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GROUPS OF PIECEWISE-LINEAR HOMEOMORPHISMS WITH IRRATIONAL SLOPES

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ABSTRACT. Let F be the group of piecewise-linear homeomorphisms of the unit interval. F has many interesting countable discrete subgroups, some of which have cohomological finiteness properties. Many subgroups of piecewise-linear homeomorphisms with irrational slopes and irrational singularities are finitely generated, finitely presented and are of type FP_∞. This is shown by constructing contractible posets upon which the various subgroups act and then by understanding the complexity of the classifying space of the poset, which is an Eilenberg-Maclane space for the subgroup.

1. Introduction. Groups of piecewise-linear homeomorphisms have proven to be interesting examples of countable groups. The first such group was used in the construction of an early example of a finitely-presented infinite simple group by Thompson [**6**]. Brown and Geoghegan [**3**] showed later that it has a subgroup which is a finitely-presented, infinite-dimensional, torsion-free group. They also showed that subgroup to be of type FP_∞ by building a $K(G,1)$ complex for the group which had only finitely many cells (in fact, two) in each dimension. The group that Brown and Geoghegan studied can be described as the subgroup of the group of all piecewise-linear homeomorphisms of the interval $[0,1]$ where every element of the group has only finitely many singularities, each singularity lies in the dyadic numbers, $\mathbf{Z}[1/2]$, and the slope of the homeomorphisms away from the singularities lie in $\{2^i, i \in \mathbf{Z}\}$. Brown [**2**] later studied generalizations of this group where the singularity set and slope set are respectively $\mathbf{Z}[1/p]$ and $\{p^i, i \in \mathbf{Z}\}$. He showed that all these groups are also of type FP_∞. Stein [**5**] studied such groups for singularity set $\mathbf{Z}[1/p_1, 1/p_2, \dots, 1/p_k]$ with slope group $\{p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}, i_j \in \mathbf{Z}\}$ and showed that all these groups are of type FP_∞.

I study another class of groups of piecewise-linear homeomorphisms where the singularities can be at irrational points, for instance in $\mathbf{Z}[\sqrt{2}]$. Many of these groups are also of type FP_∞.

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In order to conveniently name these groups, we use the following notation. If P is a multiplicative subgroup of $\mathbf{R}^+$, and A is a $\mathbf{Z}P$ submodule of $\mathbf{R}$ such that $P \cdot A = A$, then for $l \in A$, we let $F(l, A, P)$ be the group of piecewise-linear homeomorphisms of $[0, l]$ satisfying the following:

i) Each homeomorphism has only finitely many singularities.

ii) Each singularity lies in A.

iii) Away from the singularities, the slopes lie in P.

Of the subgroups with irrational breakpoint sets, perhaps the easiest to understand is the group $F(1, \mathbf{Z}[\sqrt{2}], \{(\sqrt{2}+1)^i\})$. I will first construct a poset upon which this group acts and then use a notion of subdivision to understand the poset further. Then I will show the finiteness properties of the group by studying the structure of the classifying space of the poset.

2. Construction of the poset. For convenience, we denote $\sqrt{2}+1$ by ω. The poset upon which the group $F(1, \mathbf{Z}[\sqrt{2}], \{\omega^i\})$ acts will be the set of all piecewise-linear homeomorphisms from intervals which have lengths $a + b\omega$ for some positive integers a and b to the unit interval satisfying:

a) There are only finitely many singularities.

b) Each singularity lies in $\mathbf{Z}[\sqrt{2}]$.

c) Away from the singularities, the slopes are elements of ω^i.

The partial order on this set comes from the relationship of expansion.

If $f : [0, a_1 + b_1\omega] \to [0, 1]$, we obtain simple expansions of f by precomposing f with a piecewise-linear map s of one of the following types:

$$s_1 \text{ has slope } \begin{cases} 1 & \text{on } [0, i + j\omega] \\ \omega^{-1} & \text{on } [i + j\omega, i + (j+1)\omega] \\ 1 & \text{on } [i + (j+1)\omega, a + b\omega] \end{cases}$$

or

$$s_2 \text{ has slope } \begin{cases} 1 & \text{on } [0, i + j\omega] \\ \omega^{-1} & \text{on } [i + j\omega, i + 1 + (j+2)\omega] \\ 1 & \text{on } [i + 1 + (j+2)\omega, a + b\omega] \end{cases}$$

for some integers $0 \le i \le a$, $0 \le j \le b$.

Maps of the form of s_1 expand an interval of length one in the domain of f to an interval of length ω in the domain of g. Maps of the form of s_2 expand an interval of length ω in the domain of f to an interval of length $2\omega + 1$ in the domain of g. (Note that $\omega^2 = 2\omega + 1$.) If we have $f : [0, a_1 + b_1\omega] \to [0, 1]$ and $s : [0, a_2 + b_2\omega] \to [0, a_1 + b_1\omega]$, then we call $f \circ s = g$ a *simple expansion* of f. Note that either $a_2 = a_1 - 1$ and $b_2 = b_1 + 1$ or $a_2 = a_1 + 1$ and $b_2 = b_1 + 1$ depending upon whether the expansion is of the type of s_1 or s_2. We extend the relation of simple expansion to say f is an *expansion* of g if g can be obtained from f by a finite sequence of simple expansions. This gives us a partial order on the poset X by having $f < g$ if g is an expansion of f.

3. Subdivision with $\{\omega^i\}$. One useful way to understand elements of the poset X is by understanding them as linear interpolations of subdivisions. In particular, we would like to understand a particularly nice class of subdivisions.

