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ALMOST FLAT ABELIAN GROUPS

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1. Introduction. While investigating the relationship between group-theoretic properties of an abelian group A and ring-theoretic properties of its endomorphism ring, $E(A)$, it often becomes necessary to impose restrictions on A to avoid obvious counter-examples. This is frequently done by considering abelian groups A which, as a left $E(A)$-module, belong to a given class $\mathcal{C}$ of modules. Perhaps the most frequently used choices for $\mathcal{C}$ are the classes of cyclic, finitely generated, projective, or flat left $E(A)$-modules. Although all these choices yield interesting results, only the condition that A is a flat $E(A)$-module does not impose immediate restrictions on the ring-structure of $E(A)$, see, e.g., [**2**] and [**4**]. R.S. Pierce emphasized the importance of the flatness condition on A in a talk, which he gave in 1989 at the University of Connecticut, and raised several questions concerning these groups [**8**]: Is the class of torsion-free groups which are flat as $E(A)$-module closed with respect to quasi- or near-isomorphism? What is the relationship between the flatness of A over $E(A)$, and the flatness of $\mathbf{Q}A = \mathbf{Q} \otimes_{\mathbf{Z}} A$ as a $\mathbf{Q} \otimes_{\mathbf{Z}} E(A)$-module?

It is the goal of this paper to give answers to Pierce's questions. Example 2.9 shows that the class of abelian groups A which are flat as $E(A)$-modules need not be closed under quasi-isomorphism. This observation leads to the introduction of a new class of abelian groups. We say that A is *almost flat as an $E(A)$-module* if $\operatorname{Tor}^1_{E(A)}(M, A)$ is a bounded abelian group for all right $E(A)$-modules M. Theorem 2.4 and Corollary 2.5 establish that the class of torsion-free abelian groups, which are almost flat over their endomorphism ring, is closed under quasi-isomorphism. Furthermore, the arguments used in the proof of Corollary 2.5 can be adopted to show that the class of torsion-free abelian groups, which are flat as $E(A)$-modules, is closed under near-isomorphism (Corollary 2.7).

It remains to investigate how the class of abelian groups A, which are

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almost flat as $E(A)$-modules, is related to the class of groups considered in Pierce's second question. We address this problem in Section 3 for torsion-free groups of finite rank. Its main result gives a description of almost $E(A)$-flat abelian groups in terms of the structure of the groups A/NA and $\mathrm{Tor}\,^1_{E(A)}(E(A)/N, A)$, where N is the nilradical of $E(A)$ (Theorem 3.4). This characterization allows one to construct an example of a torsion-free group A of finite rank, which is not almost flat over $E(A)$, but has the property that $\mathbf{Q}\otimes_{\mathbf{Z}} A$ is a flat $\mathbf{Q}\otimes_{\mathbf{Z}} E(A)$-module.

2. Almost flat R-modules. A left R-module A is almost flat if $\mathrm{Tor}\,^1_R(M, A)$ is a bounded group for all right R-modules M.

Proposition 2.1. *The following conditions are equivalent for a left R-module A:*

a) *A is almost flat.*

b) *There exists a nonzero integer n with $n\mathrm{Tor}\,^1_R(M, A) = 0$ for all right R-modules M.*

Proof. It remains to show that a) implies b). Suppose that condition b) fails. For each right R-module M choose a smallest positive integer n_M with $n_M\mathrm{Tor}\,^1_R(M, A) = 0$. Since b) fails, $\sup\{n_M \mid M \text{ is a right } R\text{-module}\} = \infty$. For each positive integer k, we can find a right R-module M_k with $n_{M_k} \geq k$. Since $\mathrm{Tor}\,^1_R$ commutes with direct sums, $\mathrm{Tor}\,^1_R(\oplus_{k=1}^{\infty} M_k, A)$ cannot be a bounded abelian group, which is not possibly by a). □

Before we turn to the main results of this section, we give two lemmas which establish some technical, but nevertheless very useful properties of almost flat modules. We call two R-modules M and N *quasi-isomorphic* if there are R-module-morphisms $\alpha : M \to N$ and $\beta : N \to M$ and a nonzero integer m such that $\alpha\beta = m\mathrm{id}_N$ and $\beta\alpha = m\mathrm{id}_M$.

Lemma 2.2. *Let A and A' be quasi-isomorphic left R-modules.*

a) *If A is almost flat, then so is A'.*

b) *If* $A = B \oplus C$ *is almost flat, then so is* B.

c) *The following conditions are equivalent*:

i) A *is almost flat.*

ii) $\ker(\alpha \otimes \mathrm{id}_A)$ *is a bounded group for all monomorphisms* $\alpha : M \to N$ *of right* R*-modules.*

Proof. a) Suppose $\alpha : A \to A'$ and $\beta : A' \to A$ are R-module morphisms with $\alpha\beta = m\mathrm{id}_{A'}$ and $\beta\alpha = m\mathrm{id}_A$ for some nonzero integer m. Let M be a right R-module, and observe that the induced map $\alpha_* : \mathrm{Tor}^1_R(M, A) \to \mathrm{Tor}^1_R(M, A')$ satisfies $m\mathrm{Tor}^1_R(M, A') \subseteq \alpha_*(\mathrm{Tor}^1_R(M, A))$. Since $\mathrm{Tor}^1_R(M, A)$ is bounded, we obtain that $\mathrm{Tor}^1_R(M, A')$ is bounded too.

b) Since $\mathrm{Tor}^1_R(M, -)$ commutes with direct sums, a bound for $\mathrm{Tor}^1_R(-, A)$ is one for $\mathrm{Tor}^1_R(-, B)$ too.

