat{g}(\xi)(\hat{f}(\xi) + \hat{v}(\xi))| \\ &\geq |\hat{g}(\xi)|(|\hat{v}(\xi)| - |\hat{f}(\xi)|) \\ &\geq |\hat{g}(\xi)|(1 - ||f||_X) \\ &> 0 \quad \text{by (1).} \end{aligned}$$

Thus, w is invertible in B. Since $\hat{f}(x) = \hat{v}(x) = 0$, $\hat{w}(x) = 0$. Since $\hat{g}(y) \neq 0$, $\hat{f}(y) \neq 0$ and $\hat{v}(y) = 0$, we have $\hat{w}(y) \neq 0$. Therefore w satisfies (3).

Now either w has a zero of finite order at x or else it does not. If the order of the zero at x is finite, then x lies in the closure of an interpolating subsequence of the zero sequence of w [**16**, Theorem 5.3]. In this case $w = bc$, where b is an interpolating Blaschke product which vanishes at x, but (since $\hat{w}(y) \neq 0$) not at y. Since w is invertible in B, b cannot vanish on $M(B)$, and hence is also invertible in B. Thus, b separates the points x and y and is the desired interpolating Blaschke product.

Next we assume that w has a zero of infinite order at x. This means that for any n we can write w as the product of n inner functions, invertible in B, all vanishing at x. At least one of these factors must have the property that the absolute modulus of the value at y is larger than $|\hat{w}(y)|^{1/n}$. Thus, replacing w by one of these factors for an appropriate n, we may assume that for any $\varepsilon > 0$ there exists w_ε inner and invertible in B, vanishing at x and $|\hat{w}_\varepsilon(y)| > 1 - \varepsilon$. By the argument on page 429 of Garnett [**8**], we see that for every $\varepsilon > 0$ there exists an interpolating Blaschke product B_ε with

$$|B_\varepsilon(z)| < 1/4 \quad \text{if} \quad |w_\varepsilon(z)| < 1/4, \qquad z \in \Delta$$

and there is an $\eta(\varepsilon) \to 0$ $(\varepsilon \to 0)$ such that

$$|B_\varepsilon(z)| \geq 1 - \eta(\varepsilon) \quad \text{if} \quad |w_\varepsilon(z)| \geq 1 - \varepsilon, \qquad z \in \Delta.$$

Here we choose ε so small that $1 - \eta(\varepsilon) > 1/2$ and use the Corona Theorem. Since $w_\varepsilon(x) = 0$ and $|w_\varepsilon(y)| > 1 - \varepsilon$, then

$$|\hat{B}_\varepsilon(x)| \leq 1/4 \quad \text{while} \quad |\hat{B}_\varepsilon(y)| \geq 1/2.$$

Now w_ε is inner and invertible in B, so $|\hat{w}_\varepsilon| = 1$ on $M(B)$. Thus, $|\hat{B}_\varepsilon| \geq 1/2$. So B_ε is an interpolating Blaschke product in B which separates the points x and y, and is invertible in B. □

For a subset F of L^∞, we write $[F]$ for the smallest Douglas algebra containing F. The following is a consequence of the Chang-Marshall theorem.

Lemma 2.1. *Let B and D be Douglas algebras. Then $M([B \cup D]) = M(B) \cap M(D)$.*

Proof. By the Chang-Marshall theorem, there are sets of inner functions $\{\phi_\alpha\}_{\alpha\in\Gamma}$ and $\{\psi_\beta\}_{\beta\in\Lambda}$ such that $B=[H^\infty,\bar{\phi}_\alpha;\alpha\in\Gamma]$ and $D=[H^\infty,\bar{\psi}_\beta;\beta\in\Lambda]$. Then $[B\cup D]=[H^\infty,\bar{\phi}_\alpha,\bar{\psi}_\beta;\alpha\in\Gamma,\beta\in\Lambda]$. Hence,

$$\begin{aligned} M([B\cup D]) &= \{x\in M(H^\infty); |\phi_\alpha(x)|=1, |\psi_\beta(x)|=1 \text{ for } \alpha\in\Gamma,\beta\in\Lambda\} \\ &= \{x\in M(H^\infty); |\phi_\alpha(x)|=1 \text{ for } \alpha\in\Gamma\} \\ &\quad \cap \{x\in M(H^\infty); |\psi_\beta(x)|=1 \text{ for } \beta\in\Lambda\} \\ &= M(B)\cap M(D). \qquad \square \end{aligned}$$

As an application of Theorem 2.1, we have the following theorem.

Theorem 2.2. *Let B and D be Douglas algebras. Then*

$$\begin{aligned} [\hat{D}]_{M(B)} &= \{f\in C(M(B)); f|_{M([B\cup D])}\in \hat{D}|_{M([B\cup D])}\} \\ &= [H^\infty, \hat{\bar{\psi}}; \bar{\psi}\in D, \\ &\qquad \bar{\psi} \textit{ is an interpolating Blaschke product}]_{M(B)}. \end{aligned}$$

To prove this, we give two elementary lemmas which will be used several times in this paper.

Lemma 2.2. *Let Y be a compact Hausdorff space, and let E be a closed subset of Y. Let $\mathcal{A}$ be a subset of $C(Y)$. If $\mathcal{A}|_E$ is a closed subalgebra of $C(E)$, then*

(i) $\{f\in C(Y); f|_E\in\mathcal{A}|_E\}=\mathcal{A}+\{f\in C(Y); f=0$ *on* $E\}$ *is a closed subalgebra of* $C(Y)$.

(ii) *For* $g\in C(Y)$, $||g+\{f\in C(Y); f|_E\in\mathcal{A}|_E\}||_Y=||g+\mathcal{A}||_E$.

Proof. (i) is trivial. For (ii), we have

$$\begin{aligned} ||g+\{f\in C(Y); f|_E\in\mathcal{A}|_E\}||_Y &\le ||g+\{f\in C(Y); f|_E\in A|_E\}||_E \\ &= ||g+\mathcal{A}||_E. \end{aligned}$$

The dual space of the quotient space $C(Y)/\{f\in C(Y); f|_E\in\mathcal{A}|_E\}$ coincides with the set of bounded Borel measures μ on Y such that

$\int_Y f\,d\mu = 0$ for every $f \in C(Y)$ with $f|_E \in \mathcal{A}|_E$. By (i), $f = h + c$, where $h \in \mathcal{A}$ and $c \in C(Y)$ with $c = 0$ on E. Hence the closed support set of μ is contained in E. Therefore, the dual spaces of $C(Y)/\{f \in C(Y); f|_E \in \mathcal{A}|_E\}$ and $C(E)/\mathcal{A}|_E$ coincide. By this fact, we have (ii). □

The lemma below follows from the fact that X is the Shilov boundary for every Douglas algebra.

Lemma 2.3. *Let D be a Douglas algebra, and let E be a closed subset with $X \subset E \subset M(D)$. Then $\hat{D}|_E$ is a closed subalgebra of $C(E)$.*

Proof. This fact follows from $||f||_X = ||\hat{f}||_E$ and $(fg)^\wedge = \hat{f}\hat{g}$ on E for $f, g \in D$. □

Proof of Theorem 2.2. Let

$$\mathcal{A} = [H^\infty, \hat{\bar{\psi}}; \bar{\psi} \in D, \psi \text{ is an interpolating Blaschke product}]_{M(B)};$$
$$\mathcal{B} = \{f \in C(M(B)); f|_{M([B\cup D])} \in \hat{D}|_{M([B\cup D])}\}.$$

Then trivially we have $\mathcal{A} \subset [\hat{D}]_{M(B)}$. By Lemmas 2.1 and 2.3, $\hat{D}|_{M([B\cup D])}$ is a closed subalgebra of $C(M[B \cup D])$. Hence, by Lemma 2.2, $\mathcal{B}$ is a closed subalgebra of $C(M(B))$. Since $\hat{D}|_{M(B)} \subset \mathcal{B}$, we get $[\hat{D}]_{M(B)} \subset \mathcal{B}$.