Definition 3.1. An *ω-regular subdivision of level* 0 of an interval $[0, A + B\omega]$ is a sequence $(0 = c_1, c_2, \dots , c_n = A + B\omega)$ such that for each $1 \le i \le n$, $c_i = c_{i-1} + 1$ or $c_i = c_{i-1} + \omega$.

Given an ω-regular subdivision $(0 = c_1, \dots , c_n = A + B\omega)$ of level k, we construct one of level $k + 1$ as follows:

For some $0 \le i < n$, we replace the pair $\dots , c_i, c_{i+1}, \dots$ with one of the following sets of four points:

1. $\dots , c_i, c_i + \omega^{-2}\Delta, c_i + (\omega^{-2} + \omega^{-1})\Delta, c_{i+1}, \dots$
2. $\dots , c_i, c_i + \omega^{-1}\Delta, c_i + (\omega^{-2} + \omega^{-1})\Delta, c_{i+1}, \dots$
3. $\dots , c_i, c_i + \omega^{-1}\Delta, c_i + (2\omega^{-1})\Delta, c_{i+1}, \dots$

where $\Delta = c_{i+1} - c_i$.

We say a sequence $(0 = c_1, \dots , c_n = A + B\omega)$ is an *ω-regular subdivision* of $[0, A + B\omega]$ if it is an ω-regular subdivision of $[0, A + B\omega]$ of level n for some n.

The various possibilities for subdividing listed above in the definition of ω-regular subdivisions correspond to various ways of dividing an interval of length ω^i into pieces of lengths ω^{i-1} and ω^{i-2}. The three ways of dividing the interval of length Δ above all come from the fact that $\omega^{-2} + 2\omega^{-1} = 1$. The interval of length Δ becomes subdivided into three intervals. There are two "long" ones of length $\omega^{-1}\Delta$ and one "short" one of length $\omega^{-2}\Delta$. The first way of subdividing arranges the new intervals in the order "short, long, long"; the second arranges them in the order, "long, short, long" and the last in the other order, "long, long, short."

We will also be interested mostly in subdivisions which are not only ω-regular, but furthermore have only two interval lengths occurring in the subdivision such that the two interval lengths are adjacent powers of ω. Given any ω-regular subdivision S, we can subdivide to get another ω-regular subdivision S' with intervals only of two lengths which are adjacent powers in the following way. We find the interval of shortest length in S, say of length ω^{-N}. Then we subdivide all of the longer intervals which are longer than ω^{-N+1} repeatedly into pieces using any of the subdivision arrangements above. Eventually, we will have S' with intervals only of lengths ω^{-N+1} and ω^{-N}.

We also have a standard way of subdividing ω-regular subdivisions of this type into finer ones also of this type. If the lengths of intervals occurring in a subdivision S are ω^{-N} and ω^{-N+1}, then we call the intervals of length ω^{-N} "short" intervals and the ones of length ω^{-N+1} "long" ones. We can subdivide all the long intervals in S into intervals of length ω^{-N} and ω^{-N-1} to get a new subdivision S'. In S', we now have the long intervals of length ω^{-N} and the short intervals of length ω^{-N-1}. So, from now on, we will be dealing principally with ω-regular subdivisions with intervals of only two lengths which are adjacent powers of ω. For such subdivisions, we can talk about a *long-short pair* which is a pair of integers (a, b) such that there is an initial segment of some subdivision which contains a long segments and b short segments.

Thus, we have the following notation:

$$\begin{array}{c} l \\ \downarrow \\ l\ l\ s \end{array}$$

is used to mean that a long interval in S was subdivided into two long intervals in S' followed by a short one. Similarly, we have

$$\begin{array}{c} l \\ \downarrow \\ l\ s\ l \end{array}, \quad \begin{array}{c} l \\ \downarrow \\ s\ l\ l \end{array} \quad \text{and} \quad \begin{array}{c} s \\ \downarrow \\ l \end{array}$$

for the other possible ways of subdividing a long or short interval in S to a set of intervals in S'.

The connection between ω-regular subdivisions of the unit interval and elements of $F(1, \mathbf{Z}[\sqrt{2}], \{\omega^i\})$ comes from the following theorem. This will let us understand elements of the subgroups as interpolations of ω-regular subdivisions.

Theorem 3.1. *If S and S' are ω-regular subdivisions of the unit interval with the same number of points, then the affine interpolation of S and S' belongs to $F(1, \mathbf{Z}[\sqrt{2}], \{\omega^i\})$. Conversely, if $f \in F(1, \mathbf{Z}[\sqrt{2}], \{\omega^i\})$, then f is the affine interpolation of two ω-regular subdivisions of* $[0, 1]$.

Proof. $\Rightarrow$. Each interval l in an ω-regular subdivision has length ω^i for some i and has endpoints in $\mathbf{Z}[\sqrt{2}]$. Thus, the singularity set will lie in $\mathbf{Z}[\sqrt{2}]$. Since the slopes of the interpolation are ratios of lengths of intervals, and both the domain and range intervals have lengths, say, ω^{-n_i} and ω^{-m_i}, their ratios $\omega^{-m_i+n_i}$ are powers of ω.

$\Leftarrow$. So, given $f \in F(1, \mathbf{Z}[\sqrt{2}], \{\omega^i\})$, we first want to construct an ω-regular subdivision of $[0, 1]$ containing all the singularities of f. This is accomplished with the following series of lemmas.

