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A CHARACTERIZATION OF INNER PRODUCT SPACES BASED ON ORTHOGONAL RELATIONS RELATED TO HEIGHT'S THEOREM

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ABSTRACT. We study two orthogonal relations in a real normed space related to the height's theorem and some characterizations of inner product spaces are obtained.

Orthogonal relations in real normed spaces have been studied with some detail (see [**2**]) in relation with characterizations of inner product spaces and the study of orthogonal additive mappings (see [**1, 6, 7**]). The most classical orthogonal relation in a normed space $(E, || \, ||)$ is the Pythagorean relation $\perp^P$ defined through Pythagora's theorem:

$$(1) \qquad x \perp^P y \quad \text{if} \quad ||x+y||^2 = ||x||^2 + ||y||^2.$$

Let us note that in inner product spaces Pythagoras theorem is equivalent to the height's theorem: the height over the hypotenuse is the geometric mean of the two divisions of the hypotenuse determined by the foot of the height (i.e., the height divides the triangle into two homothetic pieces). This observation has motivated the formulation of another orthogonal relation $\perp^H$ alternative to (1), i.e., if x and y are the legs, $x-y$ the hypotenuse and $1/(||x||^2+||y||^2)(||y||^2x+||x||^2y)$ the foot of the height, we define

$$x \perp^H y \quad \text{if} \quad \left\| \frac{||y||^2 x + ||x||^2 y}{||x||^2+||y||^2} \right\| = \left[\left\| \frac{||y||^2(x-y)}{||x||^2+||y||^2} \right\| \cdot \left\| \frac{||x||^2(x-y)}{||x||^2+||y||^2} \right\| \right]^{1/2},$$

or, equivalently,

$$(2) \qquad x \perp^H y \quad \text{if} \quad ||x-y|| \, ||x|| \, ||y|| = || \, ||y||^2 x + ||x||^2 y||.$$

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For inner product spaces, (1) and (2) are equivalent to the usual orthogonality $\langle x, y\rangle = 0$, but in general in real normed spaces (1) and (2) have no connection. This is shown in the following

Example 1. Consider the real linear space $E = \{f \mid f : [0,1] \to \mathbf{R}, f$ is continuous$\}$ endowed with the norm $||f||_\infty = \sup\{|f(x)| \mid x \in [0,1]\}$. If we take $f(x) = x$ and $g(x) = 2x - (\sqrt{65}/4)x^2$, then $f \perp^P g$ but $f \not\perp^H g$, i.e., (2) does not hold when (1) is true. On the other hand, taking $f(x) = 1$ and $g(x) = -2kx + k$ we have that $f \perp^H g$ but $f \not\perp^P g$, i.e., (1) does not follow from (2).

This situation allows us to study relation (2) in some detail, and this is precisely the aim of the present note (made after the authors realized that (2) has not been analyzed previously). In what follows, $(E, ||\ ||)$ *will always be a real normed space and* $\dim E \geq 2$.

Note that, for $x, y \neq 0$, relation (2) may be reformulated in the form

$$(3) \qquad x \perp^H y \quad \text{if} \quad \frac{||x-y||}{||x||\,||y||} = \left\| \frac{x}{||x||^2} + \frac{y}{||y||^2} \right\|,$$

and, obviously, on the spheres, i.e., when $||x|| = ||y||$ then (3) reduces to the well-known James orthogonality: $x \# y$ whenever $||x-y|| = ||x+y||$ (following example 1, the consideration of $f(x) = x$ and $g(x) = x - x^2$ yields $f \# g$ but (2) does not hold; if we consider $f(x) = 1$ and $g(x) = (-14 - 8\sqrt{3})x^2 + (12 + 8\sqrt{3})x$, then $f \perp^H g$ but $f \# g$ is not true).

The next lemma contains some elementary facts concerning the relation $\perp^H$ which can be proved easily.

Lemma 1. *In a real normed space* $(E, ||\ ||)$ *with* $\dim E \geq 2$, *the relation* $\perp^H$ *defined by* (2) *satisfies the following conditions for all* x, y *in* E *and for all* a *in* $\mathbf{R}$:

(i) $0 \perp^H y$;

(ii) $x \perp^H y$ *if and only if* $y \perp^H x$;

(iii) *If* $x \perp^H y$, *then* $ax \perp^H ay$;

(iv) *If* $x \perp^H y$ *and* $x, y \neq 0$, *then* x *and* y *are linearly independent*;

(v) *If the norm derives from an inner product* $\langle , \rangle$ *then the relation* $x \perp^H y$ *is equivalent to the usual orthogonality* $\langle x, y\rangle = 0$;

(vi) $x \perp^H -y$, $x, y \neq 0$, *if and only if* $x/||x||^2 \perp^H y/||y||^2$.