It remains to show that $\mathcal{B} \subset \mathcal{A}$. To prove this, we use Bishop's theorem [**7**, p. 60]. Let $S \subset M(B)$ be an antisymmetric set for $\mathcal{A}$. By the Chang-Marshall theorem $M(D) = \{x \in M(H^\infty); |\hat{\psi}(x)| = 1$ for interpolating Blaschke products ψ with $\bar{\psi} \in D\}$. Hence, if $S \cap M(D) \neq \varnothing$, then for any interpolating Blaschke product ψ with $\bar{\psi} \in D$, the facts that $\bar{\psi}, \hat{\bar{\psi}} = \bar{\hat{\psi}} \in \mathcal{A}$ and S is antisymmetric imply $\hat{\psi}$ is a constant on S and that constant must be of modulus one. Since this is true for any interpolating Blaschke product ψ with $\bar{\psi} \in D$, we see that $S \cap M(D) \neq \varnothing$ implies $S \subset M(D)$. Hence, we have either $S \cap M(D) = \varnothing$ or $S \subset M(D)$.

If $S \cap M(D) = \varnothing$, we claim S is actually a one-point set. If not, we may choose two distinct points x and y in S. By Theorem 2.1, there is an interpolating Blaschke product ψ with $\bar{\psi} \in D$ such that $\hat{\psi}(x) \neq \hat{\psi}(y)$. Thus, either $\operatorname{Re}\hat{\psi}(x) \neq \operatorname{Re}\hat{\psi}(y)$ or $\operatorname{Im}\hat{\psi}(x) \neq \operatorname{Im}\hat{\psi}(y)$. Since $\operatorname{Re}\hat{\psi} = (\hat{\psi} + \hat{\bar{\psi}})/2 \in \mathcal{A}$ and $\operatorname{Im}\hat{\psi} = (\hat{\psi} - \hat{\bar{\psi}})/2i \in \mathcal{A}$, the antisymmetry of S implies this is impossible. Thus, S must be a single point. In this case, of course, we have for any $f \in \mathcal{B}$ that $f|_S \in \mathcal{A}|_S$. If $S \subset M(D)$ and $f \in \mathcal{B}$, then by Lemma 2.2 $f = \hat{h}|_{M(B)} + c$, where $h \in D$ and $c \in C(M(B))$ satisfies $c = 0$ on $M(B) \cap M(D)$. Since $S \subset M(B) \cap M(D)$, clearly $c|_S \in \mathcal{A}|_S$. Since $h \in D$, the Chang-Marshall theorem and the fact that $S \subset M(D)$ imply $\hat{h}|_S \in \mathcal{A}|_S$. Thus, $f|_S \in \mathcal{A}|_S$ and Bishop's theorem yields the result. □

3. Bourgain algebras. In this section we present some results which are used in the following sections. Throughout this section let $\mathcal{A}$ be a closed subalgebra of L^∞, let $\mathcal{B} = [H^\infty \cup \mathcal{A}]$ denote the Douglas algebra generated by $\mathcal{A}$, and let E be a closed subset of $M(\mathcal{B})$ with $X \subset E \subset M(\mathcal{B})$. Then $\mathcal{A}|_E$ is a closed subalgebra of $C(E)$ (see Lemma 2.3). We discuss the Bourgain algebra $(\mathcal{A}|_E)_b$ relative to $C(E)$. We denote by $\mathcal{A}_b$ the Bourgain algebra of $\mathcal{A}$ relative to L^∞. For weakly convergent sequences, we have the following.

Lemma 3.1. *Let $\{f_n\}_n$ be a sequence in L^∞ and let E be a closed subset with $X \subset E \subset M(H^\infty)$. Then $||f_n|| = ||\hat{f}_n||_E$, and $f_n \to 0$ weakly on X if and only if $\hat{f}_n \to 0$ weakly on E.*

Proof. Suppose that $f_n \to 0$ weakly on X. Then $\{f_n\}_n$ is sup-norm bounded and $f_n \to 0$ pointwise on X. Since $\hat{f}_n(x) = \int_X f_n \, d\mu_x$, $x \in E$, by the Lebesgue dominated convergence theorem it is clear that $\hat{f}_n \to 0$ weakly on E. The other statements above are easily proved. □

Here we consider a particular open subset U of E satisfying the following condition:

$(*)$ for every sequence $\{f_n\}_n$ in $\mathcal{A}$ with $f_n \to 0$ weakly, $\hat{f}_n \to 0$ uniformly on each compact subset of U.

We denote by E_0 the union set of all open subsets U of E which satisfy condition $(*)$. We note that E_0 might be empty.

Lemma 3.2. *E_0 satisfies $(*)$; that is, E_0 is the largest open subset of E which satisfies condition $(*)$.*

Proof. Let $\{f_n\}_n \subset \mathcal{A}$ with $f_n \to 0$ weakly, and let K be a compact subset of E_0. Then there exist open subsets $U_1, \dots, U_k$ satisfying $(*)$ with $K \subset \{U_j; 1 \le j \le k\}$. Then one can find open subsets V_j, $1 \le j \le k$, such that

$$V_j \subset U_j, \quad \operatorname{cl} V_j \subset U_j \quad \text{and} \quad K \subset \cup\{V_j; 1 \le j \le k\}.$$

By condition $(*)$, $\hat{f}_n \to 0$ uniformly on V_j for each j, hence on K. □

In the rest of this paper, E_0 plays an important role. We note that E_0 depends on $\mathcal{A}$.

Proposition 3.1. *Let $\mathcal{A}$ be a closed subalgebra of L^∞, $\mathcal{B} = [H^\infty \cup \mathcal{A}]$, and $X \subset E \subset M(B)$. Then $(\hat{\mathcal{A}}|_E)_b \supset \{f \in C(E); f = 0$ on $E \backslash E_0\}$.*

Proof. Let $f \in C(E)$ with $f = 0$ on $E\backslash E_0$ and let $\{f_n\}_n \subset \mathcal{A}$ with $f_n \to 0$ weakly. We may assume that $||f_n||_E = 1$ for every n. Given $\varepsilon > 0$, there exists a compact subset F of E_0 such that $|f| < \varepsilon$ on $E\backslash F$. Then we have

$$||f\hat{f}_n||_E \le \varepsilon + ||f\hat{f}_n||_F.$$

By Lemma 3.2, $||f\hat{f}_n||_E \to 0$. Since $||f\hat{f}_n + \hat{\mathcal{A}}||_E \le ||f\hat{f}_n||_E$, we get $f \in (\hat{\mathcal{A}}|_E)_b$.

Proposition 3.2. *Let $\mathcal{A}$ be a closed subalgebra of L^∞, $\mathcal{B} = [H^\infty \cup \mathcal{A}]$ and $X \subset E \subset M(B)$. Let ψ be an inner function with $\bar{\psi} \in \mathcal{A}_b$. If $|\hat{\psi}| = 1$ on $E\backslash E_0$, then $\hat{\bar{\psi}} \in (\hat{\mathcal{A}}|_E)_b$.*

