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NONPROBABILISTIC COMPUTATION OF THE POISSON BOUNDARY FOR AN ÉTALÉE MEASURE ON A SEMI-SIMPLE LIE GROUP

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0. Introduction. In [**6**], Furstenberg proved the existence of a boundary representation for μ-harmonic functions on T, a connected semi-simple Lie group with finite center and no compact factors. In the same paper he also computed the boundary in the case μ was absolutely continuous with respect to Haar measure on T. In proving both these results he used probability theory (the Martingale theorem). In [**1**] Azencott generalized these results to the case of an étalée measure again using probability theory.

In [**2**] it was shown how to construct the boundary using nonprobabilistic techniques. These results were developed further in [**3**] applied to μ and T as in Furstenberg's situation with the additional hypothesis that μ was supported on T. The boundary was shown to be a compact homogeneous space of T.

In this paper we prove Azencott's generalization of Furstenberg's result (see Theorem 2.7) for the case of μ, a spread out étalée measure on T a semi-simple, connected Lie group, with finite center and no compact factors. The techniques we use are those developed in [**2**] and [**3**] and are nonprobabilistic in nature.

1. Review of basic concepts. In this section we review the basic definitions and constructions involved in the Poisson boundary (B, T, ω). For a more detailed discussion, see [**2**] and [**3**]. Let T be a locally compact Hausdorff topological group. The right uniformly continuous functions on T which are bounded form a Banach algebra, $\mathbf{R}$, with respect to pointwise multiplication and the supremum norm. Let P be the Gelfand space of $\mathbf{R}$. Then (P, T) is a flow and P has a semi-group structure such that if $p \in P$, then $l_p : P \to P : q \to p \cdot q$ is continuous, and if $t \in T$, then $r_t : P \to P : q \to q \cdot t$ is continuous.

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The set T is a dense open subset of P, consequently every measure on T is also a measure on P. Let $M(P)$ be the regular Borel probability measures on P. It is possible to define convolution of two measures α and β in $M(P)$ such that $l_\alpha : M(P) \to M(P) : \beta \to \alpha \cdot \beta$ is continuous and if $\mu \in M(T)$, then $r_\mu : M(P) \to M(P) : \beta \to \beta \cdot \mu$ is continuous also. We say $\nu \in M(P)$ is idempotent if $\nu^2 = \nu \cdot \nu = \nu$. Given $\mu \in M(T)$ (the regular Borel probability measures on T) we say $f \in \mathbf{R}$ is μ-harmonic if $\int f(t' \cdot t)\, d\mu(t') = f(t)$, $f\mu = f$. The set of μ-harmonic functions form a Banach space with respect to the sup-norm. It can be shown that $(M(P), T)$ is a flow. In [**2**] it is shown that there is an idempotent measure $\nu \in M(P)$ such that $\nu \cdot \mu = \nu$ and if $B = \overline{\mathrm{ex}}\,(\overline{\mathrm{cnv}}\,(\overline{\nu T}))$ (the extreme points of the closed convex hull of the orbit closure of ν in $M(P)$) and if $\omega \in M(B)$ such that $b(\omega) = \nu$ (where $b : M(B) \to \overline{\mathrm{ex}}\,(\overline{\mathrm{cnv}}\,(\overline{\nu T}))$ is the barycentric mapping) then $C(B)$ is isometrically isomorphic as a Banach-space with $\mathbf{H}_\mu$, and the isomorphism is given by right convolution of ω with f in $C(B)$. That is,

$$\mathbf{R}_\omega : C(B) \to \mathbf{H}_\mu : f \to \int_B f(x \cdot t)\, d\omega(x)$$

is an isometry from $C(B)$ onto $\mathbf{H}_\mu$. The set (B, T, ω) is called the Poisson flow of μ and ω is called the representing measure.

2. The Poisson boundary for an étalée measure on a connected semi-simple Lie group with finite center. In this section we give a nonprobabilistic proof of Furstenberg's result using ideas developed in [**3**] and [**2**].

Definition 2.1. A measure μ is *spread out* (étalée) if either of the following equivalent conditions is satisfied:

(i) there exists an integer n such that μ^n is not singular with respect to Haar measure m on T;

(ii) the set $\sum_\mu$ is not empty where $\sum_\mu$ is the set of elements t of T for which there exists an integer p such that μ^p dominates a multiple of m on some neighborhood of t.

Remark 2.2. It can be shown that $\sum_\mu$ contains an open neighborhood whence $\sum_{\mu}^{-1} \cdot \sum_{\mu}$ contains an open neighborhood of e in T (see [**6**] or [**1**]).

Remark 2.3. Let (X,T) be an arbitrary factor of (B,T) the Poisson flow and ω the representing measure induced on X from B.