Lemma 1 (Main Lemma). *For any finite set F in $\mathbf{Z}[\sqrt{2}] \cap [0, A+B\omega]$, there exists an ω-regular subdivision of $[0, A+B\omega]$ containing F.*

It will be enough to show that we can find an ω-regular subdivision of the interval containing an arbitrary point of $\mathbf{Z}[\sqrt{2}]$ since we can subdivide successively to get all the points in the finite set.

We postpone the proof of this lemma until we understand some facts about subdivisions through the following series of lemmas.

Lemma 2. *For any positive interval length l in $\mathbf{Z}[\sqrt{2}]$, there exists $N \in \mathbf{Z}$ such that $l = \frac{m}{\omega^N} + \frac{n}{\omega^{N+1}}$ for some $m, n \in \mathbf{Z}^+$.*

Proof. Since $l \in \mathbf{Z}[\sqrt{2}]$, $l = m_0(\omega^1) + n_0(\omega^0)$ where m_0 and n_0 are not necessarily positive integers. If we have m_0 and n_0 both positive, then we are done. Otherwise, we begin an iterative process.

At each stage, we subdivide the longer m_i intervals of length ω^N into $2m_i$ intervals of length ω^{N-1} and m_i intervals of length ω^{N-2} using any of the "long, short, long" type subdivisions as above. Thus, we have

$$\begin{aligned} l &= \frac{m_i}{\omega^N} + \frac{n_i}{\omega^{N+1}} = \frac{2m_i}{\omega^{N+1}} + \frac{m_i}{\omega^{N+2}} + \frac{n_i}{\omega^{N+1}} \\ &= \frac{2m_i + n_i}{\omega^{N+1}} + \frac{m_i}{\omega^{N+2}} = \frac{m_{i+1}}{\omega^{N+1}} + \frac{n_{i+1}}{\omega^{N+2}}. \end{aligned}$$

Thus, $m_{i+1} = 2m_i + n_i$ and $n_{i+1} = m_i$. We iterate this process until both m_i and n_i are positive.

Claim. This happens in a finite number of steps. We have a Markov chain with the following transition matrix:

$$\begin{pmatrix} m_i \\ n_i \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m_{i+1} \\ n_{i+1} \end{pmatrix}.$$

The eigenvalues for this matrix are ω and $-\omega^{-1}$. The corresponding eigenvectors are

$$\begin{pmatrix} \omega \\ 1 \end{pmatrix} \quad \text{for the } +\omega \text{ eigenvalue}$$

and

$$\begin{pmatrix} -\omega^{-1} \\ 1 \end{pmatrix} \quad \text{for the } -\omega^{-1} \text{ eigenvalue.}$$

The flow on the plane is illustrated in Figure 1. The attracting eigendirections are labeled 'A' and the repelling eigendirections are labeled 'R.'

So the attracting direction lies in the first quadrant for initial pairs (m, n) which correspond to positive lengths. That is, anything to above

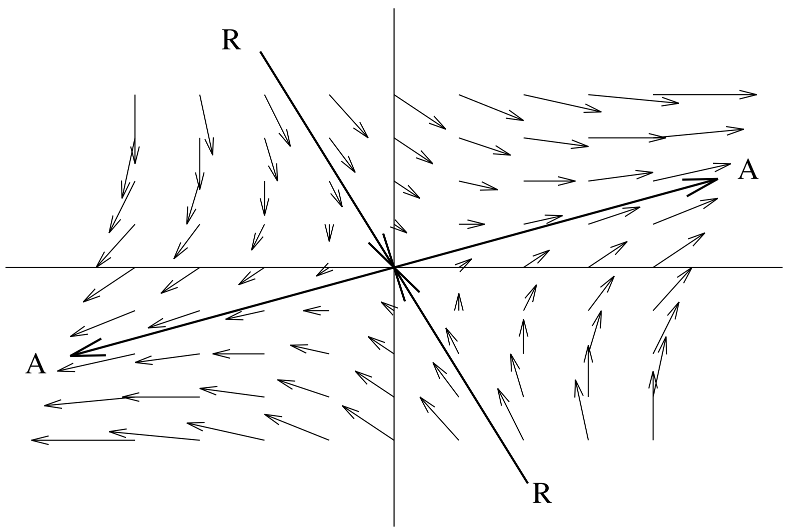

FIGURE 1.

or to the right of the line $y = -\omega^{-1}x$ will eventually have images in the first quadrant, so thus we have the claim and Lemma 2. □

Now unfortunately it is not so simple, because even now that we know that every length $l \in \mathbf{Z}[\sqrt{2}]$ can be written as $l = m/\omega^N + n/\omega^{N+1}$ for some $m, n, N \in \mathbf{Z}^+$, we do not have an explicit subdivision which realizes the combination of m intervals of length ω^{-N} and n intervals of length ω^{-N-1}. In fact, it may not be possible to subdivide for instance $[0, 1]$ in such a way. For example, say $l = 1/\omega^{12} + 50/\omega^{13}$. Then there is no possible way to subdivide $[0, 1]$ to obtain one "long" piece of length ω^{12} and 50 "short" pieces of length ω^{13}. In particular, there is no way to get three "short" pieces in a row. However, we can subdivide further to reach an expression for l which can actually be obtained from $[0, 1]$ as the initial segment of some ω-regular subdivision. This I will show by showing that long-short pairs are obtainable if they are sufficiently close to the positive eigendirection. So we define what it means for a long-short pair to be obtainable and then through a series of lemmas, we show that we can always find an obtainable subdivision containing an arbitrary point of $\mathbf{Z}[\sqrt{2}]$.