c) i) $\Rightarrow$ ii). An exact sequence $0 \to M \stackrel{\alpha}{\to} N$ induces an exact sequence $\mathrm{Tor}^1_R(N/\alpha(M), A) \stackrel{\Delta}{\to} M \otimes_R A \stackrel{\alpha\otimes\mathrm{id}_A}{\to} N \otimes_R A$. Since the group $\mathrm{Tor}^1_R(N/\alpha(M), A)$ is bounded by some positive integer n, the same n is a bound for $\ker(\alpha \otimes \mathrm{id}_A) = \mathrm{im}\Delta$.

ii) $\Rightarrow$ i). Let M be a right R-module and F a free right R-module such that $M \cong F/U$ for some submodule U of F. Then $\mathrm{Tor}^1_R(M, A)$ is bounded since it fits in an exact sequence $0 \to \mathrm{Tor}^1_R(M, A) \to U \otimes_R A \to F \otimes_R A$. □

Lemma 2.3. *Let* S *be a ring and* R *a subring of* S *such that* S/R *is bounded as an abelian group by some nonzero integer* n*. The kernel of the natural epimorphism* $\sigma_{M,L} : M \otimes_R L \to M \otimes_S L$ *is bounded by* n^2 *for all right* S*-modules* M *and left* S*-modules* L*.*

Proof. Define a map $\tau : M \otimes_S L \to M \otimes_R L$ by $\tau(x \otimes y) = (xn) \otimes (ny)$ for all $x \in M$ and $y \in L$. Since $\tau\sigma_{M,L} = n^2\mathrm{id}_{M\otimes_R L}$, we obtain $n^2\ker\sigma_{M,L} = 0$. □

We are now able to prove the main result of this section:

Theorem 2.4. *Let $R \subseteq S$ be quasi-equal rings whose additive groups are torsion-free. The following conditions are equivalent for a left S-module A:*

a) *A is almost flat over S.*

b) *A is almost flat over R.*

Proof. a) $\Rightarrow$ b). Let n be a nonzero integer with $nS \subseteq R$, and denote the inclusion $R \subseteq S$ by ι. Suppose that A is almost flat as an S-module, and consider a right R-module M. We choose a projective resolution $0 \to U \stackrel{\alpha}{\to} P \stackrel{\beta}{\to} M \to 0$ of M. It induces the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & P \otimes_R R & \xrightarrow{\mathrm{id}_{P \otimes \iota}} & P \otimes_R S & \longrightarrow & P \otimes_R (S/R) \\
 & & \uparrow{\scriptstyle \alpha \otimes \mathrm{id}_R} & & \uparrow{\scriptstyle \alpha \otimes \mathrm{id}_S} & & \\
 & & U \otimes_R R & \xrightarrow{\mathrm{id}_U \otimes \iota} & U \otimes_R S & \longrightarrow & U \otimes_R (S/R) \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & \mathrm{Tor}\,^1_R(M,S) & &
\end{array}$$

whose rows and columns are exact. Since $\mathrm{Tor}\,^1_R(P,S) = 0$, we have $\ker(\alpha \otimes \mathrm{id}_S) \cong \mathrm{Tor}\,^1_R(M,S)$. We observe that $n\ker(\alpha \otimes \mathrm{id}_S) = 0$ in the following way: If $x \in \ker(\alpha \otimes \mathrm{id}_S)$, then $nx = (\mathrm{id}_U \otimes \iota)(y)$ for some $y \in U \otimes_R R$ since $n(S/R) = 0$. Consequently, $0 = (\alpha \otimes \mathrm{id}_S)(\mathrm{id}_U \otimes \iota)(y) = (\mathrm{id}_P \otimes \iota)(\alpha \otimes \mathrm{id}_R)(y)$ yields $y = 0$.

The given projective resolution of M induces a projective resolution $0 \to V \stackrel{\sigma}{\to} P \otimes_R S \stackrel{\beta \otimes \mathrm{id}_S}{\to} M \otimes_R S \to 0$ of the right S-module $M \otimes_R S$ in which $V = \mathrm{im}\,(\alpha \otimes \mathrm{id}_S) = \ker(\beta \otimes \mathrm{id}_S)$ and $\sigma : V \to P \otimes_R S$ is the inclusion map. We apply the functor $- \otimes_S A$ to the last sequence and $- \otimes_R A$ to the given projective resolution of M to obtain the rows of the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Tor}\,^1_S(M \otimes_R S, A) & \longrightarrow & V \otimes_S A & \xrightarrow{\sigma \otimes \mathrm{id}_A} & P \otimes_R S \otimes_S A \\
 & & & & \uparrow{\scriptstyle (\alpha \otimes \mathrm{id}_S \otimes \mathrm{id}_A)\lambda^U} & & {\scriptstyle \lambda_P}\uparrow{\scriptstyle l} \\
0 & \longrightarrow & \mathrm{Tor}\,^1_R(M,A) & \longrightarrow & U \otimes_R A & \xrightarrow{(\alpha \otimes \mathrm{id}_A)} & P \otimes_R A
\end{array}$$

in which the maps λ_U and λ_P are induced by the natural equivalence $\lambda : - \otimes_R A \to - \otimes_R S \otimes_S A$, and $\alpha \otimes \mathrm{id}_S \otimes \mathrm{id}_A$ fits into the exact sequence

$$\mathrm{Tor}\,^1_R(M,S) \otimes_S A \to U \otimes_R S \otimes_S A \overset{\alpha\otimes\mathrm{id}_S\otimes\mathrm{id}_R}{\longrightarrow} V \otimes_S A \longrightarrow 0.$$