Proof. Let $\{f_n\}_n$ be a sequence in $\mathcal{A}$ with $f_n \to 0$ weakly on X. Then by Lemma 3.1, there exists $K > 0$ such that

$$(4) \qquad ||\hat{f}_n||_E \le K \quad \text{for every } n.$$

Since $\bar{\psi} \in \mathcal{A}_b$, there exists g_n in $\mathcal{A}$ such that $||\bar{\psi} f_n - g_n||_X \to 0$. Hence,

$$(5) \qquad ||f_n - g_n\psi||_X = ||\bar{\psi} f_n - g_n||_X \to 0.$$

Since f_n, g_n and ψ are all contained in $\mathcal{B}$ and $E \subset M(\mathcal{B})$, we use Lemma 3.1 to see that $||\hat{f}_n - \hat{g}_n\hat{\psi}||_E = ||f_n - g_n\psi||_X$. Hence,

$$(6) \qquad ||\hat{f}_n - \hat{g}_n\hat{\psi}||_E \to 0.$$

For $0 < r < 1$, let

$$(7) \qquad F = \{\zeta \in E; |\hat{\psi}(\zeta)| \le r\}.$$

By our assumption on ψ, F is a compact subset of E_0. By (5) and the fact that $f_n \to 0$ weakly on X, $g_n \to 0$ weakly on X. Now Lemma 3.1 implies that $\hat{g}_n \to 0$ weakly on E. Since E_0 satisfies $(*)$ by Lemma 3.2, both $\hat{f}_n$ and $\hat{g}_n$ converge to 0 uniformly on F. Hence,

$$(8) \qquad ||\hat{\bar{\psi}}\hat{f}_n - \hat{g}_n||_F \to 0.$$

Now by (4) and (7), we have

$$\begin{aligned}
||\hat{\bar{\psi}}\hat{f}_n + \hat{\mathcal{A}}||_E &\le ||\hat{\bar{\psi}}\hat{f}_n - \hat{g}_n||_E \\
&\le \left\| \hat{\bar{\psi}}\left(1 - \frac{1}{|\hat{\psi}|^2}\right)\hat{f}_n + \frac{\hat{f}_n - \hat{g}_n\hat{\psi}}{\hat{\psi}} \right\|_{E\setminus F} \\
&\quad + ||\hat{\bar{\psi}}\hat{f}_n - \hat{g}_n||_F \\
&\le K\left(\frac{1}{r^2} - 1\right) + \frac{||\hat{f}_n - \hat{g}_n\hat{\psi}||_{E\setminus F}}{r} + ||\hat{\bar{\psi}}\hat{f}_n - \hat{g}_n||_F.
\end{aligned}$$

Hence, by (6) and (8),

$$\limsup_{n\to\infty} ||\hat{\bar{\psi}}\hat{f}_n + \hat{\mathcal{A}}||_E \le K(1/r^2 - 1)$$

for every $0 < r < 1$. Therefore, $||\hat{\bar{\psi}}\hat{f}_n + \hat{\mathcal{A}}||_E \to 0$ and $\hat{\bar{\psi}} \in (\hat{\mathcal{A}}|_E)_b$. □

Proposition 3.3. *Let $\mathcal{A}$ be a closed subalgebra of L^∞, $\mathcal{B} = [H^\infty \cup \mathcal{A}]$, and $X \subset E \subset M(B)$. Suppose that both f and g belong to $(\hat{\mathcal{A}}|_E)_b$. If $f = g$ on X, then $f = g$ on $E\setminus E_0$.*

Proof. To prove our assertion, suppose not. Then there is a point x in $E\backslash E_0$ such that $|f(x)-g(x)|>\delta>0$. Take an open subset V of E such that $x\in V$ and

$$|f(x)-g(x)|>\delta \quad \text{on } V.$$

By Lemma 3.2, $E_0\cup V$ does not satisfy $(*)$. Hence, there is a sequence $\{f_n\}_n$ in $\mathcal{A}$ with $f_n\to 0$ weakly, and there is a compact subset K of $E_0\cup V$ such that $||\hat{f}_n||_K$ does not converge to 0. Since E_0 satisfies $(*)$, we may assume that $K\subset V$. Since $f,g\in(\hat{\mathcal{A}}|_E)_b$, there is a sequence $\{g_n\}_n$ in $\mathcal{A}$ such that

$$||(f-g)\hat{f}_n-\hat{g}_n||_E\to 0.$$

Since $f-g=0$ on X, $||\hat{g}_n||_E=||g_n||_X\to 0$. Now we have

$$\begin{aligned}||(f-g)\hat{f}_n-\hat{g}_n||_E &\geq ||(f-g)\hat{f}_n-\hat{g}_n||_K\\ &\geq ||(f-g)\hat{f}_n||_K-||\hat{g}_n||_K\\ &> \delta||\hat{f}_n||_K-||\hat{g}_n||_E.\end{aligned}$$

Hence, $||(f-g)\hat{f}_n-\hat{g}_n||_E$ does not converge to 0. This is a contradiction. □

Lemma 3.3. *Let $\mathcal{A}$ be a closed subalgebra of L^∞, $\mathcal{B}=[H^\infty\cup\mathcal{A}]$, and $X\subset E\subset M(B)$. If $f\in(\hat{\mathcal{A}}|_E)_b$, then there exists $g\in\mathcal{A}_b$ such that $f=g$ on X.*

Proof. Let $\{f_n\}_n\subset\mathcal{A}$ with $f_n\to 0$ weakly on X. By Lemma 3.1, $\hat{f}_n\to 0$ weakly on E. Since $f\in(\hat{\mathcal{A}}|_E)_b$ and $X\subset E$, we have

$$||ff_n-\mathcal{A}||_X\leq||f\hat{f}_n-\hat{\mathcal{A}}||_E\to 0.$$

Hence there exists g in $\mathcal{A}_b$ such that $f=g$ on X. □

We use Proposition 3.3 in the following way.

Corollary 3.1. *Let $\mathcal{A}$ be a closed subalgebra of L^∞, $\mathcal{B}=[H^\infty\cup\mathcal{A}]$, and $X\subset E\subset M(B)$. Suppose that $(\mathcal{A}_b)^\wedge|_E\subset(\hat{\mathcal{A}}|_E)_b$. If $f\in(\hat{\mathcal{A}}|_E)_b$, then there exists g in $\mathcal{A}_b$ such that $f=\hat{g}$ on $E\backslash E_0$.*

Proof. By Lemma 3.3, for $f \in (\hat{\mathcal{A}}|_E)_b$ there exists $g \in \mathcal{A}_b$ such that $f = g$ on X. By our assumption, we have $\hat{g}|_E \in (\hat{\mathcal{A}}|_E)_b$. Then Proposition 3.3 implies $f = \hat{g}$ on $E \backslash E_0$.

Proposition 3.4. *Let $\mathcal{A}$ be a closed subalgebra of L^∞, $\mathcal{B} = [H^\infty \cup \mathcal{A}]$ and $X \subset E \subset M(B)$. Let E_1 be a closed subset with $E \subset E_1 \subset M(H^\infty)$. Then*

$$(\{f \in C(E_1); f|_E \in \hat{\mathcal{A}}|_E\})_b = \{f \in C(E_1); f|_E \in (\hat{\mathcal{A}}|_E)_b\}.$$

Proof. Let $g \in C(E_1)$ where $g|_E \in (\hat{\mathcal{A}}|_E)_b$, and $\{f_n\}_n \subset \{f \in C(E_1); f|_E \in \hat{\mathcal{A}}|_E\}$ with $f_n \to 0$ weakly on E_1. By Lemma 2.2 we can write $f_n = \hat{h}_n + c_n$ where $h_n \in \mathcal{A}$ and $c_n \in C(E_1)$ with $c_n = 0$ on E. Since $X \subset E$, $h_n \to 0$ weakly on X. Hence, by Lemma 3.1, $h_n \to 0$ weakly in $\mathcal{A}$. Then, by Lemma 2.2,

$$\begin{aligned} ||gf_n + \{f \in C(E_1); f|_E \in \hat{\mathcal{A}}|_E\}||_{E_1} &= ||gf_n + \hat{\mathcal{A}}||_E \\ &= ||g\hat{h}_n + \hat{\mathcal{A}}||_E. \end{aligned}$$

Since $g|_E \in (\hat{A}|_E)_b$, we have $||g\hat{h}_n + \hat{\mathcal{A}}||_E \to 0$. Hence $g \in (\{f \in C(E_1); f|_E \in \hat{\mathcal{A}}|_E)_b$. The converse inclusion also follows from the above equations by considering the case $f_n = \hat{h}_n$. □

4. Bourgain algebras and Douglas algebras. As stated in the introduction, Ghatage, Sun and Zheng proved that $(H^\infty)_b = (H^\infty)_{bb} = H^\infty + C(\bar{\Delta})$ relative to $C(M(H^\infty))$. In this section we give a generalization of their theorem to Douglas algebras. For Douglas algebras B and D with $D \subset B$, we have $M(B) \subset M(D)$. By Lemma 2.3, $\hat{D}|_{M(B)}$ is a closed subalgebra of $C(M(B))$. First we determine the Bourgain algebra $(\hat{D}|_{M(B)})_b$ relative to $C(M(B))$. Recall that D_b denotes the Bourgain algebra of D relative to L^∞.