Definition 3.2. An expression for a length $l \in \mathbf{Z}[\sqrt{2}]$ of the form $l = m/\omega^N + n/\omega^{N+1}$ for some $m, n, N \in \mathbf{Z}$ is called *obtainable at level* N if there is an ω-regular subdivision S of $[0, A + B\omega]$ such that there is an initial segment in the subdivision S which has exactly m intervals of length ω^{-N} and exactly n intervals of length ω^{-N-1} and no other intervals.

A length l is called *obtainable* if there is some expression of the form above for which l is obtainable at level N.

Similarly, we say a long-short pair (a, b) is obtainable at level N if there is an ω-regular subdivision of $[0, A + B\omega]$ with an initial segment containing a long intervals (of length ω^{-N}) and b short ones.

Furthermore, we say that a long-short pair (a, b) is *wholly obtainable at level* N if it is obtainable at level N and if the preimage of the initial segment for the long-short pair in the previous stage of the subdivision contains an integral number of long and short intervals.

For example, for the ω-regular subdivisions of the interval $[0, 1]$ the long-short pairs which are obtainable at level 1 are $(0, 0)$, $(1, 0)$, $(1, 1)$,(2,0) and (2,1). Of these, only the long-short pairs (2,1) and (0,0) are wholly obtainable at level 1.

Lemma 3. *If a long-short pair* (a, b) *is obtainable at level* n*, for some* $n > 2$ *with at least* 4 *intervals of either size following the initial segment, then there is a series* $(a_0, b), \ldots, (a_0 + 5, b)$ *of at least* 5 *obtainable long-short pairs with the same number of short intervals as* (a, b)*.*

Proof. We know that (a, b) is obtainable so we look at an ω-regular subdivision with a longs and b shorts in some initial segment. By hypothesis, there are at least four intervals following the (a, b) pair so we look at the last part of the initial segment containing a longs and b shorts.

Case 1. The last piece in the initial segment came from a subdivision of the form

$$\begin{array}{c} s \\ \downarrow \\ l \end{array}$$

the previous stage.

Consider the possible neighbors to the short interval in the previous stage. We know that there is at least one interval following the short interval in the previous stage since there are four intervals following it in the last stage.

Subcase A. There were three shorts in a row. This never happens after the second iteration since the only way a short can arise is from a subdivision coming from a long. Each of these results is at least either a long interval to the right or left of the short. Thus, there is always at least one neighbor of a short which is a long and it is impossible to have three short intervals in a row.

Subcase B. The short in the previous stage was preceded by a short and followed by a long. In this case the subdivision from the previous stage to the current stage could be rearranged to look like:

$$\begin{array}{ccccc} \cdots\cdots\cdots & s & s & l & \cdots\cdots\cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \underbrace{\cdots\cdots\cdots}_{(a-2 \text{ longs, } b \text{ shorts})} & l & l & l\,l\ s & \cdots\cdots\cdots \end{array}$$

The only possible required adjustment would be to make sure that the subdivision of the long following the short from the previous stage is subdivided by the

$$\begin{array}{c} l \\ \downarrow \\ l\,l\ s \end{array}$$

as shown above.

Now it is clear that all the long-short pairs from $(a-2, b)$ to $(a+2, b)$ are obtainable so we have a sequence of five obtainable subdivisions containing (a, b).

Subcase C. The short in the previous stage was preceded by a long and followed by a short. In that case, the subdivision from the previous stage to the present one can be arranged to be of the form:

$$\begin{array}{c} \cdots\cdots\cdots \\ \downarrow \\ \underbrace{\cdots\cdots\cdots}_{(a-3 \text{ longs, } b-1 \text{ shorts})} \end{array} \qquad \begin{array}{cccc} l & s & s & \cdots\cdots\cdots \\ \downarrow & \downarrow & \downarrow & \downarrow \\ s\;l\;l & l & l & \cdots\cdots\cdots \end{array}$$

Thus we have a sequence of long-short pairs from $(a-3, b)$ to $(a+1, b)$ which are all obtainable.

Subcase D. The short in the previous stage was both preceded and followed by a long. Then we can arrange the subdivision to be of the form:

$$\begin{array}{c} \cdots\cdots\cdots \\ \downarrow \\ \underbrace{\cdots\cdots\cdots}_{(a-3 \text{ longs, } b-1 \text{ shorts})} \end{array} \qquad \begin{array}{cccc} l & s & l & \cdots\cdots\cdots \\ \downarrow & \downarrow & \downarrow & \downarrow \\ s\;l\;l & l & l\;l\;s & \cdots\cdots\cdots \end{array}$$

Then we have the sequence from $(a - 3, b)$ to $(a + 2, b)$ all obtainable.

Case 2. The last piece in the initial segment for the long-short pair (a, b) came from a long.

Subcase A. Case where the entire image of the long from the previous stage is the tail end of the initial segment for (a, b). In other words, (a, b) is wholly obtainable at level N.

We can arrange the subdivision to look like:

$$\begin{array}{c} \cdots\cdots\cdots \\ \downarrow \\ \underbrace{\cdots\cdots\cdots}_{(a-2 \text{ longs, } b-1 \text{ shorts})} \end{array} \qquad \begin{array}{cccc} l & x & y & \cdots\cdots\cdots \\ \downarrow & \downarrow & \downarrow & \downarrow \\ s\;l\;l & x' & y' & \cdots\cdots\cdots \end{array}$$

with x and y either short or long intervals, and x' and y' are their corresponding subdivisions, each either of the form l or $l\ l\ s$.