Since λ_U is an isomorphism, the kernel of the first vertical map is bounded by n by what has been shown so far. Furthermore, the group $\mathrm{Tor}\,^1_S(M \otimes_R S, A)$ is bounded by a). A standard diagram chase establishes that $\mathrm{Tor}\,^1_R(M,A)$ is a bounded abelian group.

b) $\Rightarrow$ a). Assume that A is almost flat as an R-module. Consider an exact sequence $0 \to M \overset{\alpha}{\to} N$ of right R-modules. It induces the commutative diagram

$$\begin{array}{ccc} M \otimes_S A & \xrightarrow{\alpha\otimes_S \mathrm{id}_A} & N \otimes_S A \\ \uparrow{\scriptstyle \sigma_{M,A}} & & \uparrow{\scriptstyle \sigma_{N,A}} \\ M \otimes_R A & \xrightarrow{\alpha\otimes_R \mathrm{id}_A} & N \otimes_R A \end{array}$$

in which $\ker(\alpha \otimes_R \mathrm{id}_A)$ is bounded by our assumption. Lemma 2.3 yields that the vertical maps are epimorphisms with bounded kernels. A diagram chase shows that $\alpha \otimes_S \mathrm{id}_A$ has a bounded kernel too. □

Corollary 2.5. *The class of torsion-free abelian groups which are almost flat as modules over their endomorphism ring is closed under quasi-isomorphism.*

Proof. Suppose that A is almost flat as an $E(A)$-module. Let B be a subgroup of A such that $nA \subseteq B$ for some nonzero integer n. Since $E(A)$ and $E(B)$ are quasi-equal rings, they are quasi-equal to $R = E(A) \cap E(B)$. By Theorem 2.4, B is almost flat over $E(B)$ if and only if B is almost flat over R. Lemma 2.2 yields that this holds if A is almost flat as an R-module. Another application of Theorem 2.4 completes the proof. □

Corollary 2.6. *Let R be a semi-prime ring whose additive group is torsion-free of finite rank. Every R-module with a torsion-free additive group is almost flat.*

Proof. Let M be a torsion-free R-module. The ring $\mathbf{Q}R$ contains a hereditary R-order $\overline{R} \supseteq R$. Since R and $\overline{R}$ are quasi-equal, $\mathbf{Q}M$ is an $\overline{R}$-module. We consider the $\overline{R}$-submodule $\overline{M}$ of $\mathbf{Q}M$ which is generated by M. Since M and $\overline{M}$ are quasi-equal R-modules, it suffices to show that $\overline{M}$ is almost flat as an R-module by Lemma 2.2.

In view of Theorem 2.4, it is enough to show that $\overline{M}$ is almost flat as an $\overline{R}$-module. But $\overline{R}$ is a semi-prime, two-sided Noetherian, hereditary ring, and $\overline{M}$ is a nonsingular $\overline{R}$-module. Observe that finitely generated nonsingular $\overline{R}$-modules are projective by [**1**, Theorem 4.6]. Since R is a semi-prime ring whose additive group is torsion-free of finite rank, a right R-module is nonsingular if and only if its additive group is torsion-free. Therefore, every finitely generated submodule of $\overline{M}$ is projective, and $\overline{M}$ is flat. □

Recall that torsion-free groups A and B are said to be *nearly isomorphic* if, for each nonzero integer m, there are a nonzero integer n relatively prime to m and maps $f : A \to B$ and $g : B \to A$ with $fg = n\mathrm{id}_B$ and $gf = n\mathrm{id}_A$.

Corollary 2.7. *Let A be a torsion-free abelian group which is flat as an $E(A)$-module. If B is nearly isomorphic to A, then B is flat as an $E(B)$-module.*

Proof. By Corollary 2.5, we know that B is almost flat as an $E(B)$-module. By Proposition 2.1, there is a positive integer m with $m\mathrm{Tor}\,^1_{E(B)}(-, B) = 0$. Then there is a nonzero integer n relatively prime to m and maps $f : A \to B$ and $g : B \to A$ such that $gf = n\mathrm{id}_A$ and $fg = n\mathrm{id}_B$. Observing that $f(A) \supseteq f(g(B)) = nB$, we identify A with $f(A)$ to obtain $nB \subseteq A \subseteq B$. Then, $n^2E(A)$, $n^2E(B) \subseteq R = E(A) \cap E(B)$. Let M be an arbitrary right R-module. Looking at the proof of a) $\Rightarrow$ b) of Theorem 2.4, we see that some power of n will annihilate $\mathrm{Tor}\,^1_R(M, A)$. This holds in particular in the case where M is a right $E(B)$-module. The arguments used in the proof of b) $\Rightarrow$ a) of Theorem 2.4 yield that $\mathrm{Tor}\,^1_{E(B)}(M, B)$ is annihilated by some power of n. Since this group is also annihilated by m, and $(n, m) = 1$, we have $\mathrm{Tor}\,^1_{E(B)}(M, B) = 0$. □

The next result shows that an abelian group, which is almost flat as a module over its endomorphism ring, becomes a flat module after localization at a finite set of primes. For a nonzero integer n, let $\mathbf{Z}[n^\infty]$ be the polynomial ring in one variable over $\mathbf{Z}$ evaluated at $1/n$.

We set $A[n^\infty] = \mathbf{Z}[n^\infty] \otimes_{\mathbf{Z}} A$.