Theorem 4.1. *Let B and D be Douglas algebras with $D \subset B$. Then*

$$(\hat{D}|_{M(B)})_b = \{f \in C(M(B)); f|_{M(B_b)} \in (D_b)^\wedge|_{M(B_b)}\}.$$

To prove our theorem, we need some lemmas. The following is the main result in [**10**].

Lemma 4.1. *Let D be a Douglas algebra.*

(i) *If ψ is an interpolating Blaschke product, then $\bar{\psi} \in D_b$ if and only if $Z(\hat{\psi}) \cap M(D)$ is a finite set.*

(ii) $D_{bb} = D_b$.

(iii) *If B is a Douglas algebra with $D \subset B$, then $D_b \subset B_b$.*

If the Hoffman map L_x for $x \in M(H^\infty)$ is a homeomorphism, the part $P(x)$ is called a homeomorphic part.

Lemma 4.2. *Let D be a Douglas algebra, and let ψ be an interpolating Blaschke product. If $Z(\hat{\psi}) \cap M(D)$ is a one-point set $\{x\}$, then*

(i) *$P(x)$ is a homeomorphic part*;

(ii) *$|\hat{\psi}| = 1$ on $M(D) \backslash P(x)$.*

Proof. (i) is proved in [**11**, Theorem 1.4].

(ii) Let $y \in M(D)$ with $[\hat{\psi}(y)| < 1$. To show $y \in P(x)$, we compute the pseudohyperbolic distance from x to y. Let $f \in H^\infty$ with $||f||_X = 1$ and $\hat{f}(x) = 0$. By [**1, 14**], there is a function h in D such that $f = \psi h$. Since $||h||_X = 1$, $|\hat{f}(y)| = |\hat{\psi}(y)|\,|\hat{h}(y)| \leq |\hat{\psi}(y)|$. Thus, $\rho(x, y) \leq |\hat{\psi}(y)| < 1$, so $y \in P(x)$. □

Lemma 4.3. *Let D be a Douglas algebra such that $M(D) \neq M(D_b)$. Then there exists a set of homeomorphic parts $\{P(x_\alpha)\}_\alpha$ such that $P(x_\alpha)$ is an open subset of $M(D)$ and $M(D)\backslash M(D_b) = \cup_\alpha P(x_\alpha)$.*

Proof. Let $x \in M(D)\backslash M(D_b)$. The Chang-Marshall theorem implies that there is an interpolating Blaschke product ψ with $\bar{\psi} \in D_b$ such that $|\hat{\psi}(x)| < 1$. Since $\bar{\psi} \in D_b$, Lemma 4.1 implies that $Z(\hat{\psi}) \cap M(D)$ is a finite set. By factoring, we get $\psi = \psi_1 \cdots \psi_n$, where each ψ_j is a subfactor and $Z(\hat{\psi}_j) \cap M(D)$ is a one-point set, say $\{x_j\}$. Since $|\hat{\psi}(x)| < 1$, there exists j such that $|\hat{\psi}_j(x)| < 1$. Then Lemma 4.2 says that $P(x) = P(x_j)$, $P(x)$ is a homeomorphic part and is an open subset of $M(D)$. Since $|\hat{\psi}_j| < 1$ on $P(x)$, $P(x) \subset M(D) \backslash M(D_b)$. Since this is true for any x in $M(D) \backslash M(D_b)$, we get our assertion. □

The following lemma shows that $M(D) \backslash M(D_b)$ is an open subset of $M(D)$ which satisfies condition $(*)$ for $\mathcal{A} = D$ in Section 3.

Lemma 4.4. *Let D be a Douglas algebra, and let K be a compact subset of $M(D) \backslash M(D_b)$. If $\{\hat{f}_n\}_n$ is a sequence in $\hat{D}$ with $\hat{f}_n \to 0$ weakly on $M(D)$, then $\hat{f}_n \to 0$ uniformly on K.*

Proof. By our assumption, $\{\hat{f}_n\}_n$ is sup-norm bounded on $M(D)$. Since K is compact, by Lemma 4.3 there exist disjoint homeomorphic parts $P(x_1), \ldots, P(x_k)$ in $M(D)$ such that

$$K \subset \bigcup_{j=1}^{k} P(x_j).$$

Since $P(x_j)$ is open, $K \cap P(x_j)$ is a compact subset of $P(x_j)$. Now $\hat{f}_n \circ L_{x_j} \in H^\infty$. Since $\{\hat{f}_n \circ L_{x_j}\}_n$ is sup-norm bounded and converges to 0 pointwise on Δ, $\hat{f}_n \circ L_{x_j} \to 0$, $n \to \infty$, uniformly on each compact subset of Δ. Since L_{x_j} is a homeomorphism, $\hat{f}_n \to 0$ on each compact subset of $P(x_j)$, so $\hat{f}_n \to 0$ uniformly on $K \cap P(x_j)$ for $j = 1, \ldots, k$. Hence, $\hat{f}_n \to 0$ uniformly on K. □

One can show using P. Beurling functions (in exactly the same way as in the proof of [**10**, Theorem 2]) that the following is true.

Lemma 4.5. *Let ψ be a noncontinuous interpolating Blaschke product, and let $\{x_n\}_n$ be a sequence in $Z(\hat{\psi}) \cap M(H^\infty + C)$ such that, for every n, $x_n \notin \mathrm{cl}\,\{x_k; k \neq n\}$. Then there is a sequence $\{f_n\}_n$ in H^∞ such that $\hat{f}_n(x_n) = 1$ and $f_n \to 0$ weakly in H^∞.*

Now we prove our theorem.

Proof of Theorem 4.1. To use the results in Section 3, we consider the case where $\mathcal{A} = \mathcal{B} = D$ and $E = M(B)$. Since $D \subset B$, $M(B) \subset M(D)$. First, we show that the largest open subset of $M(B)$ which satisfies $(*)$ with $\mathcal{A} = D$ is $M(B)\backslash M(B_b)$ (in the notation of Lemma 3.2, $E_0 = M(B)\backslash M(B_b)$). By Lemma 4.4, $M(B)\backslash M(B_b)$ satisfies condition $(*)$ for $\mathcal{A} = B$. Since $D \subset B$, the open subset $M(B)\backslash M(B_b)$ of $M(B)$ satisfies condition $(*)$ for D. To see that $M(B)\backslash M(B_b)$ is the largest such subset, let V be a nonvoid open subset of $M(B)$ such that V is not contained in $M(B)\backslash M(B_b)$. Since $M(B_b) \cap V \neq \varnothing$, by [**12**, Corollary 3.2] there is a nontrivial point x in $M(B_b) \cap V$. By [**16**] there exists an interpolating Blaschke product ψ such that $\hat{\psi}(x) = 0$. If x were isolated in $Z(\hat{\psi}) \cap M(B_b)$, we could factor $\psi = \psi_1\psi_2$ so that $Z(\hat{\psi}_1) \cap M(B_b) = \{x\}$. By Lemma 4.1, we have $\bar{\psi}_1 \in B_{bb} = B_b$. But $x \in Z(\hat{\psi}_1) \cap M(B_b)$ implies that ψ_1 is not invertible in B_b, a contradiction. Thus, x is not isolated and we may choose a sequence of distinct points $\{x_n\}_n$ in $Z(\hat{\psi}) \cap M(B_b) \cap V$ such that $\mathrm{cl}\,\{x_n\}_n \subset V$ and $x_n \notin \mathrm{cl}\,\{x_k; k \neq n\}$ for every n. By Lemma 4.5, there is a sequence $\{h_n\}_n$ in H^∞ such that $h_n \to 0$ weakly in H^∞ and $\hat{h}_n(x_n) = 1$ for every n. Then $||\hat{h}_n||_{\mathrm{cl}\,\{x_n\}_n} = 1$, so that V does not satisfy condition $(*)$. Thus $M(B)\backslash M(B_b)$ is the largest open subset of $M(B)$ which satisfies $(*)$ for D.