Immediately, we get that $(a - 2, b)$ through (a, b) are obtainable and so we consider the possibilities for x and y.

If x and y are both short intervals, then we have $(a+1, b)$ and $(a+2, b)$ as obtainable since x' and y' would both be long intervals.

If x is short and y is long, then we can choose to have y' be of the form $l\ l\ s$ and we have $(a+1, b)$ through $(a+3, b)$ obtainable.

If x is a long interval, then we can choose to subdivide it so that its subdivision x' is of the form $l\ l\ s$ and we have again that $(a+1, b)$ and $(a+2, b)$ are obtainable.

Subcase B. Case where the entire image of the long from the previous stage extends beyond the tail end of the initial segment for (a, b).

There are a number of subcases here which reduce to the previous case. Those are of the form where the part of the subdivision which is in the initial segment contains no longs. These are

$$\begin{array}{ccc} l & l & l \\ \downarrow & \downarrow & \downarrow \\ \underline{l}\ l\ s & \underline{l}\ s\ l & \underline{l}\ \underline{l}\ s \end{array}$$

where the underlined portion corresponds to those intervals which are part of the initial segment for (a, b). For long-short pairs which end in these kinds of subdivisions, we can consider the wholly obtainable long-short pair which corresponds to leaving off the long intervals from the tail part of the initial segment. Then we can look at the subcase A above for either $(a-1, b)$ or $(a-2, b)$ depending upon whether there are one or two longs in the last part of the initial segment. By the argument above, we will get a string of five obtainable long-short pairs including (a, b).

That leaves the possibilities for the tail part of the initial segment where there is a short piece included in the initial segment.

When the tail part of the initial segment looks like

$$\begin{array}{c} l \\ \downarrow \\ \underline{s}\ l\ l \end{array}$$

we have that (a, b), $(a+1, b)$ and $(a+2, b)$ are all immediately obtainable. We look at the next interval after the $s\ l\ l$ in the initial segment.

If it is a short interval, then it came from a long interval in the previous stage via a subdivision of the form

$$\begin{array}{c} l \\ \downarrow \\ s\ l\ l \end{array}$$

In that case we can replace that subdivision with one of the form

$$\begin{array}{c} l \\ \downarrow \\ l\ l\ s \end{array}$$

and we have that $(a+3, b)$ and $(a+4, b)$ are also obtainable.

If the interval following the $s\,l\,l$ is a long interval, then we immediately have that $(a+3, b)$ is obtainable. If the interval following that is also a long interval, then we have $(a+4, b)$ and thus a string of five; otherwise, we have a short interval coming from a subdivision of the form $l\ s\ l$. In that case we replace the

$$\begin{array}{c} l \\ \downarrow \\ l\ s\ l \end{array} \quad \text{with a} \quad \begin{array}{c} l \\ \downarrow \\ l\ l\ s \end{array}$$

and we have that $(a+4, b)$ is obtainable and thus, again, we have a string of five consecutive obtainable long-short pairs containing (a, b).

When the tail part of the initial segment looks like

$$\begin{array}{c} l \\ \downarrow \\ \underline{l}\ \underline{s}\ l \end{array}$$

we have that $(a-1, b)$, (a, b) and $(a+1, b)$ are all immediately obtainable. We look at the next interval after the $l\ s\ l$ in the initial segment.

If it is a short interval, then it came from a long interval in the previous stage via a subdivision of the form

$$\begin{array}{c} l \\ \downarrow \\ s\ l\ l \end{array}.$$

In that case, we can replace that subdivision with one of the form

$$\begin{array}{c} l \\ \downarrow \\ l\,l\;\,s \end{array}$$

and we have that $(a+2, b)$ and $(a+3, b)$ are also obtainable.

If the interval following the $l\ s\ l$ is a long interval, then we immediately have that $(a+2, b)$ is obtainable. If the interval following that is also a long interval, then we have $(a+3, b)$ and thus a string of 5; otherwise, we have a short interval coming from a subdivision of the form $l\ s\ l$. In that case we replace the

$$\begin{array}{c} l \\ \downarrow \\ l\,s\,l \end{array} \quad \text{with a} \quad \begin{array}{c} l \\ \downarrow \\ l\,l\;\,s \end{array}$$

and we have that $(a+3, b)$ is obtainable and thus again we have a string of 5 consecutive obtainable long-short pairs containing (a, b).

The only remaining possibility arises from when the last part of the initial segment is of the form

$$\begin{array}{c} l \\ \downarrow \\ \underline{s}\,\underline{l}\;\,l \end{array}$$

This is exactly analogous to the above case by replacing the

$$\begin{array}{c} l \\ \downarrow \\ s\,l\;\,l \end{array} \quad \text{with a} \quad \begin{array}{c} l \\ \downarrow \\ l\,s\,l \end{array}$$

and noticing that we again get a string of 5 obtainable subdivisions containing (a, b).

Thus we have Lemma 3. □

Lemma 4. *If (a, b) is a long-short pair which is obtainable at level N and $b > 0$, then there is a range $a_0, \ldots, a_0 + 4$ such that $(a_0, b-1), \ldots, (a_0 + 4, b-1)$ are all obtainable at level N.*

Proof. Look at the last occurrence of a short in the initial segment of subdivision which shows that (a, b) is obtainable. If we delete that short, we have some number of longs obtainable with $b-1$ shorts. Thus, by the previous lemma, we have a range of 5 obtainable long-short pairs with $b-1$ shorts. □

Lemma 5. *If a long-short pair (a, b) is obtainable at level N, then there is another long-short pair (a_i, b) which is wholly obtainable at level N with $|a_i - a| \leq 2$.*

Proof. Look at the tail of the initial segment for the long-short pair (a, b). This came from an interval in the previous stage. If that interval's image is contained in the initial segment, then (a, b) itself is wholly obtainable and we are done. Otherwise, we have one of the following possibilities for the preimage of the tail.