Corollary 2.8. *Let A be a torsion-free abelian group which is almost flat as an $E(A)$-module. Then $A[n^\infty]$ is a flat $E(A)$-module for some nonzero integer n.*

Proof. By Proposition 2.1, there is a nonzero integer n such that $n\mathrm{Tor}^1(M, A) = 0$ for all right $E(A)$-modules M. Since $A[n^\infty]$ is the direct limit of $E(A)$-modules isomorphic to A, we obtain that $\mathrm{Tor}^1(M, A[n^\infty]) \cong \varinjlim \mathrm{Tor}^1(M, A)$ is bounded by n too. On the other hand, since multiplication by n is an automorphism of $A[n^\infty]$, it induces an automorphism of $\mathrm{Tor}^1(M, A[n^\infty])$. This results in a contradiction, unless $\mathrm{Tor}^1(M, A[n^\infty]) = 0$. □

We conclude this section with an example.

Example 2.9. The class of torsion-free abelian groups which are flat as modules over their endomorphism ring is not closed under quasi-isomorphism.

Proof. Let R be the subring $\mathbf{Z}_2 + 2i\mathbf{Z}_2$ of the complex numbers. Since R is not hereditary (observe that $J(R) = 2\mathbf{Z}_2 + 2i\mathbf{Z}_2$ is not projective since it is a nonprincipal ideal in a local domain), there exists a torsion-free group A with $E(A) = R$ which is not flat as an $E(A)$-module [**4**]. The ring $\overline{R} = \mathbf{Z}_2 + i\mathbf{Z}_2$ is a maximal R-order; and $\overline{A} = \overline{R}A \subseteq \mathbf{Q}A$ is a torsion-free abelian group whose endomorphism ring is $\overline{R}$. Since R and $\overline{R}$ are quasi-equal, the same holds for A and $\overline{A}$. Finally, $\overline{A}$ is flat as an $E(\overline{A})$-module since $\overline{R}$ is hereditary. □

3. The finite rank case. The nilradical of a ring R is denoted by $N(R)$. When no confusion is possible, we just write N. A sequence $0 \to U \overset{\alpha}{\to} M \overset{\beta}{\to} K \to 0$ of right R-modules is *N-pure* if

$\alpha(U) \cap MN = \alpha(U)N$. This is equivalent to the condition that the induced sequence $0 \to U/UN \stackrel{\bar{\alpha}}{\to} M/MN \stackrel{\bar{\beta}}{\to} K/KN \to 0$ is exact. We say that a left R-module L is almost flat with respect to a class $\mathcal{C}$ of exact sequences if there is a nonzero integer m which uniformly bounds $\ker(U \otimes_R L \to M \otimes_R L)$ for all exact sequences $0 \to U \to M \to K \to 0$ whose entries are in $\mathcal{C}$. If U is a subgroup of a torsion-free group G, then U_* denotes the $\mathbf{Z}$-purification of U in G. Finally, we write T_A for the functor $- \otimes_{E(A)} A$.

Proposition 3.1. *Let A be a torsion-free abelian whose endomorphism ring has finite rank. The left $E(A)$-module $A/(NA)_*$ is almost flat with respect to N-pure exact sequences.*

Proof. To simplify the notation, we write $\overline{A}$ for $A/(NA)_*$. For a right $E(A)$-module M, we shall use the symbol $\overline{M}$ to represent M/MN. Since $\overline{A}$ is annihilated by N, for every right $E(A)$-module M, there exists a natural transformation $\sigma_M : M \otimes_{E(A)} \overline{A} \to \overline{M} \otimes_{\overline{E(A)}} \overline{A}$ which is defined by $\sigma_M(m \otimes (a + (NA)_*)) = (m + MN) \otimes (a + (NA)_*)$ for all $m \in M$ and $a \in A$. Its inverse $\overline{\tau_M}$ is induced by the map $\tau_M : \overline{M} \times \overline{A} \to M \otimes_{E(A)} \overline{A}$ given by the rule $\tau_M(m+MN, a+(NA)_*) = m \otimes (a+(NA)_*)$.

We now consider an N-pure exact sequence $0 \to U \to M \to K \to 0$ which induces the following commutative diagram whose vertical isomorphisms are given by the equivalence defined in the first paragraph of this proof:

$$\begin{array}{ccccccc}
U \otimes_{E(A)} \overline{A} & \longrightarrow & M \otimes_{E(A)} \overline{A} & \longrightarrow & K \otimes_{E(A)} \overline{A} & \longrightarrow & 0 \\
\wr\big\downarrow \sigma_U & & \wr\big\downarrow \sigma_M & & \wr\big\downarrow \sigma_K & & \\
\overline{U} \otimes_{\overline{E(A)}} \overline{A} & \longrightarrow & \overline{M} \otimes_{\overline{E(A)}} \overline{A} & \longrightarrow & \overline{K} \otimes_{\overline{E(A)}} \overline{A} & \longrightarrow & 0
\end{array}$$

Since $E(A)/N$ is a semi-prime ring, $\overline{A}$ is an almost flat $E(A)/N$-module by Corollary 2.6. Hence, there is a nonzero integer m, which is independent of the chosen sequence, such that the kernel of the first map in the bottom row of the diagram is bounded by m. Since the vertical maps are isomorphisms, it follows that the kernel of the first map of the top row also is bounded by m. $\square$

Corollary 3.2. *Let A be a torsion-free abelian group such that NA is pure in A and A/NA is a flat $E(A)/N$-module. Then the functor $-\otimes_{E(A)} A/NA$ preserves N-pure exact sequences.*

Proof. The corollary immediately follows from the commutative diagram considered in the proof of Proposition 3.1. □

We say that an abelian group A is *almost flat with respect to right ideals* if there exists a nonzero integer n such that $n\mathrm{Tor}^1(E(A)/I, A) = 0$ for all right ideals I. It is easy to see that a torsion-free group A is almost flat with respect to right ideals if and only if there is a nonzero integer n such that the natural map $n(I \otimes_{E(A)} A) \to E(A) \otimes_{E(A)} A$ is a monomorphism for all right ideals I. The arguments in the proof of Corollary 2.5 can be used to show that the class of abelian groups, which are almost flat with respect to ideals, is closed under quasi-isomorphism.