Theorem 2.4 [**3**]. *Let $p \in P$ be such that $\omega p \in B$ and $(t_\alpha), (s_\alpha)$ nets in T such that $t_\alpha \to p$ and $s_\alpha \to s$ in T, and $t_\alpha s_\alpha t_\alpha^{-1} \to t \in \sum_{\mu}^{-1} \cdot \sum_{\mu}$. Then*

(i) *$\omega p \in X$ and*

(ii) *$(\omega p)s = \omega p$.*

2.5. Notation and review. For a detailed discussion of these topics, see [**7**] or [**8**]. Let T be an analytic semi-simple Lie group with finite center and no compact factors.

Let $\underline{t}$ be the Lie algebra of T, $\underline{t} = \underline{k} + \underline{p}$ a Cartan decomposition of $\underline{t}$, $\underline{a}$ a maximal abelian subspace of $\underline{p}$ and Δ the roots of the pair $(\underline{t}, \underline{a})$. Order Δ and let Δ^+ be the positive elements of Δ.

For $\lambda \in \Delta$ set $\underline{t}^\lambda = \{Y \in \underline{t} \mid [H,Y] = \lambda(H)Y, H \in \underline{a}\}$ and define the Lie subalgebras $n^\pm$ by $n^\pm = \sum_{\pm\lambda\in\Delta^+} \underline{t}^\lambda$, and let $K, A, N^\pm$ be the analytic subgroups of T corresponding to $\underline{k}, \underline{a}, n^\pm$, respectively. Let $\underline{a}^+ = \{H \in \underline{a} \mid \lambda(H) > 0, \lambda \in \Delta^+\}$ and A^+ the positive Weyl Chamber. Then K is compact, $KA^+K = T = KAN^+ = KAN$.

Definition 2.6. Let $M = \{k \in K | kak^{-1} = a, a \in A\}$. In other words, M is the *centralizer* of A in K. Let M_0 be the *connected component of the identity* in M. Let $P = NAM$. We can now state Furstenberg's result.

Theorem 2.7. *Let μ be spread out in $M(T)$, T a semi-simple Lie group. Let M_0, M, A, N and K be as above, and let X be a metric-factor of (B, T, ω). Then*

(i) *There exists $x_0 \in X$ such that $x_0 \cdot s = x_0$, $s \in M_0AN = P_0$*

(ii) *K acts transitively on X, whence K acts transitively on B.*

(iii) *G/P_0 is a finite extension of G/P.*

(iv) *The Poisson boundaries of μ correspond to G/P', P' a subgroup of P and $P' \supset P_0$.*

Lemma 2.8. *There exist sequences (k_n) in K and (a_n) in A^+ such that $\omega k_n a_n \to x_0 \in X$, with $\overline{x_0 T} = X$.*

Proof. Let (t_n) be a sequence in T such that $\omega t_n \to x \in X$ and $\overline{xT} = X$.

Write $t_n = k_n \cdot a_n \cdot l_n$ with $(k_n), (l_n) \subset K$ and $(a_n) \subset A^+$. We may assume that $l_n \to l \in K$ and $\omega k_n a_n \to \rho \in M(X)$. Then $\omega t_n = \omega k_n a_n l_n \to \rho l$. Hence, $\lim \omega k_n a_n = \rho = x l^{-1} \in X$. Set $x_0 = x l^{-1}$. Then $\overline{x_0 T} = \overline{x_0 l^{-1} T} = \overline{xT} = X$. □

Remark 2.9. Recall that $\sum_\mu$ contains an open set and hence $\sum_\mu^{-1} \cdot \sum_\mu$ contains a neighborhood O of e (the identity of T).

There exist arbitrarily small open neighborhoods U of e in T for which $U = U_K U_A U_N$ and $U_K \subseteq K$, $U_A \subseteq A$ and $U_N \subseteq N$ (U_K, U_A, U_N all open and containing e).

For the rest of this section, we let $(a_n), (k_n)$ denote fixed sequences in A^+ and K, respectively, such that $\omega k_n a_n \to x_0 \in X$ with $\overline{x_0 T} = X$. We shall also assume that $k_n \to k \in K$. We want to show $x_0 K = X$.

Lemma 2.10. *There exists a neighborhood U_A of e in A such that*

$$x_0 \cdot U_A = x_0.$$

Proof. Let $k_n \to k$. Then $\mathrm{ad}\,(k_n) \to \mathrm{ad}\,(k)$ (where $[\mathrm{ad}\,(k)](t) = ktk^{-1}$). Now $\mathrm{ad}\,(k)$ is continuous on T and $\mathrm{ad}\,(k)(e) = e$.

Therefore there exists a neighborhood U of e in T such that $[\mathrm{ad}\,(k)](U) \subseteq O$ and $U = U_K \cdot U_A \cdot U_N$ where U_K, U_A and U_N are neighborhoods of e in K, A and N, respectively. Then $k \cdot U_A \cdot k^{-1} \subseteq O$.