Theorem 1. *For all x in E and for all $t > 0$, there exists y in E such that $||y|| = t$ and $x \perp^H y$.*

Proof. Fixed x in E and $t > 0$ define $f : \{z \in E \mid ||z|| = t\} \to \mathbf{R}$ by

$$f(z) = \left\| \frac{x}{||x||^2} + \frac{z}{t^2} \right\| - \frac{||x - z||}{||x|| t}.$$

Then f is a continuous function satisfying the property

$$f\left(t \frac{x}{||x||} \right) = -f\left(-t \frac{x}{||x||} \right)$$

and therefore it is immediate to show that there must exist y in E such that $||y|| = t$ and $f(y) = 0$, i.e., $x \perp^H y$. □

This result says, in particular, that the relation $\perp^H$ is not trivial in the sense that there are (always!) nonzero orthogonal elements.

Theorem 2. *For all x, y in $E \backslash \{0\}$ which are linearly independent, there exists a $t_0 > 0$ such that $x + t_0 y \perp^H x - t_0 y$.*

Proof. Define $g : [0, \infty] \to \mathbf{R}$ to be the function:

$$\begin{aligned} g(t) &= 2t||y|| \, ||x + ty|| \, ||x - ty|| \\ &\quad - || \, ||x + ty||^2 (x - ty) + ||x - ty||^2 (x + ty)||. \end{aligned}$$

Then g is continuous and $g(0) = -2||x||^2 < 0$. If we consider the function $G : (0, \infty) \to \mathbf{R}$ given by

$$G(t) = \frac{||x + ty|| \, ||x - ty||}{t} \left\| \frac{x - ty}{||x - ty||^2} + \frac{x + ty}{||x + ty||^2} \right\|,$$

then G is continuous, $\lim_{t \to 0+} G(t) = +\infty$ and

$$\begin{aligned} &\lim_{t \to +\infty} G(t) \\ &\quad = \lim_{t \to +\infty} \left\| \frac{1}{t} x + y \right\| \cdot \left\| \frac{1}{t} x - y \right\| \left\| \frac{(1/t)x - y}{||(1/t)x - y||^2} + \frac{(1/t)x + y}{||(1/t)x + y||^2} \right\| \\ &\quad = 0 < 2||y||. \end{aligned}$$

Thus there exists a $t_1 > 0$ such that $G(t_1) < 2||y||$, and for such t_1 we also have $g(t_1) > 0$. Therefore, there exists $t_0 > 0$ such that $g(t_0) = 0$, i.e., $x + t_0 y \perp^H x - t_0 y$. $\square$

Using the above result, we can exhibit an interesting characterization of inner product spaces.

Theorem 3. *Let* $(E, || \, ||)$ *be a real normed space with* $\dim E \geq 2$. *If, for all* $x, y \in E \backslash \{0\}$ *the following implication holds:*

$$x \perp^H y \quad \textit{implies} \quad x \perp^P y \quad \textit{and} \quad \frac{x}{||x||^2} \perp^P \frac{y}{||y||^2}, \tag{4}$$

then E *is an inner product space.*

Proof. Let x and y be two linearly independent vectors in E such that $||x|| = ||y|| = 1$. By Theorem 2, we know that there exists $t_0 > 0$ such that $x + t_0 y \perp^H x - t_0 y$. Thus, by (4) we have $x + t_0 y \perp^P x - t_0 y$ and $(x + t_0 y)/||x + t_0 y||^2 \perp^P (x - t_0 y)/||x - t_0 y||^2$, i.e.,

$$\begin{aligned} ||x + t_0 y||^2 + ||x - t_0 y||^2 &= ||x + t_0 y + x - t_0 y||^2 \\ &= 4||x||^2 = 4, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \frac{1}{||x + t_0 y||^2} + \frac{1}{||x - t_0 y||^2} &= \left\| \frac{x + t_0 y}{||x + t_0 y||^2} \right\|^2 + \left\| \frac{x - t_0 y}{||x - t_0 y||^2} \right\|^2 \\ &= \left\| \frac{x + t_0 y}{||x + t_0 y||^2} + \frac{x - t_0 y}{||x - t_0 y||^2} \right\|^2 \\ &= \frac{||x + t_0 y - (x - t_0 y)||^2}{||x + t_0 y||^2 ||x - t_0 y||^2} \end{aligned}$$