$\underline{l}\ \underline{l}\ s$	Then $(a-2, b)$ is wholly obtainable.
$\underline{l}\ l\ s$	Then $(a-1, b)$ is wholly obtainable.
$\underline{l}\ \underline{s}\ l$	Then $(a+1, b)$ is wholly obtainable.
$\underline{l}\ s\ l$	Then $(a-1, b)$ is wholly obtainable.
$\underline{s}\ \underline{l}\ l$	Then $(a+1, b)$ is wholly obtainable.
$\underline{s}\ l\ l$	Then $(a+2, b)$ is wholly obtainable.

In every case we have some pair (a_i, b) which is wholly obtainable with a_i close enough to a. Thus, we have the lemma. □

So we know that since there are 5 contiguous obtainable long-short pairs there is thus at least one wholly obtainable long-short pair. We would like to use these to conclude that long-short pairs which are near the eigendirection are actually obtainable. For that, we consider the distances from the line $L : y = \omega^{-1}x$ to obtainable long-short pairs. Let $d((a, b), L)$ be the Euclidean distance from the point (a, b) to the line L.

Lemma 6. *There is a real number M such that the following holds: If there is a long-short pair (a, b) which is obtainable at level N and has $d((a, b), L) > M$, then there is a long-short pair (a', b') which is obtainable at level $N-1$ and also has distance $d((a', b'), L) > M$.*

Proof. Say we have such an obtainable long-short pair, (a,b), with Euclidean distance to the line L greater than M. Then the horizontal distance (in the x direction) from the pair to the line is greater than $M\sqrt{2\omega+2}$. We know by the previous lemma that there is a wholly obtainable pair within horizontal distance 2 of (a,b), so there is a wholly obtainable pair with horizontal distance greater than $M\sqrt{2\omega+2}-2$ from the line. The Euclidean distance from L for that wholly obtainable pair is greater than $M(\sqrt{2\omega+2}-2)/\sqrt{2\omega+2}$. Since that pair is wholly obtainable, it is the image of a long-short pair which is obtainable at level $N-1$. Since the Markov process of subdivision contracts distances to the line L by a factor ω, there must be a long-short pair obtainable at level $N-1$ which has Euclidean distance at least $\omega M(\sqrt{2\omega+2}-2)/\sqrt{2\omega+2}$ from L.

So for M to be large enough for the lemma to be true, we need:

$$\begin{aligned}\omega\frac{M(\sqrt{2\omega+2})-2}{\sqrt{2\omega+2}} &> M\\ \omega M\sqrt{2\omega+2}-2\omega &> M\sqrt{2\omega+2}\\ \omega M\sqrt{2\omega+2}-M\sqrt{2\omega+2} &> 2\omega\\ (\omega-1)\sqrt{2\omega+2}M &> 2\omega\\ M &> \frac{2\omega}{(\omega-1)\sqrt{2\omega+2}}.\end{aligned}$$

So Lemma 6 holds for

$$M > \frac{2\omega}{(\omega-1)\sqrt{2\omega+2}} = \sqrt{1+\frac{\sqrt{2}}{2}} = 1.3065\ldots. \qquad \square$$

Lemma 6 gives the following consequence:

Lemma 7. *If a long-short pair* (a,b) *is within distance* .3 *of the line* $L: y=\omega x$ *and* $(a+4)\omega^{-N}+b\omega^{-N-1}<1$, *then* (a,b) *is obtainable at level* N.

Proof. First we subdivide the unit interval to any ω-regular subdivision with interval lengths ω^{-N} and ω^{-N-1}. Since the total number of longs and shorts (A, B) in the entire subdivision lies on the line L with $A\omega^{-N} + B\omega^{-N-1} = 1$ and the pair (a, b) is within distance .3 of the line with a smaller sum $a\omega^{-N} + b\omega^{-N-1} < 1$, we know that b must be smaller than B. Thus, there are at least $b+1$ shorts in the subdivision of $[0, 1]$ and it follows from Lemma 4 that there is a range of 5 obtainable subdivisions $(a_0, b) \dots (a_0 + 4, b)$. And we know by Lemma 6 that if one of the obtainable subdivisions is at Euclidean distance greater than 1.4 from the line L, then there is also an obtainable subdivision at distance greater than 1.4 at level $N-1$ as well, and thus at all lower levels as well. But there are no obtainable subdivisions that far from the line at, for example, level 2. Thus, none of the 5 contiguous ones with b short intervals can be greater than Euclidean distance 1.4 away from the line. Thus, none can be greater than horizontal distance 1.4 $\sqrt{2\omega + 2} = 3.66\dots$ away from the line L. Since there are 5 obtainable points in a line with each separating distance exactly one, the worst case is that one of the end obtainable points is 3.66 away from the line. In that case the opposite obtainable point on the other side of the line would be distance $4 - 3.66 = .34$ away from the line. Thus, we have that all points within distance .34 are obtainable. □

This lemma also gives us a good estimate of how far the subdivision process needs to go to reach an obtainable subdivision for a fixed $l \in \mathbf{Z}[\sqrt{2}]$. That is, if $l = a + b\omega$, we compute the distance from (a, b) to the line L. Since each subdivision will move the image of (a, b) closer to L by a factor of ω, we can easily tell how many subdivisions will be sufficient to get an expression for l which is actually obtainable.