Now A is abelian, $k_n a_n a = k_n a a_n = k_n a k_n^{-1} k_n a_n$. Hence, if $a \in U_A$ and we take $(t_n) = (k_n a_n)$ and $(s_n) = a$, then $(k_n a_n) a (k_n a_n)^{-1} =$

$k_n a k_n^{-1} k_n a_n a_n^{-1} k_n^{-1} = k_n a k_n^{-1} \to kak \in O$. Therefore, by Theorem 2.4,

$$\omega p \cdot a = \omega \cdot p.$$

Lemma 2.11. *There exists a neighborhood U_M of e in M such that $x_0 \cdot 1 = x_0$ $(l \in U_M)$, whence $x_0 \cdot 1 = x_0$, $1 \in M_0$.*

Proof. As in the preceding proposition, let $k_n \to k$. Then $\mathrm{ad}\,(k)$ is continuous on T, hence there exists a neighborhood U_M of e in M such that $\mathrm{ad}\,(k)(U_M) \subset O$.

Now if $l \in M$, $k_n a_n l = k_n l a_n = k_n l k_n^{-1} k_n a_n$. Now $k_n l k_n^{-1} \to klk^{-1}$. Hence, if $l \in U_M$ and we take $(t_n) = (k_n a_n)$ and $(s_n) = (l)$, then $(k_n a_n) l (k_n a_n)^{-1} = k_n l_n^{-1} k_n a_n a_n^{-1} k_n^{-1} = k_n l k_n^{-1} \to klk \in O$. Therefore, by Theorem 2.4,

$$\omega \cdot p \cdot l = \omega \cdot p. \qquad \square$$

Lemma 2.12. *Let $Y, H \in \underline{t}$ with $[H, Y] = \lambda Y$ for some $\lambda \in \mathbf{R}$. Then $(\exp(H))(\exp(Y))(\exp(-H)) = \exp(e^{\lambda} Y)$.*

Proof. Standard. $\square$

Lemma 2.13. *Let $-\lambda \in \Delta^+$. Then there exists a neighborhood U_λ of e in $\exp[t^\lambda]$ such that $x_0 \cdot t = x_0$ for all $t \in U_\lambda$.*

Proof. Let $a_n = \exp H_n$ with $H_n \in \underline{a}^+$ for all n. Then $a_n t a_n^{-1} = \exp(e^{\lambda(H_n)} Y)$ (where $t = \exp Y$) and since $\lambda(H_n) < 0$ we have that $e^{\lambda(H_n)}$ converges to some constant c. Hence, the sequence $\nu_n = a_n t a_n^{-1}$ converges to $\nu \in T$. Let $\varphi : \underline{t}^\lambda \to T : X \to \exp(cX)$ and $\psi : \underline{t}^\lambda \to T : \psi = \mathrm{ad}\,(k) \circ \varphi$. Then φ and ψ are continuous. Also

$$\begin{aligned}
\psi(Y) &= \mathrm{ad}\,(k)(\varphi(Y)) = \mathrm{ad}\,(k)(\lim_{n\to\infty} \exp(e^{\lambda(H_n)} Y)) \\
&= \lim_{n\to\infty} \mathrm{ad}\,(k)(\exp e^{\lambda(H_n)} Y) \\
&= \lim_{n\to\infty} \mathrm{ad}\,(k)(a_n t a_n^{-1}) = \lim k a_n t a_n^{-1} k^{-1} \\
&= \lim((k_n a_n) t (k_n a_n)^{-1}).
\end{aligned}$$

Now the composition ψ of $\mathrm{ad}\,(k)$ and φ is continuous. Consequently, $\psi(X) = \lim((k_n a_n)\exp X(k_n a_n)^{-1}) \in O$, for all $X \in V_\lambda$, an open neighborhood of e in $\underline{t}^\lambda$. Now set $U_\lambda = \psi(V_\lambda)$.

Then setting $(t_n) = (k_n a_n)$ and $s = t$ in Theorem 2.4 shows that $x_0 \cdot t = x_0$, $t \in U_\lambda$. □

Remark 2.14. Note that 2.10 and 2.13 imply $x_0 \cdot s = x_0$, $s \in A \cup N^-$.

Proof of Theorem 2.7. (i) This is an immediate consequence of Lemmas 2.10, 2.11 and 2.13.

(ii) Let $H = \{t \in T \mid x_0 t = x_0\}$. Then H is a closed subgroup of T which contains A.

By Lemma 2.13 $\exp(Y) \in H$ for all $Y \in V_\lambda$ and $-\lambda \in \Delta^+$. Hence the Lie algebra of H contains that of N^- and so $N^- \subset H$. Finally, $X = \overline{x_0 T} = \overline{x_0 N^- AK} = \overline{x_0 K} = x_0 K$.

(iii) M is compact. Hence M/M_0 is finite and so is P/P_0.

(iv) In [**5**] we see G/P is strongly proximal (a result due to C. Moore). Hence, every Poisson boundary for spread out μ has G/P as a homomorphic image. Consequently, every Poisson flow of a spread out μ is G/P' for some $P' \supset P_0$. Now the mapping $\pi : G/P_0 \to G/P : gP_0 \to gP$ is finite to one, and the number of preimages of gP is the cardinality of P/P_0. Hence, every Poisson boundary is a finite extension of G/P. □

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