Lemma 3.3. *The following conditions are equivalent for a torsion-free abelian group A of finite rank and a nonzero integer n:*

a) *$n\mathrm{Tor}^1(E(A)/I, A) = 0$ for all right ideals I of $E(A)$.*

b) *For all $f_1, \dots, f_m \in E(A)$ and $a_1, \dots, a_m \in A$ with $\sum_{i=1}^m f_i(a_i) = 0$, there are $g_{ij} \in E(A)$ and $b_j \in A$ such that $\sum_i f_i g_{ij} = 0$ for all indices j and $\sum_j g_{ij}(b_j) = na_i$ for all i.*

Proof. a) $\Rightarrow$ b). Suppose that $\sum_{i=1}^m f_i(a_i) = 0$ in A, and let I be the right ideal of $E(A)$ generated by $f_1, \dots, f_m$. We choose a free $E(A)$-module F with basis $\{x_1, \dots, x_m\}$, and consider a map $\phi : F \to I$ which is defined by $\phi(\sum_{i=1}^M x_i r_i) = \sum_{i=1}^m f_i r_i$. Since $\phi(\sum_{i=1}^m x_i \otimes a_i) \in \ker(I \otimes_{E(A)} A \to IA)$, we obtain $n\sum_{i=1}^m f_i \otimes a_i = 0$ in $T_A(I)$. Let $K = \ker \phi$, and consider the induced sequence $T_A(K) \to T_A(F) \to T_A(I) \to 0$ of abelian groups. Since $n\sum_{i=1}^m x_i \otimes a_i \in \ker T_A(\phi)$, there are elements $y_j = \sum_{i=1}^m x_i g_{ij} \in K$, where $g_{ij} \in E(A)$ for all i and j, and $b_1, \dots, b_r \in A$ with the property that $n\sum_{i=1}^m x_i \otimes a_i = \sum_{j=1}^r y \otimes b_j = \sum_{i=1}^m x_i \otimes [\sum_{j=1}^r g_{ij}(b_j)]$. This shows $na_i = \sum_{j=1}^r g_{ij}(b_j)$ for all i, and $\sum_{i=1}^m f_i g_{ij} = 0$ for all j by the definition of K.

b) $\Rightarrow$ a). Let I be a right ideal of $E(A)$, and choose $\sum_{i=1}^m f_i \otimes a_i \in$

$\ker(I \otimes_{E(A)} A \to IA)$. Since $\sum_{i=1}^m f_i(a_i) = 0$, there are $g_{ij} \in E(A)$ and $b_j \in A$ as described in b). We obtain

$$n\sum_{i=1}^m f_i \otimes a_i = \sum_{i=1}^m f_i \otimes \left[\sum_{j=1}^r g_{ij}(b_j)\right] = \sum_{j=1}^r \left[\sum_{i=1}^m f_i g_{ij}\right] \otimes b_j = 0.$$

Thus, $n\mathrm{Tor}^1(E(A)/I, A) = 0$. □

Suppose that A and B are quasi-isomorphic torsion-free groups and $(N(E(A))A)_*/N(E(A))A$ is bounded. We may assume that A and B are full subgroups of a $\mathbf{Q}$-vector space V such that $nA \subseteq B$ and $nB \subseteq A$ for some nonzero integer n. Then $R = E(A) \cap E(B)$ is quasi-equal to $E(A)$ and $E(B)$. Thus, $N(E(A))A \doteq N(R)A \doteq N(R)B \doteq N(E(B))B$, and the group $(N(E(B))B)_*/N(E(B))B$ is bounded.

Theorem 3.4. *The following conditions are equivalent for a torsion-free abelian group A whose endomorphism ring has finite ranks*:

a) *A is almost flat as an $E(A)$-module.*

b) *There exists a nonzero integer n such that $n\mathrm{Tor}^1(E(A)/I, A) = 0$ for all right ideals I of $E(A)$.*

c) i) *$(NA)_*/NA$ is bounded.*

ii) *$\mathrm{Tor}^1(E(A)/N, A)$ is bounded.*

Proof. The implication a) ⇒ b) is clear by Proposition 2.1.

b) ⇒ c). It remains to verify part i) of c). Let n be a nonzero integer with $n\mathrm{Tor}^1(E(A)/I, A) = 0$ for all right ideals I of $E(A)$. By the remarks which precede the theorem, we may assume $A = A_1^{n_1} \oplus \cdots \oplus A_r^{n_r}$ where the groups A_i are pairwise nonquasi-isomorphic and strongly indecomposable. We set $B_i = A_i^{n_i}$ and $N_i = N(E(B_i))$. Then, $N = [\oplus_{i=1}^r N_i] \oplus [\oplus_{i \neq j} \mathrm{Hom}(B_i, B_j)]$ as in the proof of [**3**, Theorem 9.10]. Since NA is a fully invariant subgroup of A, we have $NA = C_1 \oplus \cdots \oplus C_r$ for $C_i = B_i \cap NA$. If we have shown that the groups $(C_i)_*/C_i$ are bounded by n for each i, then $A/NA = \oplus_{i=1}^r [B_i/C_i]$ has its torsion-subgroup bounded by n. No generality is lost if we assume $i = 1$. Let