Now we return to proving the main lemma, which was to show that we can find an ω-regular subdivision of the interval containing an arbitrary point of $\mathbf{Z}[\sqrt{2}]$.

So given an arbitrary point $a + b\omega$ of $\mathbf{Z}[\sqrt{2}] \cap [0, A + B\omega]$, we want to find an ω-regular subdivision of $[0, A + B\omega]$ containing $a + b\omega$. So we will begin by subdividing $[0, A + B\omega]$ successively into progressively finer ω-regular subdivisions with interval lengths adjacent powers of ω. We know from Lemma 2 that $a + b\omega$ can be written in a sequence of ways, $m_i\omega^{-i} + n_i\omega^{-i-1}$. As i increases, the long-short pairs (m_i, n_i) enter the first quadrant and become closer to the line $L : y = \omega^{-1}x$ since the matrix for the Markov process defined in Lemma 2 contracts by a factor of ω in the direction towards the line L (and expands by a factor of ω in the direction along the line L). Eventually, we will have the long-short pair (m_j, n_j) within distance .3 of the line L. Note that all long-short pairs (m_k, n_k) for $k > j$ will also be closer than distance .3 to the line L. So we subdivide $[0, A + B\omega]$ into pieces of length ω^{-j} and ω^{-j-1}. The question is, can we find an initial segment of this subdivision with exactly m_j long intervals and n_j short intervals.

We can assume that we are at a stage j such that there are plenty (in particular, at least 5) of intervals following the point $a + b\omega$ in the subdivision at stage j, since as long as $a + b\omega$ is not the whole interval, there will be some distance from $a+b\omega$ to the end of the interval $A+B\omega$ and the lengths of the intervals get shorter and shorter. (If the point $a + b\omega$ is the end of the interval, we are done already.)

So now we would like to show that (m_j, n_j) is an obtainable long-short pair at level j. But we know from Lemma 7 that since (m_j, n_j) is close to the line L, that it is an obtainable subdivision. Thus we can subdivide $[0, A+B\omega]$ with an ω-regular subdivision containing the given point, and thus the main lemma is proven.

So now we prove the remaining direction of the theorem using the main lemma. That is, given a homeomorphism $f \in F(1, \mathbf{Z}[\sqrt{2}], \{\omega^i\})$, we look at its set of breakpoints $B \subset \mathbf{Z}[\sqrt{2}]$. We can construct an ω-regular subdivision of $[0, 1]$ containing the first breakpoint of f and then we can successively subdivide that subdivision to obtain one containing the second breakpoint and so on until we have B_1 which is an ω-regular subdivision of $[0, 1]$ containing all the (finitely many) breakpoints of f. We consider $C_1 = f(B_1)$, the images of the breakpoints of f. These form a subdivision of $[0, 1]$ which is not necessarily ω-regular. But we can find a refined subdivision C_2 of $[0, 1]$ which is ω-regular and contains all the elements of C_1. Furthermore, since we can start with the subdivision C_1 and subdivide it to obtain C_2, we can simultaneously be subdividing the intervals in the domain in the same manner as those in the range to obtain B_2, an ω-regular refinement of the subdivision B_1 such that $f(B_2) = C_2$. Thus, we have obtained f as the linear interpolation of a pair of ω-regular subdivisions B_2 and C_2 with the same number of points.

4. The classifying space of the poset. Given a poset X, we have the general construction of the classifying space associated to it. That is, the space Y has an n-dimensional simplex for each ordered $n+1$ tuple $x_0 < x_1 < \cdots < x_n$ in the poset X. The boundary maps δ_i for the space Y are those which come from omitting the ith element from the $n+1$ tuple.

We would like to know that the poset we constructed in the previous section is directed, since by a result of Quillen [**4**] we have that if our poset is directed, its classifying space is contractible.

Given f and g in the poset, we want to find h such that $f < h$ and $g < h$. We can do this by finding a common expansion in the following way. Say we have $f : [0, a + b\omega] \to [0, 1]$ and $g : [0, c + d\omega] \to [0, 1]$, both elements of the poset X. We know by the theorem in the previous section than f and g are the linear interpolations of pairs of ω-regular subdivisions. So we have f interpolating C_1 (a subdivision of $[0, a+b\omega]$) to C_2 (a subdivision of $[0, 1]$), and likewise we have g interpolating D_1 (a subdivision of $[0, c + d\omega]$) to D_2 (a subdivision of $[0, 1]$) with all subdivisions ω-regular. Then the union $C_2 \cup D_2$ in $[0, 1]$ is not necessarily ω-regular, but there is a subdivision D' which contains the union and is ω-regular. Furthermore, we can arrange for the f and g preimages of D' in $[0, a + b\omega]$ and $[0, c + d\omega]$ respectively to also be ω-regular by taking finer subdivisions. So now we have three subdivisions (one each of $[0, a + b\omega]$, $[0, c + d\omega]$ and $[0, 1]$) all with the same number of points, say n. We can expand f and g by lengthening each of the n intervals in their respective subdivisions to length ω to obtain $h : [0, n\omega] \to [0, 1]$. And now we have that h is a common expansion of f and g so thus the poset is directed.