Now we can use the results in Section 3. By Proposition 3.1,

$$\{f \in C(M(B)); f = 0 \text{ on } M(B_b)\} \subset (\hat{D}|_{M(B)})_b. \tag{9}$$

By Theorem 2.2,

$$[(D_b)^\wedge]_{M(B)} = [H^\infty, \hat{\bar{\psi}}; \bar{\psi} \in D_b, \psi \text{ is an interpolating Blaschke product}]_{M(B)}.$$

By Lemma 4.1, $D_b \subset B_b$, so $\bar{\psi} \in D_b$ implies that $|\hat{\psi}| = 1$ on $M(B_b)$. Then, by Proposition 3.2, $\hat{\bar{\psi}} \in (\hat{D}|_{M(B)})_b$. Hence,

$$(D_b)^\wedge|_{M(B)} \subset [(D_b)^\wedge]_{M(B)} \subset (\hat{D}|_{M(B)})_b. \tag{10}$$

Combining (9), (10) and Lemma 2.2,

$$\begin{aligned}\{f \in C(M(B)); f|_{M(B_b)} &\in (D_b)^\wedge|_{M(B_b)}\} \\ &= (D_b)^\wedge|_{M(B)} + \{f \in C(M(B)); f = 0 \text{ on } M(B_b)\} \\ &\subset (\hat{D}|_{M(B)})_b.\end{aligned}$$

Next we show that

$$(11)\qquad (\hat{D}|_{M(B)})_b \subset \{f \in C(M(B)); f|_{M(B_b)} \in (D_b)^\wedge|_{M(B_b)}\}.$$

By (10), assumptions of Corollary 3.1 are satisfied (note that $E \backslash E_0 = M(B_b)$). Hence (11) follows directly from Corollary 3.1. This completes the proof. □

Using Theorem 4.1, we can prove a more general theorem.

Theorem 4.2. *Let B and D be Douglas algebras. Then*

$$\begin{aligned}([\hat{D}]_{M(B)})_b &= ([\hat{D}]_{M(B)})_{bb} \\ &= \{f \in C(M(B)); f|_{M([B\cup D]_b)} \in (D_b)^\wedge|_{M([B\cup D]_b)}\}.\end{aligned}$$

Proof. By Theorem 2.2,

$$[\hat{D}]_{M(B)} = \{f \in C(M(B)); f|_{M([B\cup D])} \in \hat{D}|_{M([B\cup D])}\}.$$

By Proposition 3.4,

$$(12)\qquad ([\hat{D}]_{M(B)})_b = \{f \in C(M(B)); f|_{M([B\cup D])} \in (\hat{D}|_{M([B\cup D])})_b\}.$$

By Theorem 4.1,

$$(13)\qquad \begin{aligned}(\hat{D}|_{M([B\cup D])})_b = \{&f \in C(M([B \cup D])); \\ &f|_{M([B\cup D]_b)} \in (D_b)^\wedge|_{M([B\cup D]_b)}\}.\end{aligned}$$

Combining (12) and (13), we have

$$(14)\qquad ([\hat{D}]_{M(B)})_b = \{f \in C(M(B)); f|_{M([B\cup D]_b)} \in (D_b)^\wedge|_{M([B\cup D]_b)}\}.$$

Here we note that $(D_b)^\wedge|_{M([B\cup D]_b)}$ is a closed algebra, for $D_b \subset [B\cup D]_b$ by Lemma 4.1. We again apply Proposition 3.4 to obtain

$$(15)\qquad ([\hat{D}]_{M(B)})_{bb} = \{f \in C(M(B)); f|_{M([B\cup D]_b)} \in ((D_b)^\wedge|_{M([B\cup D]_b)})_b\}.$$

By Theorem 4.1 and Lemma 4.1,

$$\begin{aligned}((D_b)^\wedge|_{M([B\cup D]_b)})_b &= \{f\in C(M([B\cup D]_b));\\ &\qquad f|_{M([B\cup D]_{bb})}\in (D_{bb})^\wedge|_{M([B\cup D]_{bb})}\}\\ &= \{f\in C(M([B\cup D]_b)); f|_{M([B\cup D]_b)}\\ &\qquad \in (D_b)^\wedge|_{M([B\cup D]_b)}\}.\end{aligned}$$

Hence, by (14) and (15), we get

$$\begin{aligned}([\hat{D}]_{M(B)})_{bb} &= \{f\in C(M(B)); f|_{M([B\cup D]_b)}\in (D_b)^\wedge|_{M([B\cup D]_b)}\}\\ &= ([\hat{D}]_{M(B)})_b.\end{aligned}$$

This completes the proof. □

Corollary 4.1 [**9**]. *$(H^\infty)_b = (H^\infty)_{bb} = H^\infty + C(\bar{\Delta})$ relative to $C(M(H^\infty))$.*

Proof. Using Theorem 4.2 with $B = D = H^\infty$ and Lemma 2.2(i), we have

$$\begin{aligned}(H^\infty)_b &= (H^\infty)_{bb}\\ &= \{f\in C(M(H^\infty)); f|_{M(H^\infty+C)}\in (H^\infty+C)^\wedge|_{M(H^\infty+C)}\}\\ &= H^\infty + C(\bar{\Delta}),\end{aligned}$$

because $(H^\infty)_b = H^\infty + C$. □

Corollary 4.2. *Let B and D be Douglas algebras with $B\subset D$. Then $([\hat{D}]_{M(B)})_b = [(D_b)^\wedge]_{M(B)}$.*

Proof. Since $B\subset D$, $[B\cup D]_b = D_b$. By Theorems 2.2 and 4.2,

$$\begin{aligned}([\hat{D}]_{M(B)})_b &= \{f\in C(M(B)); f|_{M([B\cup D]_b)}\in (D_b)^\wedge|_{M([B\cup D]_b)}\}\\ &= \{f\in C(M(B)); f|_{M(D_b)}\in (D_b)^\wedge|_{M(D_b)}\};\\ [(D_b)^\wedge]_{M(B)} &= \{f\in C(M(B)); f|_{M([B\cup D_b])}\in (D_b)^\wedge|_{M([B\cup D_b])}\}.\end{aligned}$$

Since $B\subset D_b$, we have $D_b = [B\cup D_b]$, hence $([\hat{D}]_{M(B)})_b = [(D_b)^\wedge]_{M(B)}$. □

Remark 4.1. The conclusion of Corollary 4.2 is not true for an arbitrary pair of Douglas algebras B and D. We give an example of a pair of B and D. Let $P(x)$ be a homeomorphic part such that $P(x) \neq \Delta$. Let

$$B = \{f \in L^\infty; f|_{\operatorname{supp}\mu_x} \in H^\infty|_{\operatorname{supp}\mu_x}\} \text{ and } D = H^\infty + C,$$

where $\operatorname{supp}\mu_x$ is the closed support set of x. Then both B and D are Douglas algebras and $D \subset B$. It can be shown that $P(x) \subset M(B)$ and $M(B_b) = M(B)\backslash P(x)$. Since $D_b = H^\infty + C$ and $[B \cup D]_b = B_b$, by Theorems 2.2 and 4.2,

$$([\hat{D}]_{M(B)})_b = \{f \in C(M(B)); f|_{M(B_b)} \in (H^\infty + C)^\wedge|_{M(B_b)}\};$$
$$[(D_b)^\wedge]_{M(B)} = \{f \in C(M(B)); f|_{M(B)} \in (H^\infty + C)^\wedge|_{M(B)}\}.$$

Hence $\{f \in C(M(B)); f = 0 \text{ on } M(B_b)\}$ is contained in $([\hat{D}]_{M(B)})_b$ but is not contained in $[(D_b)^\wedge]_{M(B)}$.