$b \in B_1$ with $kb \in C_1$ for some nonzero integer k. Then $kb = \sum_{i=1}^t f_i(a_i)$ where $f_i \in \operatorname{Hom}(B_{s_i}, B_1)$ and $a_i \in A$ for suitably chosen indices s_i. Moreover, $f_i \in N_i$ if $s_i = 1$. Set $f_0 = k\mathrm{id}_A$ and $a_0 = b$. By Lemma 3.3, we can find $g_{ij} \in E(A)$ and $y_j \in A$ such that $f_0 g_{0j} - \sum_{i=1}^t f_i g_{ij} = 0$ for all j and $na_i = \sum_j g_{ij}(y_j)$ for $i = 0, \dots, t$. Since the f_i's are elements of N, we have $kg_{0j} = \sum_{i=1}^t f_i g_{ij} \in N$. But N is a pure ideal of $E(A)$, so that $g_{0j} \in N$ for all j. Therefore, $nb = \sum_j g_{0j}(y_j) \in NA \cap B_1 = C_1$. This shows $n(C_1)_* \subseteq C_1$.

c) $\Rightarrow$ a). By Proposition 3.1, $A/(NA)_*$ has the almost flatness property with respect to N-pure sequences. Let m be a nonzero integer arising this way, and choose nonzero integers k and n with $k\mathrm{Tor}^1(E(A)/N, A) = 0$ and $n[(NA)_*/NA] = 0$. We now show that nmk is a bound for $\mathrm{Tor}^1(M, A)$ for all $E(A)$-modules M with $MN = 0$. Consider M as an $E(A)/N$-module, and choose an exact sequence $0 \to U \overset{\alpha}{\to} F \overset{\beta}{\to} M \to 0$ of $E(A)/N$-modules where F is free. As $E(A)$-modules, $FN = 0$ and $FN \cap U = 0 = UN$. Thus, the sequence is N-pure. It yields the commutative diagram

$$\begin{array}{ccccc} U \otimes_{E(A)} (NA)_*/NA & \xrightarrow{\mathrm{id}_U \otimes \sigma} & U \otimes_{E(A)} A/NA & \xrightarrow{\mathrm{id}_U \otimes \pi} & U \otimes_{E(A)} A/(NA)_* \\ & & \Big\downarrow{\scriptstyle \alpha \otimes \mathrm{id}_{A/NA}} & & \Big\downarrow{\scriptstyle \alpha \otimes \mathrm{id}_{A/(NA)_*}} \\ & & F \otimes_{E(A)} A/NA & \xrightarrow{\mathrm{id}_F \otimes \pi} & F \otimes_{E(A)} A/(NA)_* \end{array}$$

in which the rows are induced by the exact sequence $0 \to (NA)_*/NA \overset{\sigma}{\to} A/NA \overset{\pi}{\to} A/(NA)_* \to 0$ of left $E(A)$-modules.

For every $x \in \ker(\alpha \otimes \mathrm{id}_{A/NA})$, we obtain

$$(\alpha \otimes \mathrm{id}_{A/(NA)_*})(\mathrm{id}_U \otimes \pi)(x) = (\mathrm{id}_F \otimes \pi)(\alpha \otimes \mathrm{id}_{A/NA})(x) = 0.$$

Since $\ker(\alpha \otimes \mathrm{id}_{A/(NA)_*})$ is bounded by m, we have $mx \in \ker(\mathrm{id}_U \otimes \pi) = \mathrm{im}(\mathrm{id}_U \otimes \sigma)$. However, $U \otimes_{E(A)} [(NA)_*/NA]$ is bounded by n, and hence $(nm)x = 0$. Furthermore, the exact sequence $0 \to NA \overset{\lambda}{\to} A \overset{\tau}{\to} A/NA \to 0$ of left $E(A)$-modules induces the exact sequence $Y \otimes_{E(A)} NA \overset{\mathrm{id}_Y \otimes \lambda}{\to} Y \otimes_{E(A)} A \overset{\mathrm{id}_Y \otimes \tau}{\to} Y \otimes_{E(A)} A/NA \to 0$ for all right $E(A)$-modules Y. If $YN = 0$, then we obtain $(\mathrm{id}_Y \otimes \lambda)(y \otimes sa) = y \otimes sa = ys \otimes a = 0$ for all elements $y \in Y$, $s \in N$, and $a \in A$. Hence, $\mathrm{id}_Y \otimes \tau$ is an isomorphism. We denote the kernel of $\alpha \otimes \mathrm{id}_A$ by X and consider the commutative diagram

$$\begin{array}{ccccccc} & & & & U \otimes_{E(A)} A/NA & \xrightarrow{\alpha\otimes \mathrm{id}_{A/NA}} & F \otimes_{E(A)} A/NA \\ & & & & l \Big\uparrow \mathrm{id}_U \otimes \tau & & l \Big\uparrow \mathrm{id}_F \otimes \tau \\ 0 & \longrightarrow & X & \longrightarrow & U \otimes_{E(A)} A & \xrightarrow{\alpha \otimes \mathrm{id}_A} & F \otimes_{E(A)} A \end{array}$$

Since mn bounds $\ker(\alpha \otimes \mathrm{id}_{A/NA})$, we have $(mn)X = 0$. The group X also fits into the exact sequence $\mathrm{Tor}^1(F, A) \to \mathrm{Tor}^1(M, A) \to X \to 0$. Since the group $\mathrm{Tor}^1(F, A) \cong \oplus_I \mathrm{Tor}^1(E(A)/N, A)$ is bounded by k, we obtain $(knm)\mathrm{Tor}^1(M, A) = 0$.