So we have that Y is contractible, and we look at the obvious action of the group $F(1, \mathbf{Z}[\sqrt{2}], \{\omega^i\})$ (which we will call F) on the poset X and thus on the classifying space Y. Given $f : [0, 1] \to [0, 1] \in F$ and $x : [0, a + b\omega] \to [0, 1] \in X$ we define $f(x)$ to be the composition $f \circ x : [0, a + b\omega] \to [0, 1]$. Thus $f(x)$ will have slopes in $\{\omega^i\}$ and only finitely many singularities, all lying in $\mathbf{Z}[\sqrt{2}]$. The action of F on the poset X preserves the partial order, since if we have x_1 and x_2 in X with $x_1 < x_2$ via a sequence of simple expansions $x_2 = x_1 \circ s_n \circ \cdots \circ s_1$, we will have $f(x_2) = f(x_1 \circ s_n \circ \cdots \circ s_1)$ and thus $f(x_1) < f(x_2)$.

Since the action of F on Y is free and Y is contractible we have that Y/F is a $K(F, 1)$. Now we need to look at the structure of this space to understand the finiteness properties of the group F.

Unfortunately, the $K(F,1)$ has infinitely many cells in each dimension. For example, there are infinitely many 0-cells. Each 0-cell corresponds to an equivalence class of maps. Even though all maps from $[0,1]$ to $[0,1]$ (satisfying the requirements about slopes and singularities) are equivalent $\bmod F$, there are maps from other interval lengths $[0, a+b\omega]$ which are not equivalent to any maps from the unit interval to itself. Similarly, there are infinitely many 1-cells $\bmod F$ since there is one for each pair of maps f, g with $f < g$. Of these, there are not too many "essential" cells with g a simple expansion of f and there are many cells where there may be many intermediate maps, $f < f_1 < f_2 < \cdots < g$. To separate the "redundant" cells from the "essential" cells which we can filter our $K(F,1)$ in the following way.

First we need to filter the space Y by height. For $y \in Y$, we let the height $h(y)$ be the length of the largest chain $y_0 < y_1 < \cdots < y_h = y$ for $y_i \in Y$. We can filter Y by height to obtain $Y_h = \{f \in Y | h(f) \leq h\}$. We would like to use a theorem of Brown [**2**].

Theorem 4.1 (Brown). *Let X be a contractible Γ-complex with a filtration $\{X_j\}$ such that each X_j is finite* $\bmod \Gamma$. *Then if the connectivity of the pair (X_{j+1}, X_j) tends to ∞ as j tends to ∞, Γ is finitely presented and of type FP_∞.*

To apply the theorem, we need to verify the condition on the connectivity of the filtration pairs, (Y_{h+1}, Y_h) through the following lemma.

Lemma 8. *Suppose $Y_1, \ldots, Y_k$ are distinct simple contractions of a poset element Y. Then $Y_1, \ldots, Y_k$ have a lower bound in the poset if and only if the intervals which disappear from Y in the contractions of the Y_i's are disjoint. If they do have a lower bound, then they have a greatest lower bound Z which is obtained by contracting all the intervals which disappear for some Y_i's contraction.*

Proof. $\Leftarrow$. If the intervals in Y which disappear during the simple contractions are disjoint, then we can contract all those intervals successively to obtain Z which is a lower bound for all the Y_i.

$\Rightarrow$. Suppose W is a lower bound for all the Y_i. So each Y_i is an expansion of W. Think of W as a forest of $a+b$ roots where W has size $a + b\omega$, and with each root labeled either 1 or ω.

W is a lower bound for each of the Y_i, so each of them can be obtained by a sequence of elementary expansions. Each of these simple expansions can be expressed as one of:

$$\begin{array}{c}1\\ \downarrow\\ \omega\end{array}, \quad \begin{array}{c}\omega\\ \downarrow\\ \omega\,\omega\,1\end{array}, \quad \begin{array}{c}\omega\\ \downarrow\\ \omega\,1\,\omega\end{array} \quad \text{or} \quad \begin{array}{c}\omega\\ \downarrow\\ 1\,\omega\,\omega\end{array}.$$

So we expand W by all the expansions needed to obtain all the Y_i then we expand that to obtain Y. From this, each contraction from Y to any Y_i is a pruning of one or three leaves. Thus, the intervals which get contracted in Y to get the Y_i are disjoint. Furthermore, $Z > W$ since Z is obtained by pruning only the intervals necessary to go from Y to Y_i. □

Thus we have the lemma needed to ensure the connectivity condition. See Brown [**2**] for the remaining part of showing that we now have the connectivity condition; the argument is now completely analogous to the cases described there (when the slope groups are rank one and generated by $1/n$).

Thus we know that the group $F(1, \mathbf{Z}[\sqrt{2}], \{\omega\})$ is finitely presented and of type FP_∞.

These same techniques work for certain other irrational slope groups; in particular, if we have an algebraic integer λ with a subdivision rule similar to the one for ω, the argument is exactly analogous. These cases include when we have

$$\begin{array}{c}l\\ \downarrow\\ s\,l\cdots l\end{array}$$

where the long is divided into n long intervals and one short interval.

In other words, whenever λ satisfies an equation of the form

$$1 = n\lambda + \lambda^2$$

for some $n > 1$, the argument is analogous for the group $F(1, \mathbf{Z}[\lambda], \{\lambda^i\})$. These solutions for λ are of the form $(-n \pm \sqrt{n^2+4})/2$. For n even, solutions for λ will give us groups with breakpoint sets of the form

$\mathbf{Z}[\sqrt{m^2+1}]$, for $m \in \mathbf{Z}^+$. For n odd, we get groups with breakpoint sets are of the form $\mathbf{Z}[\sqrt{4m^2+4m+5}/2]$, also with $m \in \mathbf{Z}^+$.

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