whence

$$4 = ||x + t_0 y||^2 + ||x - t_0 y||^2 = 4t_0^2 ||y||^2 = 4t_0^2,$$

i.e., $t_0 = 1$ and therefore, by (5),

$$||x + y||^2 + ||x - y||^2 = 4. \tag{6}$$

Since it is well known (see [**2**]) that this version of the parallelogram law implies that $||\cdot||$ comes from an inner product, the proof is complete. □

Let us turn our attention to another orthogonal relation related to the heights theorem.

Define $h(x,y) := y + (\rho'_+(y-x,y))(x-y)/||x-y||^2$, where $\rho'_\pm(a,b) = \lim_{t\to 0^\pm}(||a+tb||^2 - ||a||^2)/2t$.

When the norms come from an inner product, $\langle,\rangle$, then $\rho'_\pm = \langle,\rangle$ and h is the usual height. Thus, we now define

$$x \perp_H y \quad \text{if} \quad ||h(x,y)||^2 = ||x - h(x,y)||\,||y - h(x,y)||,$$

or equivalently,

$$x \perp_H y \quad \text{if} \quad \left\| y + \frac{\rho'_+(y-x,y)}{||x-y||^2}(x-y)\right\|^2 = \left|1 - \frac{\rho'_+(y-x,y)}{||x-y||^2}\right| |\rho'_+(y-x,y)|.$$

In a real normed space $(E, ||\ ||)$ with $\dim E \geq 2$, the relation $\perp_H$ satisfies the following conditions for all x, y in E and for all a in $\mathbf{R}$

(i) $0 \perp_H y$ and $x \perp_H 0$.

(ii) If $x \perp_H y$, then $ax \perp_H ay$.

(iii) If $x \perp_H y$ and $x, y \neq 0$, then x and y are linearly independent.

(iv) For all x, y in $E\backslash\{0\}$ which are linearly independent, there exists $t_0 > 0$ such that $x + t_0 y \perp_H x - t_0 y$.

Concerning this last condition, note that a positive number t_0 verifies $x + t_0 y \perp_H x - t_0 y$ if and only if

$$\left\|x - \frac{\rho'_-(y,x)}{||y||^2}y\right\|^2 = ||y||^2 \left|t_0^2 - \frac{\rho'_-(y,x)^2}{||y||^4}\right|.$$

Thus, we can take

$$t_0 = \left(\left\|x - \frac{\rho'_-(y,x)}{||y||^2}y\right\|^2 + \frac{\rho'_-(y,x)^2}{||y||^2}\right)^{1/2}\frac{1}{||y||}.$$

Theorem 4. *Let* $(E, || \ ||)$ *be a real normed space with* $\dim E \geq 2$. *If, for all* x, y *in* $E \backslash \{0\}$

$$\left(x + \frac{||x||}{||y||} y\right) \perp_H \left(x - \frac{||x||}{||y||} y\right),$$

then E *is an inner product space.*

Proof. It is immediate from the assumptions that

$$\begin{aligned}\left\|x - \frac{\rho'_-(y,x)}{||y||^2} y\right\|^2 &= ||y||^2 \left\|\frac{||x||^2}{||y||^2} - \frac{\rho'_-(y,x)^2}{||y||^4}\right\| \\ &= ||x||^2 - \frac{\rho'_-(y,x)^2}{||y||^2}\end{aligned}$$

for all x, y in $E \backslash \{0\}$.

If we substitute x by $x + ty$ where t is in $\mathbf{R}^+$, we obtain

$$||x + ty||^2 - ||x||^2 = t^2 ||y||^2 + 2t\rho'_-(y,x)$$

and

$$\frac{||x + ty||^2 - ||x||^2}{2t} = \frac{t||y||^2}{2} + \rho'_-(y,x).$$

When t tends to 0, we have the equality $\rho'_+(x,y) = \rho'_-(y,x)$.

Therefore, $\rho'_+(x,y) \leq \rho'_+(y,x)$ for all x, y in $E \backslash \{0\}$. This fact implies that $\rho'_+(x,y) = \rho'_+(y,x)$ and (see [**2**]) E is an inner product space. □

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