Remark 4.2. We do not know whether $[B \cup D]_b = [B_b \cup D_b]$ for Douglas algebras B and D.

Remark 4.3. We can prove that $M(B)\backslash M([B \cup D]_b)$ is the largest open subset of $M(B)$ which satisfies condition $(*)$ for $[\hat{D}]_{M(B)}$.

5. Bourgain algebras of the disk algebra. The space of functions continuous on $\bar{\Delta}$ and analytic in Δ is called the disk algebra and is denoted by A. Let $\mathcal{A}$ be a closed subalgebra of H^∞ containing A. Recent results [**17**] imply that $\mathcal{A}_b \subset H^\infty + C$, relative to L^∞. Since $\mathcal{A}_b \cap C$ is a closed algebra between A and C, $\mathcal{A}_b \cap C = A$ or $\mathcal{A}_b \cap C = C$, see [**15**, p. 93]. For $f \in \mathcal{A}$, define f^* by $f^*(z) = (f(z) - f(0))/z$. If $f \in \mathcal{A}$ implies $f^* \in \mathcal{A}$, then $\mathcal{A}$ is said to be stable. If $\mathcal{A}$ is stable, then $f_n \to 0$ weakly in $\mathcal{A}$ implies

$$||\bar{z}f_n + \mathcal{A}||_X \leq ||\bar{z}f_n - f_n^*||_X = |f_n(0)| \to 0.$$

Thus, $\bar{z} \in \mathcal{A}_b$ and $\mathcal{A}_b \cap C = C$. We do not know whether or not $\mathcal{A}_b \cap C = C$ for every closed subalgebra $\mathcal{A}$ of H^∞ containing A. Let $C_0(\bar{\Delta}) = \{f \in C(\bar{\Delta}); f = 0 \text{ on } T\}$. Then $C_0(\bar{\Delta}) = \{f \in$

$C(M(H^\infty)); f = 0$ on $M(H^\infty + C)\}$. We first study the Bourgain algebra $(\hat{\mathcal{A}})_b$ relative to $C(M(H^\infty))$.

Theorem 5.1. *Let $\mathcal{A}$ be a closed subalgebra such that $A \subset \mathcal{A} \subset H^\infty$ and $\mathcal{A}_b \cap C = C$. Then $(\hat{\mathcal{A}})_b = (\mathcal{A}_b)^\wedge + C(\bar{\Delta})$.*

We will need to prove the lemma below first.

Lemma 5.1. *Let $\{z_n\}_n$ be a sequence in Δ with $|z_n| \to 1$. Then there exists a subsequence $\{z_{nj}\}_j$ of $\{z_n\}_n$ and a sequence $\{h_j\}_j$ in A such that $h_j \to 0$ weakly and $|h_j(z_{n_j})| \geq 1$.*

Proof. By considering a subsequence of $\{z_n\}_n$, we may assume that $z_n \to 1$. Let $f(z) = (z+1)/2$. Then $f \in A$, $f(1) = 1$ and $|f| < 1$ on $T\backslash\{1\}$. Let $\{s_j\}_j$ be a sequence of positive integers such that $s_j \to \infty$. Since $f^{s_j}(1) = 1$, we can choose $z_{n_j} \in \{z_n\}_n$ such that $|f^{s_j}(z_{n_j})| > 1/2$. Furthermore, the fact that $|f(z_{n_j})| < 1$ implies that we may choose a positive integer t_j such that

$$|f^{s_j}(z_{n_j})(1 - f^{t_j}(z_{n_j}))| \geq \frac{1}{2}.$$

Put $h_j - 2f^{s_j}(1 - f^{t_j})$. Then $h_j \in A$, $h_j(1) = 0$ and $||h_j||_T \leq 4$. Since $s_j \to \infty$, $h_j \to 0$ weakly and $|h_j(z_{n_j})| \geq 1$. □

We now return to the proof of Theorem 5.1.

Proof of Theorem 5.1. To use the results in Section 3, we take $E = M(H^\infty)$. We show that Δ is the largest open subset of $M(H^\infty)$ which satisfies $(*)$ for $\mathcal{A}$, that is, $E_0 = \Delta$. Since $\mathcal{A} \subset H^\infty$, Δ satisfies $(*)$. Let V be a nonvoid open subset of $M(H^\infty)$ which is not contained in Δ. Since Δ is dense in $M(H^\infty)$ there is a sequence $\{z_n\}_n$ in $\Delta \cap V$ such that $|z_n| \to 1$ and $\operatorname{cl}\{z_n\}_n \subset V$. Lemma 5.1 implies that V does not satisfy condition $(*)$. Hence $E_0 = \Delta$.

First we show that $(\mathcal{A}_b)^\wedge + C(\bar{\Delta}) \subset (\hat{\mathcal{A}})_b$. Since $\bar{z} \in \mathcal{A}_b$ and $|\hat{z}| = 1$ on $M(H^\infty)\backslash\Delta$, Proposition 3.2 implies that $\hat{\bar{z}} \in (\hat{\mathcal{A}})_b$. Since $\hat{z} \in \hat{\mathcal{A}} \subset (\hat{\mathcal{A}})_b$, we have $C(\bar{\Delta}) \subset (\hat{\mathcal{A}})_b$. Now by [**17**], $\mathcal{A}_b \subset H^\infty + C$, and by assumption

$\mathcal{A}_b \cap C = C$. Thus, if $f \in \mathcal{A}_b$, then $f = h + c$ with $h \in H^\infty$ and $c \in C$. Since $C \subset \mathcal{A}$, $h \in \mathcal{A}_b \cap H^\infty$ and $f \in (\mathcal{A}_b \cap H^\infty) + C$. Now $\hat{c} \in C(\bar{\Delta})$ so

$$\begin{aligned}(\mathcal{A}_b)^\wedge + C(\bar{\Delta}) &\subset (\mathcal{A}_b \cap H^\infty)^\wedge + \hat{C} + C(\bar{\Delta}) \\ &= (\mathcal{A}_b \cap H^\infty)^\wedge + C(\bar{\Delta}).\end{aligned}$$

Now let $g \in \mathcal{A}_b \cap H^\infty$ and $\{f_n\}_n$ in $\mathcal{A}$ with $f_n \to 0$ weakly. Then

$$||\hat{g}\hat{f}_n + \hat{\mathcal{A}}||_{M(H^\infty)} = ||gf_n + \mathcal{A}||_X \to 0.$$

Hence, by Lemma 3.1, $\hat{g} \in (\hat{\mathcal{A}})_b$, and hence

$$(\mathcal{A}_b)^\wedge + C(\bar{\Delta}) \subset (\mathcal{A}_b \cap H^\infty)^\wedge + C(\bar{\Delta}) \subset (\hat{\mathcal{A}})_b.$$

To prove the converse inclusion, let $h \in (\hat{\mathcal{A}}_b)$. By Corollary 3.1, $h|_{M(H^\infty+C)} \in (\mathcal{A}_b)^\wedge|_{M(H^\infty+C)}$, for $M(H^\infty+C) = M(H^\infty)\backslash\Delta$. Hence, $h \in (\mathcal{A}_b)^\wedge + C_0(\Delta) \subset (\mathcal{A}_b)^\wedge + C(\bar{\Delta})$. This completes the proof. □

Hereafter we study Bourgain algebras of the disk algebra A. Let B be a Douglas algebra with $H^\infty + C \subset B$. For $\lambda \in T$, let $M_\lambda(B) = \{x \in M(B); \hat{z}(x) = \lambda\}$. For a function f in $C(M(B))$, let

$$\omega_B(f, \lambda) = \sup\{|f(x) - f(y)|; x, y \in M_\lambda(B)\}.$$

Let

$$V_B = \{f \in C(M(B)); \{\lambda \in T; w_B(f,\lambda) > \delta\} \text{ is a finite set for every } \delta > 0\}.$$

We note that V_B is a closed subalgebra of $C(M(B))$. There are several ways to prove this fact. One way is as follows: Put $\mathcal{C} = \hat{\mathcal{C}}|_{M(B)}$. Then $\mathcal{C}$ is a C^*-subalgebra of $C(M(B))$. By [**17**, Proposition 1], $\mathcal{C}_b = V_B$, hence V_B is a closed subalgebra of $C(M(B))$. Since $f \in V_B$ implies $\bar{f} \in V_B$, V_B is a C^*-subalgebra of $C(M(B))$.