We now show that $(knm)^r\mathrm{Tor}^1(M, A) = 0$ for all right $E(A)$-modules M with $MN^r = 0$. Suppose $r > 0$ and $MN^r = 0$. We have $MN(N^{r-1}) = 0$ and $(M/MN)N = 0$. Thus, $(knm)^{r-1}\mathrm{Tor}^1(MN, A) = 0$ and $(knm)\mathrm{Tor}^1(M/MN, A) = 0$. By considering the exact sequence $\mathrm{Tor}^1(MN, A) \to \mathrm{Tor}^1(M, A) \to \mathrm{Tor}^1(M/MN, A)$, we obtain $(knm)^r\mathrm{Tor}^1(M, A) = 0$ as required.

Finally, since $E(A)$ has finite rank, we have $N^s = 0$ for some $s > 0$. Thus, N^s annihilates all right $E(A)$-modules M. By what has just been shown $(knm)^s\mathrm{Tor}^1(M, A) = 0$. □

A first application of the previous discussion is the following description of torsion-free groups with a finite rank endomorphism ring which are flat as modules over their endomorphism ring:

Theorem 3.5. *The following conditions are equivalent for a torsion-free abelian group A whose endomorphism ring has finite rank*:

a) *A is flat as an $E(A)$-module.*

b) i) *NA is pure in A.*

ii) $\mathrm{Tor}^1(E(A)/N, A) = 0$.

iii) *A/NA is a flat $E(A)/N$-module.*

Proof. a) ⇒ b). Since N is pure as a subgroup of $E(A)$, the right $E(A)$-module $E(A)/N$ has a torsion-free additive group. Thus, multiplication by a prime p of $\mathbf{Z}$ induces a monomorphism $\alpha_p : E(A)/N \to$

$E(A)/N$. Since A is a flat $E(A)$-module, the induced map $\alpha_p \otimes \mathrm{id}_A : (E(A)/N)\otimes_{E(A)} A \to (E(A)/N)\otimes_{E(A)} A$ is a monomorphism too. Since $\alpha_p \otimes \mathrm{id}_A$ is again multiplication by p, and $(E(A)/N)\otimes_{E(A)} A \cong A/NA$, we have that A/NA is torsion-free.

Condition ii) obviously holds if A is a flat $E(A)$-module.

For every right $E(A)/N$-module M there exists a natural isomorphism $\sigma_M : M \otimes_{E(A)} A \to M \otimes_{E(A)/N} A/NA$.

An exact sequence $0 \to U \xrightarrow{\alpha} M$ of right $E(A)/N$-modules induces the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & U \otimes_{E(A)} A & \xrightarrow{\alpha\otimes \mathrm{id}_A} & M \otimes_{E(A)} A \\ & & \wr\downarrow \sigma_U & & \wr\downarrow \sigma_M \\ & & U \otimes_{E(A)/N} A/NA & \xrightarrow{\alpha\otimes \mathrm{id}_{A/NA}} & M \otimes_{E(A)/N} A/NA \end{array}$$

The top row is exact since A is a flat $E(A)$-module. Consequently, the map in the bottom row has to be a monomorphism too. This shows that A/NA is a flat $E(A)/N$-module.

b) $\Rightarrow$ a). By Theorem 3.4, A is almost flat as an $E(A)$-module, and the bound for $\mathrm{Tor}^1(-,A)$ can be chosen of the form $(knm)^r$ where $k\mathrm{Tor}^1(E(A)/N, A) = 0$, n is a bound for $(NA)_*/NA$, and m is the nonnegative integer given by Proposition 3.1. It is obvious that it is possible to choose $k = n = 1$ under the conditions in b). Corollary 3.2 yields that $m = 1$ can be chosen. Therefore, $\mathrm{Tor}^1(M, A) = 0$ for all right $E(A)$-modules M. □

Corollary 3.6. *Let A be a torsion-free abelian group such that $E(A)$ has finite rank. If $E(A)/N$ is hereditary, then A is flat as an $E(A)$-module if and only if NA is pure in A and* $\mathrm{Tor}^1_{E(A)}(E(A)/N, A) = 0$.

As a further application of Theorem 3.4, we give an example of an abelian group A which is not almost as flat as an $E(A)$-module but satisfies that $\mathbf{Q}A$ is a flat $\mathbf{Q}E(A)$-module.

Example 3.7. There exists a torsion-free abelian group A which is not almost flat as an $E(A)$-module, but has the property that

$\mathrm{Tor}\,^1_{E(A)}(M, A)$ is torsion for all right $E(A)$-modules M.

Proof. Let G and H be strongly indecomposable torsion-free abelin groups of finite rank greater than 1 such that $H \subseteq G$, G/H is an infinite torsion group, and $\mathrm{Hom}\,(H, G) \cong \mathbf{Z}$. Such a pair of groups can, for instance, by constructed in the following way.

Given $n \geq 3$, choose mutually disjoint infinite subsets $S_0, \dots, S_n$ of the set of primes. Let $A_i = \langle 1/p \mid p \in S_i \text{ for } i > 0\rangle \subseteq \mathbf{Q}$ and $B_1 = \langle 1/p \mid p \in S_0 \cup S_1\rangle$. Define the groups H and G by $H = [A_1 \oplus \cdots \oplus A_n]/\langle(1, \dots, 1)\rangle_*$ and $G = [B_1 \oplus A_2 \oplus \cdots A_n]/\langle(1, \dots, 1)\rangle_*$. The inclusion map $A_1 \oplus \cdots \oplus A_n \to B_1 \oplus A_2 \oplus \cdots \oplus A_n$ induces a monomorphism from H to G. The fact that G and H are strongly indecomposable is a consequence of [**7**, Theorem 1.2].