In [**17**, Theorems 5 and 6], the Bourgain algebra of the disk algebra relative to $L^\infty = C(X)$ was shown to have the form

$$(\alpha) \qquad A_b = (H^\infty \cap V_{L^\infty}) + C \subset H^\infty + C$$

and $A_{bb} = A_b$. Hence both $\hat{A}|_{M(B)}$ and $(A_b)^\wedge|_{M(B)}$ are closed subalgebras of $C(M(B))$. We study the Bourgain algebra of $\hat{A}|_{M(B)}$ relative to $C(M(B))$. It is easy to see that $(V_{L^\infty})^\wedge|_{M(B)} \subset V_B$. Hence, by (α), we have $(A_b)^\wedge|_{M(B)} \subset V_B$. Since $(A_b)^\wedge|_X = A_b|_X$ is a closed subalgebra of $C(X)$, Lemma 2.2 (consider $\mathcal{A} = (A_b)^\wedge|_{M(B)}$, $Y = M(B)$ and $E = X$) implies that $(A_b)^\wedge|_{M(B)} + \{f \in V_B; f = 0 \text{ on } X\}$ is a closed subalgebra of $C(M(B))$, and it coincides with $\{f \in V_B; f|_X \in A_b\}$. Now we have the following theorem.

Theorem 5.2. *Let B be a Douglas algebra with $H^\infty + C \subset B$. Then $(\hat{A}|_{M(B)})_b = (A_b)^\wedge|_{M(B)} + \{f \in V_B; f = 0$ on $X\}$.*

Proof. First we prove that

$$(A_b)^\wedge|_{M(B)} + \{f \in V_B; f = 0 \text{ on } X\} \subset (\hat{A}|_{M(B)})_b.$$

Let $g \in (A_b)^\wedge|_{M(B)}$. Let $f \in A_b$ so that $g = \hat{f}$ on $M(B)$. Let $\{f_n\}_n \subset A$ such that $f_n \to 0$ weakly. Since $f \in A_b \subset H^\infty + C \subset B$, we have $||\hat{f}\hat{f}_n + \hat{A}||_{M(B)} = ||ff_n + A||_X \to 0$. Therefore, $g = \hat{f}|_{M(B)} \in (\hat{A}|_{M(B)})_b$.

Next let $f \in V_B$ with $f = 0$ on X. Since $\{\lambda \in T; \omega_B(f, \lambda) \neq 0\}$ is a countable set, we denote it by $\{\lambda_j\}_j$. Of course, $\omega_B(f, \lambda_j) \to 0$, $j \to \infty$. Take $\varepsilon > 0$ arbitrary. Then there exists j_0 such that $\omega_B(f, \lambda_j) < \varepsilon$ for $j > j_0$. Since $f = 0$ on X, by the definition of ω_B we have that $|f| < \varepsilon$ on $M_\lambda(B)$ for every $\lambda \notin \{\lambda_1, \dots, \lambda_{j_0}\}$. Since f_n is continuous on T, $\hat{f}_n$ is constant on $M_\lambda(B)$ and $\hat{f}_n|_{M_\lambda(B)} \to 0$, $n \to \infty$, for each $\lambda \in T$. Hence,

$$||f\hat{f}_n||_{M(B)} \to 0, \qquad n \to \infty.$$

Thus, we get $f \in (\hat{A}|_{M(B)})_b$.

To prove the converse inclusion, let $g \in (\hat{A}|_{M(B)})_b$. Then, by Lemma 3.3, there exists h in A_b such that $h = g$ on X. It is sufficient to prove $g - \hat{h} \in V_B$. Assume to the contrary that $g - \hat{h} \notin V_B$. Then there is a distinct sequence $\{\zeta_n\}_n$ in T and $\delta > 0$ such that $\omega_B(g - \hat{h}, \zeta_n) > \delta$ for every n. We may assume that $\zeta_n \to \zeta_0$ for some $\zeta_0 \in T$. Put $h_n(z) = ((z + \zeta_n)/2)^{k_n}$. Then $h_n \in A$ and $|h_n(\zeta_n)| = 1$. If we take $k_n \to \infty$ sufficiently fast, it is not difficult to see that $h_n \to 0$ weakly in A. By our work in the first paragraph, we know $(A_b)^\wedge|_{M(B)} \subset (\hat{A}|_{M(B)})_b$ and hence $\hat{h}|_{M(B)} \in (\hat{A}|_{M(B)})_b$ and $g - \hat{h} \in (\hat{A}|_{M(B)})_b$. Thus,

$$\begin{aligned}\delta < \omega_B(g - \hat{h}, \zeta_n) &\leq 2||(g - \hat{h})\hat{h}_n + \hat{A}||_{M_{\zeta_n}(B)} \\ &\leq 2||(g - \hat{h})\hat{h}_n + \hat{A}||_{M(B)} \\ &\to 0, \qquad n \to \infty.\end{aligned}$$

This contradiction shows that $g - \hat{h} \in V_B$. This completes the proof. □

Remark 5.1. We can prove that the largest open subset of $M(B)$ which satisfies condition $(*)$ for $\hat{A}|_{M(B)}$ is the union set of $\{x \in M(B); |f(x)| > 0\}$ for $f \in V_B$ with $f = 0$ on X.

Corollary 5.1. $(\hat{A})_b|_{M(H^\infty+C)} \neq (\hat{A}|_{M(H^\infty+C)})_b$.

Proof. By Theorem 5.1, we have

$$(\hat{A})_b|_{M(H^\infty+C)} = (A_b)^\wedge|_{M(H^\infty+C)} \subset (H^\infty + C)^\wedge|_{M(H^\infty+C)}.$$

Let ψ be an infinite Blaschke product, continuous everywhere except $\lambda = 1$. Then $\hat{\psi} \in V_{(H^\infty+C)}$. Thus, $1 - |\hat{\psi}| \in V_{(H^\infty+C)}$ and $1 - |\hat{\psi}| = 0$ on X. By Theorem 5.2, $1 - |\hat{\psi}| \in (\hat{A}|_{M(H^\infty+C)})_b$. Since $|\hat{\psi}| \notin (H^\infty + C)^\wedge$, we see that $1 - |\hat{\psi}| \notin (H^\infty + C)^\wedge|_{M(H^\infty+C)}$. Hence, $1 - |\hat{\psi}| \notin (\hat{A})_b|_{M(H^\infty+C)}$. □

Here we have the following problem.

Problem. Let B be a Douglas algebra with $B \neq L^\infty$. Is it true that $(\hat{A}|_{M(B)})_b \neq (A_b)^\wedge|_{M(B)}$?

As the last theorem of this paper, we prove the following.

Theorem 5.3. *For every Douglas algebra B,* $(\hat{A}|_{M(B)})_{bb} = (\hat{A}|_{M(B)})_b$.

To prove this, we need some lemmas. In [**17**, Theorem 6], the second author proved that $A_{bb} = A_b$ relative to L^∞. In proving this, the following fact was used (see the proof of [**17**, Theorem 7]).

Lemma 5.2. *Let ψ be a noncontinuous interpolating Blaschke product with $\psi \in V_{L^\infty}$. Then there is a sequence $\{\zeta_n\}_n$ in $Z(\hat{\psi}) \cap M(H^\infty + C)$ and a sequence $\{h_n\}$ in $H^\infty \cap V_{L\infty}$ such that $h_n \to 0$ weakly in $H^\infty \cap V_{L\infty}$ and $\hat{h}_n(\zeta_n) = 1$ for every n.*

In [**17**, Theorem 1], to prove $(H^\infty \cap V_{L\infty})_b \subset V_{L^\infty}$ we used the following fact.

Lemma 5.3. *Let $\{\lambda_j\}_j$ be a distinct sequence in T with $\lambda_j \to \lambda_0$. Then there exists a sequence $\{h_n\}_n$ in $H^\infty \cap V_{L^\infty}$ such that $h_n \to 0$ weakly, h_n is continuous on $T \backslash \{\lambda_0\}$, and $|h_n(\lambda_{n,i})| \geq 1$ for some subsequence $\{\lambda_{n,i}\}_i$ of $\{\lambda_j\}_j$.*

Proof of Theorem 5.3. It is sufficient to prove $(\hat{A}|_{M(B)})_{bb} \subset (\hat{A}|_{M(B)})_b$. We need to divide the proof into two cases, when $B = H^\infty$ and $B \supset H^\infty + C$.

Case 1. $B = H^\infty$. By Theorem 5.1 and Lemma 2.2, we have

$$(16)\qquad \begin{aligned}(\hat{A})_b &= (A_b)^\wedge + C(\bar{\Delta}) \\ &= \{f \in C(M(H^\infty)); f|_{M(H^\infty+C)} \in (A_b)^\wedge|_{M(H^\infty+C)}\}.\end{aligned}$$

To use the result in Section 3, we set $E = M(H^\infty + C)$ and $\mathcal{A} = A_b$. Since $A_b \subset H^\infty + C$, by Proposition 3.4 we have

$$(17)\qquad (\hat{A})_{bb} = \{f \in C(M(H^\infty)); f|_{M(H^\infty+C)} \in ((A_b)^\wedge|_{M(H^\infty+C)})_b\}.$$

We show that there are no nonvoid open subsets U of $M(H^\infty + C)$ which satisfy $(*)$ for A_b. Let U be an open subset of $M(H^\infty + C)$ with $U \neq \varnothing$. By the corona theorem and well-known results on interpolating sequences, there is an interpolating sequence $\{z_n\}_n$ in Δ such that $\mathrm{cl}\,\{z_n\}_n \backslash \{z_n\}_n \subset U$ and z_n converges to some point λ in T. Let ψ be the interpolating Blaschke product with zeros $\{z_n\}_n$. Then $\psi \in V_{L^\infty}$ and $Z(\hat{\psi}) \cap M(H^\infty + C) \subset U$. Hence, Lemma 5.2 implies that U does not satisfy $(*)$ for A_b.

Now we prove that

$$((A_b)^\wedge|_{M(H^\infty+C)})_b = (A_b)^\wedge|_{M(H^\infty+C)}. \tag{18}$$

Let $g \in ((A_b)^\wedge|_{M(H^\infty+C)})_b$. Then by Lemma 3.3, there exists h in $A_{bb} = A_b$ such that $g = h$ on X. Since both g and $\hat{h}$ are contained in $((A_b)^\wedge|_{M(H^\infty+C)})_b$, Proposition 3.3 implies that $g = \hat{h}$ on $M(H^\infty + C)$. Hence we get (18). By (16), (17) and (18), we have $(\hat{A})_{bb} = (\hat{A})_b$.

Case 2. We prove the theorem assuming that $B \supset H^\infty + C$. Let $f \in (\hat{A}|_{M(B)})_{bb}$, and let $\{f_n\}_n \subset A$ with $f_n \to 0$ weakly. Then $\hat{f}_n \to 0$ weakly in $(\hat{A}|_{M(B)})_b$. By Theorem 5.2,

$$(\hat{A}|_{M(B)})_b = (A_b)^\wedge|_{M(B)} + \{f \in V_B; f = 0 \text{ on } X\}, \tag{19}$$

so that $(\hat{A}|_{M(B)})_b|_X = A_b$. Thus,

$$||ff_n + A_b||_X \le ||f\hat{f}_n + (\hat{A}|_{M(B)})_b||_{M(B)} \to 0.$$

Hence, there is a g in $A_{bb} = A_b$ such that $g = f$ on X. By Theorem 5.2, to prove $f \in (\hat{A}|_{M(B)})_b$, it is sufficient to prove $\hat{g} - f \in V_B$. By (α), since $g \in A_b = (H^\infty \cap V_{L^\infty}) + C$, we have $\hat{g} \in V_B$. Therefore we need to prove $f \in V_B$. Here we use the same idea as the proof of [**17**, Theorem 1]. To prove the above fact, suppose not. Then there is a distinct sequence $\{\lambda_j\}_j$ in T and $\delta > 0$ such that

$$\omega_B(f, \lambda_j) > \delta \quad \text{for every } j. \tag{20}$$

Here we may assume that $\{\lambda_j\}_j$ is a convergent sequence in T. By Lemma 5.3, there is a sequence $\{h_n\}_n$ in $H^\infty \cap V_{L^\infty}$ such that

$$h_n \to 0 \quad \text{weakly on } X; \tag{21}$$

$$h_n \text{ is continuous at each point } \lambda_j, \qquad j = 1, 2, \dots; \tag{22}$$

(23)

$$\text{for each } n,\ |h_n(\lambda_{n,i})| \ge 1 \quad \text{for some subsequence } \{\lambda_{n,i}\}_i \text{ of } \{\lambda_j\}_j.$$

By (α), $H^\infty \cap V_{L^\infty} \subset A_b$ so that, by (19) and (21) $\hat{h}_n \to 0$ weakly in $(\hat{A}|_{M(B)})_b$. By (α) again, $(A_b)^\wedge \subset V_B$, so that by (19) $(\hat{A}|_{M(B)})_b \subset V_B$. Since $f \in (\hat{A}|_{M(B)})_{bb}$, we have

$$||f\hat{h}_n + V_B||_{M(B)} \le ||f\hat{h}_n + (\hat{A}|_{M(B)})_b||_{M(B)} \to 0, \qquad n \to \infty.$$

Take g_n in V_B such that

$$||f\hat{h}_n + g_n||_{M(B)} \to 0. \tag{24}$$

Since $g_n \in V_B$, $\omega_B(g_n, \lambda_{n,i}) \to 0$, $i \to \infty$, for each n. Then we have

$$\begin{aligned}
\delta &= \liminf_{i\to\infty} \delta - \omega_B(g_n, \lambda_{n,i}) \\
&\le \liminf_{i\to\infty} |h_n(\lambda_{n,i})|\omega_B(f, \lambda_{n,i}) \\
&\quad - \omega_B(g_n, \lambda_{n,i}) \qquad \text{by (20) and (23)} \\
&\le \liminf_{i\to\infty} \omega_B(f\hat{h}_n + g_n, \lambda_{n,i}) \qquad \text{by (22)} \\
&\le \liminf_{i\to\infty} 2||f\hat{h}_n + g_n||_{M_{\lambda_{n,i}}(B)} \\
&\le 2||f\hat{h}_n + g_n||_{M(B)} \\
&\to 0, \quad n \to \infty \qquad \text{by (24).}
\end{aligned}$$

This is the desired contradiction. Hence, $f \in V_B$. Consequently, we have $f \in (\hat{A}|_{M(B)})_b$ and $(\hat{A}|_{M(B)})_{bb} \subset (\hat{A}|_{M(B)})_b$. This completes the proof. □

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