Let $\phi : A_1 \oplus \cdots \oplus A_n \to G$ satisfy $\phi(1, \dots, 1) = 0$. By computing types in G, it is easy to see that, for $j > 1$, $\phi(A_j) \subseteq A_j + \langle(1, \dots, 1)\rangle_*$ and $\phi(A_1) \subseteq B_1 + \langle(1, \dots, 1)\rangle_*$ [**7**, Proposition 1.1]. Define $e_j \in A_1 \oplus \cdots \oplus A_n$ to have 1 in the jth-component and 0s elsewhere. It was shown in [**7**] that $\langle e_j + \langle(1, \dots, 1)\rangle_*\rangle_* = A_j e_j + \langle(1, \dots, 1)\rangle_*$. Hence, $\phi|_{A_j}$ operates as multiplication by some integer m_j for $j > 1$. Furthermore, the map $\phi|_{A_1} \in \mathrm{Hom}\,(A_1, B_1)$ is multiplication by some rational number r. Then $0 = \phi(1, \dots, 1) = (r, m_2, \dots, m_n) + \langle(1, \dots, 1)\rangle_*$ implies $r = m_2 = \cdots \in \mathbf{Z}$. This shows $\mathrm{Hom}\,(H, G) \cong \mathbf{Z}$. Clearly $G/H \cong B_1/A_1$.

Once we have such a pair of groups, then $\mathrm{Hom}\,(G, H) = 0$ since every map $\psi : G \to H$ satisfies $\psi|_H \in \mathrm{Hom}\,(H, G) \cong \mathbf{Z}$. If $\psi \neq 0$, then G and H would be quasi-isomorphic groups, which would imply that G/H is finite, a contradiction. Since $IT(G) = \mathrm{type}\,(\mathbf{Z})$ and H is not free, $\mathrm{Ext}\,(H, G) \neq 0$ as was shown in [**5**]. Let the sequence $0 \to G \to A \xrightarrow{\pi} H \to 0$ represent an element of infinite order in $\mathrm{Ext}\,(H, G)$. Such a sequence exists by [**9**]. The group A is strongly indecomposable since G is fully invariant in A, the group H is strongly indecomposable, and the sequence does not quasi-split. Let $\alpha : A \to A$ be the map which is obtained by following π by the embedding of H into G. Since $\mathrm{Hom}\,(H, G) \cong \mathbf{Z}$, the nil-radical, N, of $E(A)$ can be written as $\alpha E(A)$. It was shown in [**6**] that the following holds for an abelian group B such that $\mathbf{Q}N(E(B)) = \beta \mathbf{Q}E(B)$ for some $\beta \in E(B)$ and $N(E(B))^2 = 0$: $\mathbf{Q}B$ is flat as a $\mathbf{Q}E(B)$-module if and only if

$\operatorname{rank}(\operatorname{im}\beta) = r_0(\ker\beta)$. Since these conditions are satisfied in our case, $\mathbf{Q}A$ is a flat $\mathbf{Q}E(A)$-module. However, $(NA)_* = G$ and $NA = H$ so that $(NA)_*/NA \cong B_1/A_1$ is an infinite torsion group. By Theorem 3.4, A is not an almost flat $E(A)$-module. $\square$

We remark that modifying the definition of B_1 by setting $B_1 = \langle 1/p \mid p \in S_1\rangle + \langle 1/q^n \mid n < \omega\rangle$ for some prime $q \in S_0$ produces an example of a group A with $(NA)_*/NA \cong \mathbf{Z}(q^\infty)$ for which $\mathbf{Q}A$ is a flat $\mathbf{Q}E(A)$-module.

On the other hand, we obtain

Proposition 3.8. *Let A be an almost flat torsion-free abelian group. Then $\mathbf{Q}A$ is a flat $\mathbf{Q}E(A)$-module.*

Proof. For a right $E(A)$-module M, consider the exact sequence

$$\operatorname{Tor}^1_{E(A)}(M, A) \to \operatorname{Tor}^1_{E(A)}(M, \mathbf{Q}A) \to \operatorname{Tor}^1_{E(A)}(M, \mathbf{Q}A/A)$$

in which the first and the last term are torsion. Since $\mathbf{Q}A$ is torsion-free and divisible, the same holds for $\operatorname{Tor}^1_{E(A)}(M, \mathbf{Q}A)$. This is only possible if $\operatorname{Tor}^1_{E(A)}(M, \mathbf{Q}A) = 0$.

Let $0 \to M \to N$ be an exact sequence of right $\mathbf{Q}E(A)$-modules. Observe that the natural map $X \otimes_{E(A)} \mathbf{Q}A \to X \otimes_{\mathbf{Q}E(A)} \mathbf{Q}A$ is an isomorphism for all right $\mathbf{Q}E(A)$-modules X since $X \otimes_{E(A)} \mathbf{Q}A$ is torsion-free. Consideration of the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M \otimes_{E(A)} \mathbf{Q}A & \longrightarrow & N \otimes_{E(A)} \mathbf{Q}A \\ & & \wr\downarrow & & \wr\downarrow \\ & & M \otimes_{\mathbf{Q}E(A)} \mathbf{Q}A & \longrightarrow & N \otimes_{\mathbf{Q}E(A)} \mathbf{Q}A \end{array}$$

reveals that $\mathbf{Q}A$ is flat as a $\mathbf{Q}E(A)$-module. $\square$

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